The 83rd William Lowell Putnam Mathematical Competition 2022

A1 Determine all ordered pairs of real numbers (a, b) such that the line $y = ax + b$ intersects the curve $y = \ln(1 + x^2)$ in exactly one point.

A2 Let n be an integer with $n \geq 2$. Over all real polynomials $p(x)$ of degree n, what is the largest possible number of negative coefficients of $p(x)^2$?

A3 Let p be a prime number greater than 5. Let $f(p)$ denote the number of infinite sequences $a_1, a_2, a_3, \ldots$ such that $a_n \in \{1, 2, \ldots, p-1\}$ and $a_n a_{n+2} \equiv 1 + a_{n+1} \pmod{p}$ for all $n \geq 1$. Prove that $f(p)$ is congruent to 0 or 2 (mod 5).

A4 Suppose that $X_1, X_2, \ldots$ are real numbers between 0 and 1 that are chosen independently and uniformly at random. Let $S = \sum_{i=1}^{k} X_i/2^i$, where k is the least positive integer such that $X_k < X_{k+1}$, or $k = \infty$ if there is no such integer. Find the expected value of S.

A5 Alice and Bob play a game on a board consisting of one row of 2022 consecutive squares. They take turns placing tiles that cover two adjacent squares, with Alice going first. By rule, a tile must not cover a square that is already covered by another tile. The game ends when no tile can be placed according to this rule. Alice's goal is to maximize the number of uncovered squares when the game ends; Bob's goal is to minimize it. What is the greatest number of uncovered squares that Alice can ensure at the end of the game, no matter how Bob plays?

A6 Let n be a positive integer. Determine, in terms of n, the largest integer m with the following property: There exist real numbers $x_1, \ldots, x_{2n}$ with $-1 < x_1 < x_2 < \cdots < x_{2n} < 1$ such that the sum of the lengths of the n intervals

$$[x_1^{2k-1}, x_2^{2k-1}],\ [x_3^{2k-1}, x_4^{2k-1}], \ldots, [x_{2n-1}^{2k-1}, x_{2n}^{2k-1}]$$

is equal to 1 for all integers k with $1 \leq k \leq m$.