Density of Primes Dividing Terms in the Somos-5 Sequence
Bryant Davis, Rebecca Kotsonis, Jeremy Rouse

The Somos-5 sequence is defined by
\[ a_0 = a_1 = a_2 = a_3 = a_4 = 1 \text{ and, for } m \geq 5, \]
\[ a_m = \frac{a_{m-1}a_{m-4} + a_{m-2}a_{m-3}}{a_{m-5}}. \]

- All terms in the Somos-5 sequence are integers.
- Let \( \pi(x) \) denotes the total number of primes less than \( x \).
- Let \( \pi'(x) \) denotes the number of primes less than \( x \) that divide some term in the Somos-5 sequence.
- Some manual calculations revealed the following:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \pi(x) )</th>
<th>( \pi'(x) )</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^3 )</td>
<td>168</td>
<td>83</td>
<td>.494048</td>
</tr>
<tr>
<td>( 10^4 )</td>
<td>1229</td>
<td>588</td>
<td>.478438</td>
</tr>
<tr>
<td>( 10^5 )</td>
<td>9592</td>
<td>4539</td>
<td>.473209</td>
</tr>
<tr>
<td>( 10^6 )</td>
<td>78498</td>
<td>37075</td>
<td>.472305</td>
</tr>
<tr>
<td>( 10^7 )</td>
<td>664579</td>
<td>314485</td>
<td>.473209</td>
</tr>
</tbody>
</table>

**Theorem.** With \( P = (2, 2) \) and \( Q = (0, 0) \) on \( E \), for all \( m \geq 0 \), the following relationship is true:
\[ mP + Q = \left( \frac{a_{m}^2 - a_{m}a_{m+4} + 4a_{m}a_{m+2}a_{m+4} - a_{m}^2a_{m+1} - a_{m}^2a_{m+2}}{a_{m+2}}, \frac{3a_{m}^2}{a_{m+2}} \right). \]

- A consequence of this theorem is that, if \( p \) divides a term in the Somos-5 sequence, then \( |P| = 2 : |R| \) in the field \( \mathbb{F}_p \).
- If \( p \) does not divide a term in the Somos-5 sequence, then \( |P| = |R| \) in the field \( \mathbb{F}_p \).

**Gal's Representations.**
- Let \( K_k \) be the field obtained by adjoining to \( \mathbb{Q} \) the \( x \) and \( y \) coordinates of points \( \beta \) where \( 2^k \beta = P \).
- For each prime \( p \) (\( p \neq 2, 3, \) or \( 17 \)), there is a \( \sigma_p \in \text{Gal}(K_k/\mathbb{Q}) \) that “determines” what’s going on modulo \( p \).
- Define the map \( \rho_{E,2^k} : \text{Gal}(K_k/\mathbb{Q}) \to AGL_2(\mathbb{Z}/2^k\mathbb{Z}) \) so that each element \( \sigma_p \) maps to a matrix \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \).
- We want to determine the image of \( \rho_{E,2^k} \). Call this image \( I_k \).

**How to determine if \( p \mid \rho_{E,2^k} \).**
- Observe the subgroup of matrices modulo 8 generated by \( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \) and \( \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} \). A matrix \( M \) modulo \( 2^k \) can be in \( I_k \) if, when reduced modulo 8, it is in this subgroup.
- For all \( (\nu', M) \) in \( I_k \): \( e \equiv 0 \pmod{2} \) if and only if \( \text{det}(M) \equiv 1 \pmod{7} \).
- Chebotarev’s Density Theorem lets us know that each element \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) in \( I_k \) occurs as \( \rho_{E,2^k}(\sigma_p) \) 'equally often' for some prime \( p \).
- Our goal becomes to count the fraction of elements in \( I_k \) that have properties that make different powers of 2 divide the orders of \( P \) and \( R \).
- Recalling \( \nu = [e, f] \), define the following:
\[ I - M = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \]
\[ A = \gamma f - \delta e \]
\[ B = \alpha f - \beta e \]
\[ C = \alpha \delta - \beta \gamma. \]
- **Theorem.** If \( \text{ord}_2(A) > \text{ord}_2(B) \) and \( \text{ord}_2(C) > \text{ord}_2(B) \), then the pair \( (\nu', M) \) represents a \( \sigma_p \) of a prime \( p \) that divides a term in the Somos-5 sequence.

**Results.**

**Theorem (D-K-R).** The density of primes dividing some term of the Somos-5 sequence is
\[ \lim_{x \to \infty} \frac{\pi'(x)}{\pi(x)} = \frac{5087}{10752}. \]