A Two-Parameter Trigonometric Series
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Let us consider the following dual questions.

A. \textit{Given a function } $f(x)$, \textit{find its Fourier series}

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left( a_k \cos kx + b_k \sin kx \right).$$

B. \textit{Given a trigonometric series } $\sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$, \textit{find a function } $f(x)$ \textit{such that the given trigonometric series is its Fourier series (then we can find the sum of the given trigonometric series)}.

Upon learning the theory of Fourier series, we know questions of type A are straightforward but questions of type B are not, for a given trigonometric series may not even be a Fourier series, as

$$\sum_{k=2}^{\infty} \frac{1}{\ln k} \sin kx$$

shows, let alone to find a such function. Nevertheless, our experiences in doing many type A exercises are crucially helpful when tackling type B problems. For instance, if you are asked to find the sum of the trigonometric series

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \cos(kx) \quad (x \in [0, \pi]),$$

your experiences of calculating the Fourier series of $f(x) = x^2$ on $[-\pi, \pi]$, which gives you

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{k=1}^{\infty} (-1)^k \frac{\cos kx}{k^2},$$

and of $f(x) = x$ on $[0, \pi]$, which gives you

$$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2},$$

will help you to figure out that for $x \in [0, \pi]$,

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \cos(kx) = \frac{1}{12}(3x^2 - 6\pi x + 2\pi^2). \quad (1)$$
However, in most such problems, there is only one parameter involved. In this short note, we are going to obtain the sum of a two-parameter trigonometric series, namely the sum of

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \left( 1 - \cos \left( \frac{2k \pi}{l} \right) \right) \cos \left( \frac{2k \pi}{l} n \right)$$

for integer parameters $l$ and $n$, by carefully locating a function $f(x)$ and calculating its Fourier series. How did we find this function? By trying many type A exercises.

First, let us consider the general term $a_n$ for the periodic sequence

$$1, 0, 0, \ldots, 0, 1, 0, 0, \ldots, 0, 1, 0, \ldots$$

(2)

If $l = 2$, we know $a_n = \frac{1}{2} ((-1)^n + 1)$, $n = 0, 1, 2 \ldots$ For the general case, we show

$$a_n = \frac{1}{l} \sum_{k=0}^{l-1} \exp \left( 2\pi i \frac{n}{l} k \right).$$

(3)

In fact, it is a direct corollary of the following simple lemma.

**Lemma.**

$$\sum_{k=0}^{l-1} \exp \left( 2\pi i \frac{n}{l} k \right) = \begin{cases} l, & \text{if } l \mid n; \\ 0, & \text{otherwise}. \end{cases}$$

**Proof.** If $l \mid n$, $\exp(2\pi i \frac{n}{l} k) = 1$, and we get

$$\sum_{k=0}^{l-1} \exp \left( 2\pi i \frac{n}{l} k \right) = l.$$

If $l \nmid n$, $\exp((2\pi i \frac{n}{l} k) \neq 1$, and we have

$$\sum_{k=0}^{l-1} \exp \left( 2\pi i \frac{n}{l} k \right) = \sum_{k=0}^{l-1} \left( \exp \left( 2\pi i \frac{n}{l} \right) \right)^k = \frac{1 - (\exp(2\pi i \frac{n}{l}))^l}{1 - \exp(2\pi i \frac{n}{l})} = 0. \blacksquare$$

Next, let us consider $a_n$ from a different angle. Define a continuous function $f(x)$ on $[0, l]$ as

$$f(x) = \begin{cases} -x + 1, & \text{for } 0 \leq x \leq 1; \\ 0, & \text{for } 1 \leq x \leq l - 1; \\ x + 1 - l, & \text{for } l - 1 \leq x \leq l. \end{cases}$$

It is easy to see that $f(x)$ is the linear interpolation of the sequence in (2). Now, extend this $f(x)$, first evenly to $[-l, l]$, then periodically to the whole real line $R$. Denote the extended function as $\tilde{f}(x)$. The famous Dirichlet-Jordan theorem (see [1]) assures the convergence of its Fourier series to $\tilde{f}(x)$; hence $a_n = \tilde{f}(n)$. 
Now, let us calculate the Fourier series of the even function \( \tilde{f}(x) \). By the Euler-Fourier formulas, we have

\[
\alpha_0 = \frac{2}{l} \int_0^l \tilde{f}(x) \, dx = \frac{2}{l} \left( \int_0^l (1 - x) \, dx + \int_0^l (x + 1 - l) \, dx \right)
= \frac{2}{l} \left( \frac{1}{2} + \frac{1}{2} \right) = \frac{2}{l},
\]

and

\[
\alpha_m = \frac{2}{l} \int_0^l \tilde{f}(x) \cos \left( \frac{m\pi}{l} x \right) \, dx
= \frac{2}{l} \left\{ \int_0^1 (-x + 1) \cos \left( \frac{m\pi}{l} x \right) \, dx + \int_{l-1}^l (x + 1 - l) \cos \left( \frac{m\pi}{l} x \right) \, dx \right\}. \tag{4}
\]

