2022 Session A

A1. Determine all ordered pairs of real numbers (a, b) such that the line $y = ax+b$ intersects the curve $y = \ln(1 + x^2)$ in exactly one point.

Answer: Those (a, b) for which

- $|a| \geq 1$, or
- $a = b = 0$, or
- $0 < |a| < 1$ and $b < \ln[2(1 - \sqrt{1 - a^2})/a^2] + \sqrt{1 - a^2} - 1$, or
- $0 < |a| < 1$ and $b > \ln[2(1 + \sqrt{1 - a^2})/a^2] - \sqrt{1 - a^2} - 1$.

Solution: Notice that upon reflection about the y-axis, the curve $y = \ln(1+x^2)$ is unchanged, and the line $y = ax + b$ becomes the line $y = -ax + b$. Thus, (a, b) is a solution if and only if $(-a, b)$ is a solution, so we need only verify the claim for $a \geq 0$.

Define $h(x) = \ln(1 + x^2) - ax - b$, with derivative $h'(x) = 2x/(1 + x^2) - a$; we must determine the pairs (a, b) for which $h(x) = 0$ for exactly one x.

By L'Hôpital's rule, $\lim_{x\to\pm\infty}[\ln(1 + x^2) - b]/x = 0$, so it follows that $\lim_{x\to\pm\infty} h(x)/x = -a$. Thus, if $a > 0$, then $\lim_{x\to-\infty} h(x) = \infty$ and $\lim_{x\to\infty} h(x) = -\infty$. Also, $x^2 - 2x + 1 = (x - 1)^2 \geq 0$ implies that $h'(x) \leq 1 - a$, with equality only when $x = 1$.

If $a \geq 1$, then $h'(x) \leq 0$, with equality only if $a = 1$ and $x = 1$. It follows that h is strictly decreasing, and from this, the limits we proved, and the continuity of h, we conclude that h takes on every value (including 0) exactly once. Thus, (a, b) is a solution for all $a \geq 1$ and all b.

If $0 < a < 1$, then $h'(x) = 0$ for two values of x, which from the quadratic formula are $x_\pm = (1 \pm \sqrt{1 - a^2})/a$. Notice that $h'(x) < 0$ for $x < x_-$ and $x > x_+$, and $h'(x) > 0$ for $x_- < x < x_+$. Thus, h takes on every value between $h(x_-)$ and $h(x_+)$ three times, takes on the values $h(x_-)$ and $h(x_+)$ twice each, and takes on all other values once. Since $h(x_-) < h(x_+)$, the solutions are those (a, b) for which $h(x_-) > 0$ or $h(x_+) < 0$; equivalently, $b < \ln(1 + x_-^2) - ax_-$ or $b > \ln(1 + x_+^2) - ax_+$, which upon substitution yield the formulas claimed above.

If $a = 0$, then $h'(x) < 0$ for $x < 0$ and $h'(x) > 0$ for $x > 0$; also, $\lim_{x\to\pm\infty} h(x) = \infty$. It follows that $h(x)$ takes on every value greater than $h(0)$ twice, takes on the value $h(0)$ once, and does not take on values less than $h(0)$. Since $h(0) = -b$, we conclude that $(0, 0)$ is the only solution with $a = 0$.

A2. Let n be an integer with $n \geq 2$. Over all real polynomials $p(x)$ of degree n, what is the largest possible number of negative coefficients of $p(x)^2$?

Answer: $2n - 2$.

