CONSTANT VECTOR CURVATURE IN THREE-DIMENSIONS

ALBANY THOMPSON   CALIFORNIA STATE UNIVERSITY, SAN BERNARDINO REU PROGRAM

CONTACT INFORMATION
CENTRAL WASHINGTON UNIVERSITY
THOMPSONALBANY@GMAIL.COM

INTRODUCTION

This research examines a the property of constant vector curvature ($cvc(\epsilon)$) on three-dimensional model spaces with a positive definite inner product. It has been proven that constant vector curvature is well defined in three-dimensions [2]. The main result of this research is that all three-dimensional model spaces with positive definite inner products have $cvc(\epsilon)$ for some ϵ.

BACKGROUND

Definition: $M = (V, \langle\cdot,\cdot\rangle, R)$ is called a model space if V is a vector space, $\langle\cdot,\cdot\rangle$ is an inner product, and R is a 4-tensor with the same symmetries as the Riemann curvature tensor.
Let $M = (V, \langle\cdot,\cdot\rangle, R)$ be a model space. M is said to have:

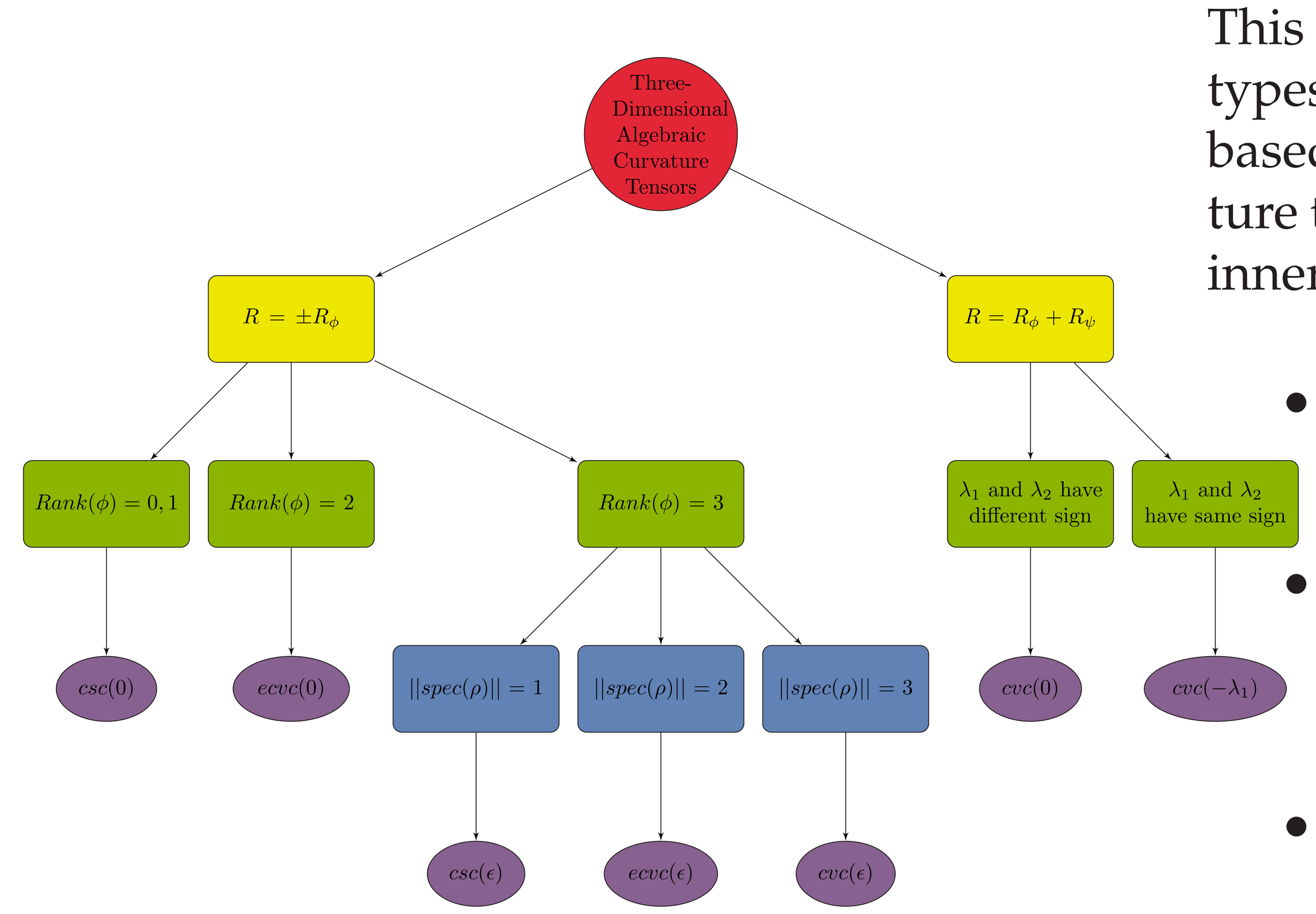
- Constant Sectional Curvature ϵ ($csc(\epsilon)$) if all non-degenerate 2-planes have section curvature ϵ.
- Constant Vector Curvature ϵ ($cvc(\epsilon)$) if for all $v \neq 0 \in V$ there exists $w \in V$ such that the section curvature of the 2-plane spanned by v and w is ϵ.
- Extremal constant vector curvature ϵ if M has $cvc(\epsilon)$ and ϵ is a bound on all sectional curvatures.

