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A Calculation of $\int_0^\infty e^{-x^2}\,dx$

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If you are looking for a way to tie together different strands of material your students see in their second semester of a standard three-semester calculus course, the result in the title will serve you very well. Of course, the result is

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

Students find it surprising (where does the π come from?) and meaningful (when they learn its relationship to the bell curve). You can also use this significant improper integral as an entry into a discussion of approximations of definite integrals over finite regions.

There are numerous ways to evaluate the integral in the title. In just a few minutes in the library, I located two evaluations based on Wallis' infinite product expansion of π, [**8**] and [**5**]; a calculation using contour integration in the complex plane, [**1**]; a reduction to Euler's integral of the first kind, [**2**]; two evaluations using differentiation under the integral sign, [**7**] and [**9**]; and a calculation based on solids of revolution, [**3**]. You can find the standard computation based on a double integral in polar coordinates in almost any statistics or calculus book. The integral is commonly associated with Gauss, although he credits Laplace with its discovery and publication in 1805. Euler was working with similar integrals thirty years earlier, but he seems to have missed its exact formulation. By 1813, the result appears to have been well known. See [**6**] for a more detailed history.

Few of the techniques for evaluating this integral are easily accessible to a begining student. Here is a technique that students can appreciate. Although the method may not be new (the main idea is already in [**3**] and [**4**]) our variation of it is sufficiently elementary that it can be presented to students.

Let I denote the value of the integral of interest. A comparison with e^{-x} confirms that I is finite. Let's start by finding the volume V of the solid of revolution made by revolving the graph of e^{-x^2} for $x \in [0, \infty)$ about the y-axis (Figure 1).

Using the usual technique of nested cylindrical shells, we obtain

$$V = \int_0^\infty 2\pi x e^{-x^2}\,dx = \pi.$$

Now let's compute V a second time. This time take cross sections parallel to the x-axis (Figure 2).

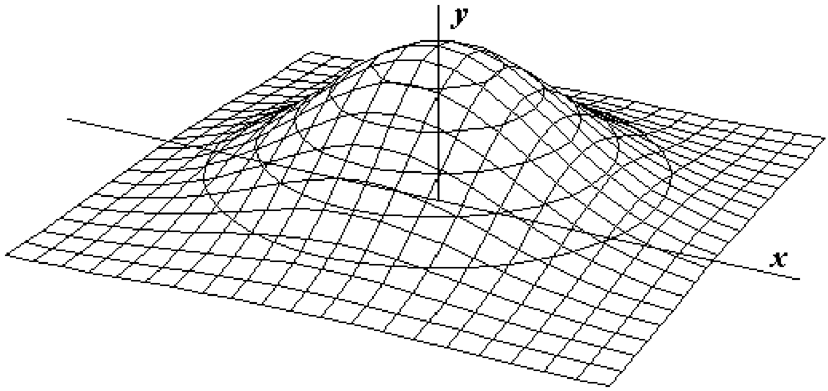

Figure 1. Graph of e^{-x^2} rotated about y-axis.

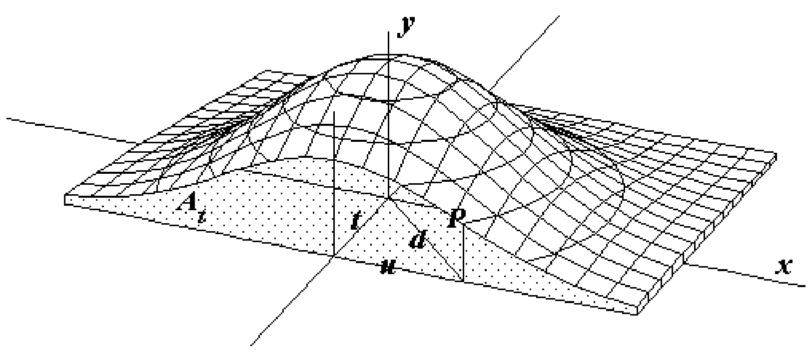

Figure 2. Cross section at distance t from xy-plane.

Denoting the area of the cross section at distance t from the x-axis by A_t, we get $V = 2\int_0^\infty A_t\,dt$. In order to compute A_t, first find the equation for the graph of the cross section. Consider the point P on the cross section. Since the surface is radially symmetric, we can find P's height by rotating back to the value $d = \sqrt{u^2+t^2}$ on the x-axis. Thus, P's height is $e^{-(\sqrt{u^2+t^2})^2}$, and

$$A_t = 2\int_0^\infty e^{-\left(u^2+t^2\right)}\,du = 2\int_0^\infty e^{-u^2}e^{-t^2}\,du = 2e^{-t^2}\int_0^\infty e^{-u^2}\,du = 2e^{-t^2}I.$$

Therefore,

$$V = 2\int_0^\infty A_t\,dt = 2\int_0^\infty 2e^{-t^2}I\,dt = 4I\int_0^\infty e^{-t^2}\,dt = 4I^2.$$

Equating the two expressions for V, we obtain $\pi = 4I^2$ and the result follows.

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A Simple Introduction to *e*

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The idea for this note was suggested by Dimitric's recent paper [**1**] that discusses a way to introduce the Euler number ***e*** in a heuristic way. Given the volume and breadth of literature on the subject, what follows is a very simplistic way of introducing ***e*** that is easily understood by freshmen.

Let A_2 denote the area under the hyperbola $y = \frac{1}{x}$ between $x = 1$ and $x = 2$. Then (Figure 1) A_2 is greater than the area $\frac{1}{2}$ of the "inner" rectangle having height $\frac{1}{2}$, and

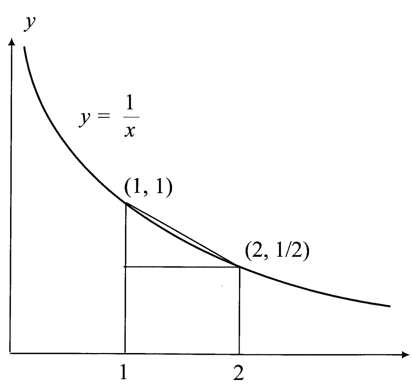

Figure 1.

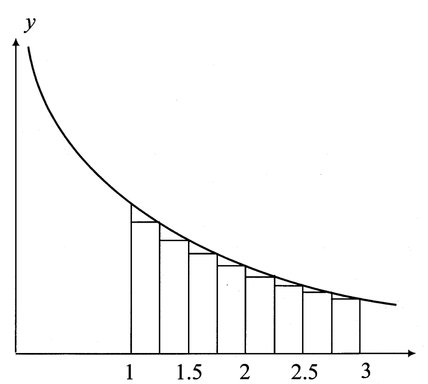

Figure 2.

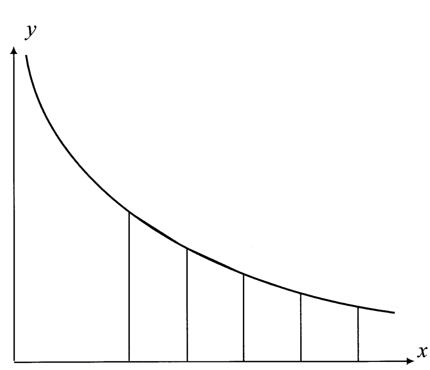

Figure 3.