Solution: Let R be a large real number and define

$$p(x) = Rx^n - x^{n-1} - x^{n-2} - \cdots - x^2 - x + R = R(x^n + 1) - \frac{x^n - x}{x - 1}.$$

Then

$$p(x)^2 = R^2(x^{2n} + 2x^n + 1) - 2R(x^n + 1)\frac{x^n - x}{x - 1} + \frac{(x^n - x)^2}{(x - 1)^2}.$$

Thus we see that the coefficients of x^{2n}, x^n and x^0 will be quadratic polynomials in R with positive leading coefficient, and hence will be positive for all sufficiently large R. Since

$$(x^n + 1)\frac{x^n - x}{x - 1} = \frac{x^{2n} - x}{x - 1} - x^n = x^{2n-1} + x^{2n-2} + x^{n+1} + x^{n-1} + \cdots + x,$$

we see that the coefficients of all powers of x other than 0, n, and $2n$ take the form $-2R + c$, where c depends on the power but not on R. Thus for large enough R (in fact $2R > n - 2$ suffices), these coefficients will all be negative. Thus, $p(x)$ has $2n - 2$ negative coefficients for large R.

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where without loss of generality we assume $a_n > 0$, and suppose $p(x)^2$ has $2n - 1$ negative coefficients. Since

$$p(x)^2 = a_n^2 x^{2n} + 2a_n a_{n-1} x^{2n-1} + \cdots + 2a_0 a_1 x + a_0^2,$$

the coefficients of x^{2n} and x^0 must both be nonnegative. Thus all other coefficients must be negative. In particular a_0 and a_1 must have opposite signs (and be nonzero). Thus, since $n \geq 2$, there is a largest integer k with $k < n$ such that $a_k > 0$. Then the coefficient of x^{n+k} is

$$2a_k a_n + \sum_{m=k+1}^{n-1} a_m a_{n+k-m}.$$

The first term is positive since a_k and a_n are both positive, and all the terms in the sum are nonnegative since maximality of k implies a_m and a_{n+k-m} are both nonpositive. Thus, the coefficient of x^{n+k} is also positive, a contradiction.

Thus, the maximum is $2n - 2$.

A3. Let p be a prime number greater than 5. Let $f(p)$ denote the number of infinite sequences $a_1, a_2, a_3, \ldots$ such that $a_n \in \{1, 2, \ldots, p-1\}$ and $a_n a_{n+2} \equiv 1 + a_{n+1} \pmod p$ for all $n \geq 1$. Prove that $f(p)$ is congruent to 0 or 2 (mod 5).

Solution 1: For $n \geq 1$, we must have $a_{n+1} \not\equiv -1 \pmod p$, since a_n and a_{n+2} are not divisible by p. Therefore, we can cancel factors of a_n and (for $n \geq 2$) $a_n + 1$ to derive the following (mod p) congruences:

$$\begin{aligned}
a_3 &\equiv \frac{1+a_2}{a_1}, \\
a_4 &\equiv \frac{1+a_3}{a_2} \equiv \frac{1+a_1+a_2}{a_1 a_2}, \\
a_5 &\equiv \frac{1+a_4}{a_3} \equiv \frac{1+a_1}{a_2}, \\
a_6 &\equiv \frac{1+a_5}{a_4} \equiv a_1, \\
a_7 &\equiv \frac{1+a_6}{a_5} \equiv a_2.
\end{aligned}$$

Since a_n and a_{n+1} determine a_{n+2}, by induction the sequence is periodic modulo p with period 5, and hence periodic with period 5; in other words, $a_{n+5} = a_n$ for all $n \geq 1$.

Next, we assert that since 5 is prime, the period-5 infinite sequences $(a_1, a_2, a_3, \ldots)$, $(a_2, a_3, a_4, \ldots)$, $(a_3, a_4, a_5, \ldots)$, $(a_4, a_5, a_1, \ldots)$, and $(a_5, a_1, a_2, \ldots)$ are all distinct unless $a_1 = a_2 = a_3 = a_4 = a_5$. For example, if the third and fifth sequences are identical, then $a_3 = a_5 = a_2 = a_4 = a_1$. Therefore, each non-constant period-5 cycle that meets the conditions of the problem corresponds to 5 infinite sequences that meet the conditions, so the total number of allowed sequences that are non-constant is a multiple of 5.