If ϕ is a symmetric bilinear form, then define the algebraic curvature tensor

$$R_\phi(x,y,z,w) = \phi(x,w)\phi(y,z) - \phi(x,z)\phi(y,w).$$

Let $v, w \in V$ and $\{e_1, ..., e_n\}$ be an orthonormal basis for V. The Ricci tensor is a symmetric bilinear form (ρ) defined in the following way:
$\rho(v,w) = \sum_{i=1}^{n} \langle e_i, e_i\rangle R(v, e_i, e_i, w)$.
It is always possible to find a basis for V such that the eigenvalues of the Ricci tensor are diagonalized and that $R_{1221} \geq R_{1331} \geq R_{2332}$. In this event all other curvature tensor entries are 0. For every curvature tensor in three-dimensions there is either exactly one symmetric form such that $R = R_\phi$ or two symmetric forms such that $R = R_\phi + R_\psi$. [1]

CLASSIFICATION OF CURVATURE TENSORS IN 3-D



This flowchart can be used to classify all of the types of constant curvature in three-dimensions based on the properties of a model space's curvature tensor for model spaces with positive definite inner products.

- $||R_{ijji}||$ refers to the number of distinct values of the three nonzero curvature entries.
- In the $R = \pm R_\phi$ case $\epsilon = R_{1331}$ and thus refers to the only possible cvc value for a particular model space.
- In the $R = R_\phi + R_\psi$ case $-\lambda_1 = R_{1331}$.

$cvc(\epsilon)$ EXAMPLE

Consider a curvature tensor with $R_{1221} = 7, R_{1331} = 4$, and $R_{2332} = 3$. The values of this model space's Ricci tensor can be found below:
$\rho_{11} = R_{1221} + R_{1331} = 11$
$\rho_{22} = R_{1221} + R_{2332} = 10$
$\rho_{33} = R_{1331} + R_{2332} = 7$
In this case $R = \pm R_\phi$. Also, since $Rank(\phi) = 3$ and $||R_{ijji}|| = 3$ this model space has $cvc(\epsilon)$ where $\epsilon = 4$.