It is easy to check that

\[
\int_0^1 \cos \left( \frac{m\pi}{l} x \right) \, dx = \frac{l}{m\pi} \sin \left( \frac{m\pi}{l} \right)
\]

and

\[
\int_{l-1}^l \cos \left( \frac{m\pi}{l} x \right) = \frac{l}{m\pi} (-1)^{m+2} \sin \left( \frac{m\pi}{l} \right),
\]

and by integrating by parts, we also have the following:

\[
\int_0^1 x \cos \left( \frac{m\pi}{l} x \right) \, dx = \frac{l}{m\pi} \left( \sin \left( \frac{m\pi}{l} \right) + \frac{l}{m\pi} \cos \left( \frac{m\pi}{l} \right) - \frac{l}{m\pi} \right),
\]

\[
\int_{l-1}^l x \cos \left( \frac{m\pi}{l} x \right) \, dx = \frac{l}{m\pi} \left\{ (-1)^{m+1} (l - 1) \sin \left( \frac{m\pi}{l} \right) + \frac{l}{m\pi} (-1)^m \left( 1 - \cos \left( \frac{m\pi}{l} \right) \right) \right\}.
\]

Now putting all these into (4), we get that

\[
\alpha_m = \begin{cases} 
0, & \text{for } m \text{ odd;} \\
\frac{l}{k^2\pi^2} \left( 1 - \cos \left( \frac{2k\pi}{l} \right) \right), & \text{for } m = 2k.
\end{cases}
\]

Therefore,

\[
\tilde{f}(x) = \frac{\alpha_0}{2} + \sum_{m=1}^{\infty} \alpha_m \cos \left( \frac{m\pi}{l} x \right)
= \frac{1}{l} + \frac{l}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \left( 1 - \cos \left( \frac{2k\pi}{l} \right) \right) \cos \left( \frac{2k\pi}{l} x \right).
\]
Thus,
\[ a_n = \tilde{f}(n) = \frac{1}{l} + \frac{l}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \left(1 - \cos \left(\frac{2k\pi}{l}\right)\right) \cos \left(\frac{2k\pi}{l}n\right). \]

Comparing the above with (3), we have the following result.

**Theorem.**
\[ \sum_{k=1}^{\infty} \frac{1}{k^2} \left(1 - \cos \left(\frac{2k\pi}{l}\right)\right) \cos \left(\frac{2k\pi}{l}n\right) = \frac{\pi^2}{l^2} \sum_{k=1}^{l-1} \exp \left(2\pi i \frac{n}{k}\right), \]

or equivalently,
\[ \sum_{k=1}^{\infty} \frac{1}{k^2} \left(1 - \cos \left(\frac{2k\pi}{l}\right)\right) \cos \left(\frac{2k\pi}{l}n\right) = \begin{cases} \frac{l-1}{l^2}\pi^2, & \text{for } l \mid n; \\ \frac{1}{l^2}\pi^2, & \text{otherwise}. \end{cases} \] (5)

**Remarks.**

1. Letting \( n = 0 \) and \( l = 2 \), we get the well-known identity
\[ \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^2} = \frac{\pi^2}{12}, \]
which in turn implies the other well-known identity
\[ \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \]

2. Letting \( n = 0 \) in the theorem, and using the fact
\[ \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \]
we can easily get
\[ \sum_{k=1}^{\infty} \frac{1}{k^2} \cos \left(\frac{2k\pi}{l}\right) = \frac{\pi^2}{6} - \frac{l-1}{l}\pi^2, \]
which, in essence, is a different version of what we got in (1). In contrast to there, where we need two Fourier expansions, here only one function \( f(x) \) is needed to derive a much more general result (5).

3. The lemma, although simple, is useful in classical number theory. For instance, by this very lemma, the number of solutions of \( f(x_1, \ldots, x_n) \equiv N \mod l \), \( 0 \leq x_j \leq l-1 \), can be expressed as
\[ \frac{1}{l} \sum_{x_1=0}^{l-1} \ldots \sum_{x_n=0}^{l-1} \sum_{k=0}^{l-1} \exp \left(2\pi i \frac{f(x_1, \ldots, x_n) - N}{l}k\right). \]

For this and related rich discussions in classical number theory, see [2].
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References

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