It remains only to show that the number of allowed constant sequences – those with $a_n = c$ for some $c \in \{1, 2, \ldots, p-1\}$ and all $n \geq 1$ – is congruent to 0 or 2 modulo 5. The number of such sequences is the number of such c for which $c^2 \equiv 1 + c \pmod p$. Multiplying by 4 and then adding $1 - 4c$ to each side yields $(2c-1)^2 \equiv 5 \pmod p$. Since $p > 5$ is prime, we can divide by 4 modulo p, so every solution of the latter congruence is a solution of the former congruence. And by the same properties of p, there are either 0 or 2 square roots of 5 modulo p, and if there are 2, each yields a corresponding value of c. (Each of these values of c is nonzero because $(-1)^2 \not\equiv 5 \pmod p$.)

Solution 2: We say that an ordered pair of integers (a, b) has property P if $a, b \in \{1, 2, \ldots, p-2\}$ and $a + b \neq p - 1$. We claim that if (a_1, a_2) has property P, then it is part of a unique infinite sequence $a_1, a_2, a_3, \ldots$ that satisfies the conditions of the problem. The justification uses the following lemma.

Lemma. *The conditions in the problem statement on a_n, a_{n+1}, and a_{n+2} imply that*

$$\begin{aligned}
a_{n+2} \equiv -1 \pmod p &\iff a_n + a_{n+1} \equiv -1 \pmod p \qquad \textit{and} \\
a_n \equiv -1 \pmod p &\iff a_{n+1} + a_{n+2} \equiv -1 \pmod p.
\end{aligned}$$

Proof. The first equivalence follows from $a_n(a_{n+2}+1) \equiv 1 + a_{n+1} + a_n \pmod p$, and the fact that we can divide by a_n modulo p since a_n is not a multiple of p. The second equivalence is proved similarly. □

If (a_n, a_{n+1}) has property P, then $a_n + a_{n+1} \not\equiv -1 \pmod{p}$. Since a_n and $1 + a_{n+1}$ are not multiples of p, there is a unique $a_{n+2} \in \{1, \ldots, p-1\}$ such that $a_n a_{n+2} \equiv 1 + a_{n+1} \pmod{p}$, and $a_{n+2} \neq p-1$ by the first part of the lemma. Further, by the second part of the lemma, $a_{n+1} + a_{n+2} \neq p-1$. Thus, (a_{n+1}, a_{n+2}) has property P, and is uniquely determined by (a_n, a_{n+1}) and the conditions in the problem statement. Our claim follows by induction on n.

Next, suppose that the sequence $a_1, a_2, a_3, \ldots$ that satisfies the conditions of the problem. For $n \geq 1$, since a_n and a_{n+2} are not multiples of the prime number p, neither is $a_n a_{n+2}$, so $a_{n+1} \not\equiv -1 \pmod{p}$. In particular, a_2, a_3, and a_4 are not congruent to -1 modulo p. Then by the first part of the lemma, $a_1 + a_2$ and $a_2 + a_3$ are not congruent to -1 modulo p. Then by the second part of the lemma, $a_1 \not\equiv -1 \pmod{p}$. In particular, (a_1, a_2) must have property P.

Therefore, $f(p)$ is the number of ordered pairs (a_1, a_2) that have property P. There are $p-2$ possible values of a_1, and for each such value, there are $p-3$ values of a_2 consistent with property P. Then $f(p) = (p-2)(p-3) = p^2 - 5p + 6$, and $f(p) \equiv p^2 + 1 \pmod{5}$. Since $p \neq 5$ and p is prime, p is congruent to 1, 2, 3, or 4 modulo 5. In all cases, $p^2 \equiv \pm 1 \pmod{5}$, and the conclusion of the problem follows.

A4. Suppose that $X_1, X_2, \ldots$ are real numbers between 0 and 1 that are chosen independently and uniformly at random. Let $S = \sum_{i=1}^{k} X_i/2^i$, where k is the least positive integer such that $X_k < X_{k+1}$, or $k = \infty$ if there is no such integer. Find the expected value of S.

