Odd-like (Even-like) Functions on \((a, b)\)

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The integral \(\int_0^{\pi/2} \cos(\pi \sin^2 \theta) \, d\theta\), which appeared in the famous Ramanujan’s Notebooks [1, p. 56], appears difficult to evaluate and, therefore, served to motivate this note.

For a continuous function \(f(x)\) on \([-a, a]\), calculus students know that
\[
\int_{-a}^{a} f(x) \, dx = 0
\]
when \(f(x)\) is an odd function (i.e., \(f(-x) = -f(x)\) for all \(x\)). Therefore, it would be useful for students to know when a function on \((a, b)\) can be translated to describe an odd function on a corresponding interval \((-c, c)\).

Suppose \(f(x)\) is defined on the interval \((a, b)\). Then (see Figure 1), \(f(a + b - x) = -f(x)\) for all \(x\) in \((a, b)\) is equivalent to \(g(x) = f(x + \frac{a+b}{2})\) being an odd function on the interval \((-\frac{b-a}{2}, \frac{b-a}{2})\), since
\[
g(-x) = f\left(-x + \frac{a+b}{2}\right) = f\left[a + b - \left(x + \frac{a+b}{2}\right)\right]
= -f\left(x + \frac{a+b}{2}\right) = -g(x)
\]
for all \(x \in (-\frac{b-a}{2}, \frac{b-a}{2})\).

![Figure 1](image)

**Definition.** A function \(f(x)\) is odd-like on the interval \((a, b)\) if \(f(a + b - x) = -f(x)\) for all \(x\) in \((a, b)\). (Thus, \(f(x)\) is odd if and only if it is odd-like on \((-a, a)\) for all \(a > 0\). And \(f(x)\) is odd on \((-a, a)\) if and only if it is odd-like on \((-a, a)\).)
If \( f(x) \) is an integrable, odd-like function on \((a, b)\), then the translated function 
\( g(x) = f(x + \frac{a+b}{2}) \) is an odd function on \((-\frac{b-a}{2}, \frac{b-a}{2})\) and so
\[
\int_{a}^{b} f(x) \, dx = \int_{\frac{a+b}{2}}^{b} g(x) \, dx = 0.
\]
This also can be verified directly by letting \( x = a + b - u \) and observing that
\[
\int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(a + b - u) \, du = \int_{b}^{a} -f(u) \, du = -\int_{a}^{b} f(x) \, dx.
\]

**Examples.** Each function below is odd-like on the indicated interval.

(i) \( \cos nx \) on \((0, \pi)\) with \( n \) odd, and \( \sin nx \) on \((0, \pi)\) with \( n \) even.
(ii) \( \frac{\sin(mx)}{\sin(nx)} \) on \((0, \pi)\) for positive integers \( m \) and \( n \), when \( m + n \) is odd.
(iii) \( \cos(n\pi \sin^2 x) \) and \( \cos(n\pi \cos^2 x) \) on \((0, \pi/2)\) when \( n \) is odd.
(iv) \( \sin(n\pi \sin^2 x) \) and \( \sin(n\pi \cos^2 x) \) on \((0, \pi/2)\) when \( n \) is even.

The odd-like property for (i) and (ii) can be verified directly. For \( \cos(n\pi \sin^2 x) \) in (iii), observe that
\[
\cos(n\pi \sin^2 \left(\frac{\pi}{2} - x\right)) = \cos(n\pi \cos^2 x) = \cos(n\pi (1 - \sin^2 x)) = \cos(n\pi - n\pi \sin^2 x) = -\cos(n\pi \sin^2 x).
\]
Verification of the remaining functions follows similarly. Each of the above functions has \( \int_{a}^{b} f(x) \, dx = 0 \) on the indicated interval \((a, b)\). Thus,
\[
\int_{0}^{\pi/2} \cos(\pi \sin^2 \theta) \, d\theta = 0
\]
is the value of the integral that motivated this note.

For any given function \( f(x) \) on \((a, b)\), we can construct an odd-like function on \((a, b)\). Figure 2 illustrates why \( F(x) = f(a + b - x) - f(x) \) is an odd-like function on \((a, b)\).
Example. The function $\ln\left(\frac{\pi-x}{x}\right) = \ln(\pi - x) - \ln x$ is odd-like on $(\frac{\pi}{4}, \frac{3\pi}{4})$ since $\ln\left(\frac{\pi-(\pi-x)}{x}\right) = -\ln\left(\frac{\pi-x}{x}\right)$. Thus, $\int_{\frac{3\pi}{4}}^{\pi/4} \ln\left(\frac{\pi-x}{x}\right) dx = 0$.