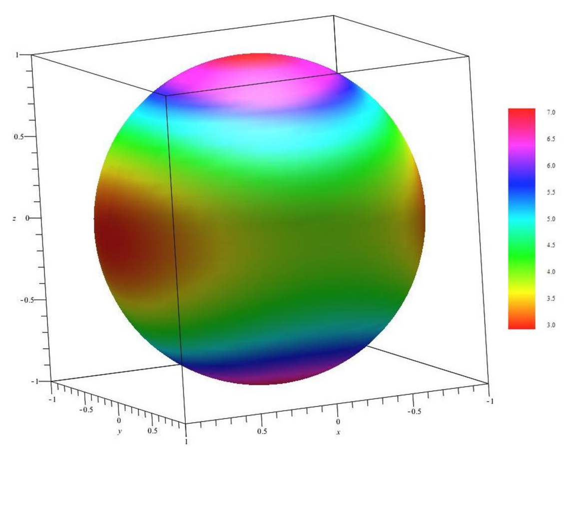
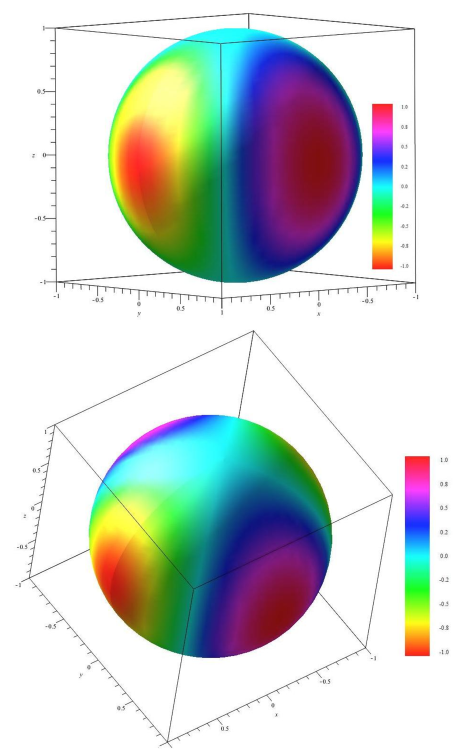


Figure 3: Coloration of the unit sphere according to the curvature value of the plane normal to the vector that begins at the origin and ends at a point on the sphere with the given curvature tensor values.

$cvc(0)$ EXAMPLE

Consider a curvature tensor with $R_{1221} = 1, R_{1331} = 0$, and $R_{2332} = -1$. The Ricci tensor values of this model space are as follows:
$\rho_{11} = -1$, $\rho_{22} = 0$, $\rho_{33} = 1$ Since $\rho_{22} = \rho_{11} + \rho_{33}$ the $R = R_\phi + R_\psi$. Also, because ρ_{11} and ρ_{33} have different signs this model space has $cvc(0)$.



In these images the light blue area shows the distribution of the planes with sectional curvature 0. The second image illustrates two great circles of planes with sectional curvature 0.

Figure 2: Views of the unit sphere colored for the $cvc(0)$ condition.

$ecvc(\epsilon)$ EXAMPLE

Consider a curvature tensor with $R_{1221} = 9, R_{1331} = 1$, and $R_{2332} = 1$. No single value of this model space's Ricci tensor is equal to the sum of the other two, thus this curvature tensor falls into the $\pm R_\phi$ category. Since $||R_{ijji}|| = 2$ this model space has $ecvc(1)$.

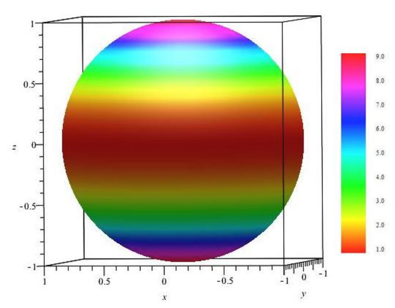


Figure 1: Coloration of the unit sphere for the $ecvc(1)$ condition.

CONCLUSION

Every model space with a positive definite inner product in three-dimensions has consent vector curvature(ϵ) for some ϵ. If the values of the curvature tensor are known then it is possible to find the value of ϵ and in some cases the even stronger results of external vector curvature or constant sectional curvature. For every ϵ there exists a model space in 3-dimensions with $cvc(\epsilon)$ and $ecvc(\epsilon)$ where ϵ is either bound.

OPEN QUESTIONS

- Is constant vector curvature well defined in dimensions greater than 3 or when the inner product of the model space is nondegenerate?
- To what extent to the eigenvalues of the Ricci tensor effect the cvc condition in higher dimensions or other situations?

REFERENCES

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2. Mumford, Kelci, *The Study of the Constant Vector Curvature Condition for Model Spaces in Dimension Three with Positive Definite Inner Product*, CSUSB REU, (2013).
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