Answer: $2\sqrt{e} - 3$.

Solution 1: The approach is a "first-step analysis" that expresses the expected value for the sum starting with X_1 in terms of the expected value for the sum starting with X_2. Let $F(x)$ denote the expected value of the quantity that is S if $X_1 \le x$, and 0 if $X_1 > x$. The problem statement asks for $F(1)$. Since $0 \le S \le 1$, it follows that $|F(y) - F(x)| \le |y - x|$, and in particular F is continuous.

Conditioning on $X_1 = u$,

$$F(x) = (1-x) \cdot 0 + \int_0^x (u/2 + F(u)/2)\, du = \frac{x^2}{4} + \frac{1}{2}\int_0^x F(u)\, du.$$

Since F is continuous, the right side is differentiable, and then so is F. This implies the differential equation

$$F'(x) - F(x)/2 = x/2, \quad F(0) = 0.$$

The simplest particular solution for the nonhomogeneous component is $F_p(x) = -x - 2$ (which is found by substituting $ax + b$ and solving). The general homogeneous solution to $F' - F/2 = 0$ is $F_h(x) = ce^{x/2}$, so in all the solution has the form

$$F(x) = -x - 2 + ce^{x/2}.$$

Plugging in the initial condition gives $c = 2$, so finally, $F(1) = -3 + 2\sqrt{e}$.

Solution 2: This approach calculates the expected value by summing an explicit series conditioned on the position of the first increase in the sequence (note that such a position exists with probability 1). We will also need the order statistics for uniform random variables: the expected value of the j-th largest among $X_1, \ldots, X_n$ is $\frac{n+1-j}{n+1}$.

Now suppose that the first increase occurs at position k, so $X_1 \ge X_2 \ge \cdots \ge X_k$ and $X_k < X_{k+1}$. Denote this event by I_k. The probability of this occurring is $P(I_k) = \frac{1}{k!} \cdot \frac{k}{k+1}$ due to symmetry and independence; the first factor imposes the monotonic order on $X_1, \ldots, X_k$, and the second factor comes from the fact that X_{k+1} can lie in any of the $k+1$ possible order positions (relative to $X_1, \ldots, X_k$) except for the smallest (i.e. the $(k+1)$-st largest).

We next calculate the expected value of each X_j for $1 \le j \le k$, conditional on the event I_k. The claim is that

$$E[X_j \mid I_k] = 1 - \frac{j(k+1)}{k(k+2)}.$$

This is shown by conditioning on the k possible order positions for X_{k+1}. In particular, if it is in one of the j largest positions, then the expected value of X_j is shifted down to the $(j+1)$-st largest position, namely $\frac{k+2-j-1}{k+2}$, but if it is in a smaller position, the expected value of X_j is $\frac{k+2-j}{k+2}$. Overall,

$$E\left[X_j \mid I_k\right] = \frac{j}{k} \cdot \frac{k+1-j}{k+2} + \frac{k-j}{k} \cdot \frac{k+2-j}{k+2},$$

which simplifies to the claimed expression.

We can now evaluate the expected value of the sum in the case of I_k. Using the geometric series, and its relative

$$\sum_{j=1}^{k} jx^j = \frac{x\left(kx^{k+1} - (k+1)x^k + 1\right)}{(x-1)^2},$$

we find that

$$\begin{aligned} E[S \mid I_k] &= \sum_{j=1}^{k} \frac{1}{2^j} E\left[X_j \mid I_k\right] = \sum_{j=1}^{k} \frac{1}{2^j}\left(1 - \frac{j(k+1)}{k(k+2)}\right) \\ &= 1 - \frac{1}{2^k} - \frac{k+1}{k(k+2)} \cdot \frac{\frac{1}{2}\left(\frac{k}{2^{k+1}} - \frac{k+1}{2^k} + 1\right)}{\frac{1}{4}} \\ &= 1 + \frac{1}{k2^k} - \frac{2(k+1)}{k(k+2)}. \end{aligned}$$