Proposition. If $f(x)$ is an integrable function and $f(x) \neq -f(a + b - x)$ for any $x$ in $(a, b)$, then

$$\int_{a}^{b} \frac{f(x)}{f(x) + f(a + b - x)} dx = \int_{a}^{b} \frac{f(a + b - x)}{f(x) + f(a + b - x)} dx = \frac{b - a}{2}.$$

Proof. The function $\frac{f(x) - f(a + b - x)}{f(x) + f(a + b - x)}$ is an odd-like function on $(a, b)$, and so the first equality holds. (This also follows from the substitution $x = a + b - u$.) And since $\frac{f(x)}{f(x) + f(a + b - x)} = 1 - \frac{f(a + b - x)}{f(x) + f(a + b - x)}$, it follows that $2 \int_{a}^{b} \frac{f(x)}{f(x) + f(a + b - x)} dx = b - a$. 

Examples.

(i) $\int_{\pi/2}^{3\pi/2} \frac{\ln x}{\ln x + \ln(4\pi - x)} dx = \pi$.

(ii) $\int_{\pi/2}^{3\pi/2} \frac{\sin^r x}{\sin^r x + \cos^r x} dx = \frac{\pi}{12}$ for any real number $r$.

(iii) $\int_{\pi/2}^{3\pi/2} \frac{x^r}{x^r + (\pi - x)^r} dx = \frac{\pi}{6}$ for any real number $r$.

The counterpart of odd-like functions are even-like functions defined as follows.

Definition. A function $f(x)$ is even-like on the interval $(a, b)$ if $f(a + b - x) = f(x)$ for all $x$ in $(a, b)$. (Thus, a function $f(x)$ is even if and only if it is even-like on $(-a, a)$ for all $a > 0$. And $f(x)$ is even on $(-a, a)$ if and only if it is even-like on $(-a, a)$.)

The reasoning for odd-like functions can be used to show that $f(x)$ is even-like on $(a, b)$ if and only if $g(x) = f(x + \frac{a+b}{2})$ is an even function on $(-\frac{b-a}{2}, \frac{b-a}{2})$. If $f(x)$ is an integrable, even-like function on $(a, b)$, then $g(x)$ is even on $(-\frac{b-a}{2}, \frac{b-a}{2})$ and so

$$\int_{a}^{b} f(x) dx = \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} g(x) dx = 2 \int_{0}^{\frac{b-a}{2}} g(x) dx = 2 \int_{a-b}^{b} f(x) dx.$$

Note also that for every function $f(x)$, the function $F(x) = f(a + b - x) + f(x)$ is even-like on $(a, b)$.

Examples.

(i) $\cos nx$ is even-like on $(0, \pi)$ when $n$ is even, and $\sin nx$ is even-like on $(0, \pi)$ when $n$ is odd.

(ii) $\ln(x(4\pi - x)) = \ln x + \ln(4\pi - x)$ is even-like on the interval $(\pi, 3\pi)$.

Students may find it instructive to prove the following.
Exercises.

1. If $f(x)$ is differentiable and odd-like on $(a, b)$, then $f'(x)$ is even-like on $(a, b)$.
2. If $f(x)$ is differentiable and even-like on $(a, b)$, then $f'(x)$ is odd-like on $(a, b)$.
3. If $f(x)$ is odd-like (even-like) on $(a, b)$ and $n$ is a positive even number, then the $n$th derivative $f^{(n)}(x)$ is even-like (odd-like) on $(a, b)$.
4. If $f(x)$ is a continuous, odd-like function on $(a, b)$, then any antiderivative of $f(x)$ is even-like on $(a, b)$.
5. If $f(x)$ is a continuous, even-like function on $(a, b)$, then there is exactly one antiderivative of $f(x)$ that is odd-like on $(a, b)$.
6. The set of all odd-like functions on $(a, b)$ is a vector space, as is the set of all even-like functions on $(a, b)$.

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Reference


Keyboard Inequalities

Monte Zerger (mjzerger@adams.edu) observes that the $F$, $F^\#$, and $G$ keys on the piano demonstrate mathematical truth. The Pythagoreans (“all is number”) found that the ratio of the frequencies in a perfect fifth is 3 to 2 and in a fourth is 4 to 3. We do not hear these when we strike piano keys because, to avoid having to have twelve pianos around the house, each tuned to a different key, we have adopted well-tempered tuning where the ratio of frequencies for any half-tone is $2^{1/12}$ to 1. So, in any octave we have

<table>
<thead>
<tr>
<th>Interval</th>
<th>Well-tempered Tuning</th>
<th>Pythagorean Tuning</th>
<th>Mean of the numbers 1 and 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F : C$</td>
<td>$2^{5/12}$ = 1.335...</td>
<td>4/3 = 1.333...</td>
<td>Harmonic: 4/3</td>
</tr>
<tr>
<td>$F^# : C$</td>
<td>$2^{6/12}$ = 1.414...</td>
<td>729/512 = 1.424...</td>
<td>Geometric: $\sqrt{2}$</td>
</tr>
<tr>
<td>$G : C$</td>
<td>$2^{7/12}$ = 1.498...</td>
<td>3/2 = 1.5</td>
<td>Arithmetic: 3/2</td>
</tr>
</tbody>
</table>

The three keys are telling us that

Harmonic mean $\leq$ Geometric mean $\leq$ Arithmetic mean.