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A Simple Introduction to *e*

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The idea for this note was suggested by Dimitric's recent paper [**1**] that discusses a way to introduce the Euler number ***e*** in a heuristic way. Given the volume and breadth of literature on the subject, what follows is a very simplistic way of introducing ***e*** that is easily understood by freshmen.

Let A_2 denote the area under the hyperbola $y = \frac{1}{x}$ between $x = 1$ and $x = 2$. Then (Figure 1) A_2 is greater than the area $\frac{1}{2}$ of the "inner" rectangle having height $\frac{1}{2}$, and

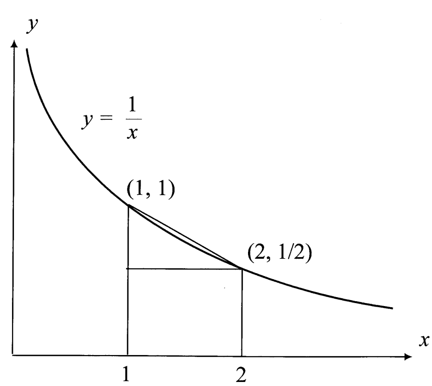

Figure 1.

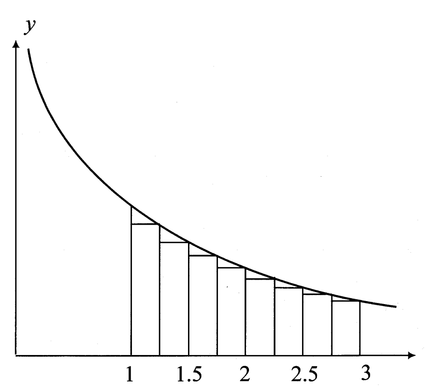

Figure 2.

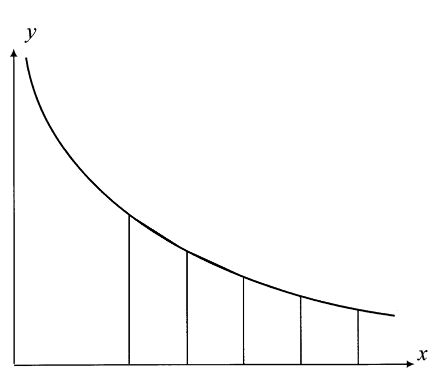

Figure 3.

A_2 is less than the area $\frac{1}{2}(1+\frac{1}{2}) = \frac{3}{4}$ of the trapezoid having heights 1 and $\frac{1}{2}$. [The line segment joining points (1, 1) and (2, $\frac{1}{2}$) has equation $y = -\frac{x}{2} + \frac{3}{2}$, and $-\frac{x}{2} + \frac{3}{2} > \frac{1}{x}$ for $x \in (1, 2)$.] Thus, $\frac{1}{2} < A_2 < \frac{3}{4}$. Now (Figure 2) let A_3 be the area under the hyperbola $y = \frac{1}{x}$ between $x = 1$ and $x = 3$. If we divide the interval [1, 3] into eight equal-sized subintervals, then the sum of the area of the "inner" rectangles is $\frac{1}{4}(\frac{4}{5} + \frac{4}{6} + \cdots \frac{4}{12}) \approx 1.0199$. Therefore $A_3 = 1$.

It is intuitively obvious (without the benefit of the Fundamental Theorem of Calculus) that the area under the hyperbola from $x = 1$ on increases continuously as the value of the right-hand endpoint increases. Hence, somewhere strictly between 2 and 3, there lies a unique number $\boldsymbol{e}$ for which the area under the hyperbola from $x = 1$ to $x = \boldsymbol{e}$ is exactly 1. For $x \in [1, \frac{5}{2}]$, students can verify (Figure 3) that the sum of the three trapezoidal areas is

$$\frac{1}{2} \times \frac{1}{2}\left[\left(1+\frac{2}{3}\right)+\left(\frac{2}{3}+\frac{1}{2}\right)+\left(\frac{1}{2}+2\right)\right] = \frac{56}{60} < 1,$$

and so $\boldsymbol{e} > 2.5$.

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A Surface Useful for Illustrating the Implicit Function Theorem

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While teaching multi-variable calculus last year, I stumbled upon a surface that can be used to make the content of the Implicit Function Theorem concrete and visual. The folium of Descartes, defined by the equation $x^3 + y^3 - 3xy = 0$, is a classic curve often used to illustrate various techniques in single variable calculus. We construct our surface by setting $z = h(x, y) = x^3 + y^3 - 3xy$, so that the level set at $z = 0$ is the folium. Most of the level sets of the defining function for this surface are smooth curves, but there are two points where the hypotheses of this theorem break down, and the level sets at these points display interesting singularities. The surface can also be used to illustrate the complementarity of two and three dimensional graphs for studying a function of two variables.

A curve is locally smooth at a point P if the curve does not intersect itself at P and the direction of the tangent line varies continuously there. The Implicit Function Theorem asserts that a level set of a function $z = f(x, y)$ is locally a smooth curve at a point $P(a, b)$ if $\mathbf{grad}\ f(a, b) \neq \mathbf{0}$. Here $\mathbf{grad}\ f(a, b)$ denotes the vector $(f_x(a, b), f_y(a, b))$ and $\mathbf{0}$ is the vector $(0, 0)$. Many students have trouble appreciating the significance of this theorem. A study of the surface $z = h(x, y)$ sheds some light on this fundamental result.