Finally, summing over k gives

$$\begin{aligned} E[S] &= \sum_{k\geq 1} P(I_k) E[S \mid I_k] \\ &= \sum_{k\geq 1} \frac{1}{k!}\frac{k}{k+1}\left(1 + \frac{1}{k2^k} - \frac{2(k+1)}{k(k+2)}\right) \\ &= \sum_{k\geq 1} \frac{1}{(k+1)!}\left(k + \frac{1}{2^k} - \frac{2(k+1)}{k+2}\right) \\ &= \sum_{k\geq 1} \left(\frac{(k+1)(k+2) - 3(k+2) + 2}{(k+2)!} + \frac{1}{2^k(k+1)!}\right) \\ &= (e-1) - 3(e-2) + 2\left(e - \frac{5}{2}\right) + 2\left(\sqrt{e} - 1 - \frac{1}{2}\right) = 2\sqrt{e} - 3. \end{aligned}$$

Solution 3: Let $c_j = 2^{-j}$ if $X_1 \geq \cdots \geq X_j$, and let $c_j = 0$ otherwise. Then $S = \sum_{j=1}^{\infty} c_j X_j$. The probability that $c_j > 0$ is $1/j!$, and the expected value of X_j conditioned on $c_j > 0$ is the (unconditioned) expected value of the minimum of $X_1, \dots, X_j$, which is $1/(j+1)$ [because the cumulative distribution function of the minimum is $1 - (1-x)^j$]. By Tonelli's theorem (which applies because $c_j X_j$ is nonnegative), the expected value of the sum is the sum of the expected values:

$$E[S] = \sum_{j=1}^{\infty} E[c_j X_j] = \sum_{j=1}^{\infty} \frac{2^{-j}}{(j+1)!} = 2\sum_{n=2}^{\infty} \frac{(1/2)^n}{n!} = 2(e^{1/2} - 1 - 1/2) = 2\sqrt{e} - 3.$$

A5. Alice and Bob play a game on a board consisting of one row of 2022 consecutive squares. They take turns placing tiles that cover two adjacent squares, with Alice going first. By rule, a tile must not cover a square that is already covered by another tile. The game ends when no tile can be placed according to this rule. Alice's goal is to maximize the number of uncovered squares when the game ends; Bob's goal is to minimize it. What is the greatest number of uncovered squares that Alice can ensure at the end of the game, no matter how Bob plays?

Answer: 290.

Solution: After k tiles have been placed, $2k$ squares will be covered by tiles. The uncovered squares will form at most $k+1$ blocks of one or more consecutive squares, whose total length will therefore be $2022 - 2k$ squares.

Claim 1. Alice can always ensure that there are at least 290 uncovered squares at the end.

Proof. Alice can use the following strategy when there is at least one uncovered block of length $L \geq 3$ squares: Alice picks such a block, and covers the second and third squares of that block, breaking it into a block of length 1 and (if $L > 3$) a block of length $L - 3$.

Suppose Alice is able to place m tiles according to this strategy, but not $m + 1$. Then after $2m-1$ (if Bob is unable to place an mth tile) or $2m$ tiles have been placed, all remaining uncovered blocks have length 1 or 2. At this point, Alice has created at least m blocks of length 1, and there are at most $2m+1$ uncovered blocks. Thus, the total number of uncovered squares is at most $2(2m + 1) - m = 3m + 2$. Since at most $4m$ squares are covered at this point, $7m + 2 \geq 2022$, and hence $m \geq 2020/7 > 288$. Thus, the game reaches a point with at least 289 uncovered blocks of length 1, none of which can be covered subsequently. Since the number of uncovered squares is always even, at the end of the game there are at least 290 uncovered squares.

Claim 2. Bob can always ensure that there are at most 290 uncovered squares at the end.

Proof. Bob can use the following strategy when there is at least one uncovered block of length $L \geq 4$ squares: Bob picks such a block, and covers the third and fourth squares of that block, breaking it into a block of length 2 and (if $L > 4$) a block of length $L - 4$

Let D be the difference between the number of uncovered blocks with length other than 2 and the number of uncovered blocks with length 2. At the start of the game, $D = 1$. Placing a tile can increase the number of uncovered blocks by 1, or it can cover a block of length 2, but not both. Thus, neither player can increase D by more than 1 by placing a tile. If Bob is able to place a tile according to the strategy above, then D decreases by at least 1. Therefore, $D \leq 2$ for as long as Bob is able to follow the strategy.

Suppose Bob is able to place m tiles according to this strategy, but not $m + 1$. Then after $2m$ (if Alice is unable to place an $(m + 1)$st tile) or $2m + 1$ tiles have been placed, all remaining uncovered blocks have length at most 3. At this point, let n_1, n_2, and n_3 be the number of uncovered blocks with lengths 1, 2, and 3, respectively. Then the total number of uncovered blocks is $n_1 + n_2 + n_3 \leq 2m + 2$, and since either $4m$ or $4m + 2$ squares are covered by tiles, $n_1 + 2n_2 + 3n_3 + 4m \leq 2022$. Double the first inequality in the previous sentence, add it to the second inequality, and eliminate m to get $3n_1 + 4n_2 + 5n_3 \leq 2026$. Also, at this point $n_1 + n_3 - n_2 = D \leq 2$. Multiply this inequality by 4 and add it to the previous inequality to get $7n_1 + 9n_3 \leq 2034$. Since uncovered blocks of length 2 or 3 will have a tile placed in them before the game ends, the number of uncovered squares at the end will be $n_1 + n_3 \leq (7n_1 + 9n_3)/7 \leq 2034/7 < 291$.

A6. Let n be a positive integer. Determine, in terms of n, the largest integer m with the following property: There exist real numbers $x_1, \ldots, x_{2n}$ with $-1 < x_1 < x_2 < \cdots < x_{2n} < 1$ such that the sum of the lengths of the n intervals

$$[x_1^{2k-1}, x_2^{2k-1}],\ [x_3^{2k-1}, x_4^{2k-1}], \ldots,\ [x_{2n-1}^{2k-1}, x_{2n}^{2k-1}]$$

is equal to 1 for all integers k with $1 \le k \le m$.

Answer: $m = n$.

Solution: Note that the given condition can be rewritten as

$$\sum_{j=1}^{2n} (-1)^j x_j^{2k-1} = 1.$$

We will show that $x_j = -\cos(j\pi/(2n+1))$ works for k up to $m = n$. To see this, let $\omega = e^{2i\pi/(2n+1)}$. Then ω is a primitive $(2n+1)$-st root of unity, so for all integers a that are not multiples of $2n+1$,

$$\sum_{j=0}^{2n} \omega^{aj} = 0.$$

It follows that for $k = 1, \ldots, n$,

$$\sum_{j=0}^{2n} \left(\frac{\omega^j + \omega^{-j}}{2} \right)^{2k-1} = 0,$$

since the binomial expansion of the $(2k-1)$-st power is a linear combination of ω^{aj} for odd integers j from $-2k+1$ to $2k-1$, none of which are multiples of $2n+1$. Since $(\omega^j + \omega^{-j})/2 = \cos\left(2j\pi/(2n+1)\right)$, we compute

$$\begin{aligned}
1 &= 1 - \sum_{j=0}^{2n} \cos^{2k-1}\left(\frac{2j\pi}{2n+1} \right) \\
&= -\sum_{j=1}^{n} \cos^{2k-1}\left(\frac{2j\pi}{2n+1} \right) - \sum_{j=n+1}^{2n} \cos^{2k-1}\left(\frac{2j\pi}{2n+1} \right) \\
&= -\sum_{j=1}^{n} \cos^{2k-1}\left(\frac{2j\pi}{2n+1} \right) + \sum_{j=n+1}^{2n} \cos^{2k-1}\left(\frac{(2n+1-2j)\pi}{2n+1} \right) \\
&= -\sum_{j=1}^{n} \cos^{2k-1}\left(\frac{2j\pi}{2n+1} \right) + \sum_{j=1}^{n} \cos^{2k-1}\left(\frac{(2j-1))\pi}{2n+1} \right) \\
&= \sum_{\ell=1}^{2n} (-1)^{\ell-1} \cos^{2k-1}\left(\frac{\ell\pi}{2n+1} \right) \\
&= \sum_{\ell=1}^{2n} (-1)^{\ell} x_\ell^{2k-1},
\end{aligned}$$

for $k = 1, \ldots, n$.

We will give two proofs that n is the maximum possible value of m.

Proof 1. Define $x_0 = -1$ and $x_{2n+1} = 1$. Let $f(x)$ be the $\{-1,1\}$-valued function that equals 1 on the intervals $[x_0, x_1], [x_2, x_3], \ldots, [x_{2n}, x_{2n+1}]$ and -1 on the intervals $(x_1, x_2), (x_3, x_4), \ldots, (x_{2n-1}, x_{2n})$. Write $f(x) = f_e(x) + f_o(x)$, where f_e and f_o are the even and odd parts: $f_e(x) = (f(x)+f(-x))/2$ and $f_o(x) = (f(x)-f(-x))/2$. Both of these are therefore $\{-1,0,1\}$-valued functions. Note that $f_e(x) = 1$ for $x \in [x0, \min(x_1, -x_{2n})]$ and $x \in [\max(-x_1, x_{2n}), x_{2n+1}]$. The hypothesis implies that

$$\int_{-1}^{1} x^\ell f(x)\,dx = 0$$

for even ℓ up to $2m-2$. Since the contribution of f_o cancels by symmetry, we see that

$$\int_{-1}^{1} x^\ell f_e(x)\,dx = 0$$

for even ℓ up to $2m-2$, and by symmetry this also holds for all odd ℓ. Thus, it holds for $\ell = 0, 1, \ldots, 2m-1$. By a sign change of f_e, we will mean a transition as we increase x from an interval where $f_e = \pm 1$ to one where $f_e = \mp 1$, possibly with an interval where $f_e = 0$ between them. For any sign change, say it occurs at the upper endpoint of the first interval. We claim that the integral condition above implies that the function $f_e(x)$ has at least $2m$ sign changes. (Since $f_e(\pm 1) = 1$, it has an even number of sign changes. If it has fewer then $2m$ sign changes, let $P(x)$ be the monic polynomial with simple roots exactly where the sign changes occur. Then $P(x)f_e(x) \geq 0$ for all x, and it is strictly positive near $x = \pm 1$, but we compute $\int_{-1}^{1} P(x)f_e(x)\,dx = 0$, a contradiction.) But since f_e is $-1, 0, 1$-valued, each sign change requires at least 2 "jumps" of size 1 in f_e, and f_e can jump by 1 only at the points $\pm x_k$ for $k = 1, \ldots, 2n$. Thus, the number of jumps of size 1 is at least $4m$ and at most $4n$, and hence $m \leq n$.

Proof 2. Look at the polynomial $p(x) = (x+1)\prod_{j=1}^{2n}[x - (-1)^j x_j]$. The condition in the problem statement is that the sum of the $(2k-1)$-st powers of the roots of this polynomial is zero for $k = 1, 2, \ldots, m$. By induction on Newton's identities, it follows that the $(2k-1)$-st elementary symmetric function of the $2n+1$ roots is zero for $k = 1, 2, \ldots, \min(m, n+1)$. If $m > n$, this would imply that $p(x) = xq(x^2)$ for some polynomial q. Then since -1 is a root of p, so would be 1, which would violate the hypothesis that $x_1, \ldots, x_{2n}$ are strictly between -1 and 1.