CLASSROOM CAPSULES

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Classroom Capsules consists primarily of short notes (1–3 pages) that convey new mathematical insights and effective teaching strategies for college mathematics instruction. Please submit manuscripts prepared according to the guidelines on the inside front cover to the Editor, Michael K. Kinyon, Indiana University South Bend, South Bend, IN 46634.

Self-Integrating Polynomials

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I walked into a colleague's office one day as he was grading calculus papers and he showed me the following mistake that one of his students made:

$$\int_0^1 3x^2 + 2x + 3\,dx = \big(3(1)^2 + 2(1) + 3\big) - \big(3(0)^2 + 2(0) + 3\big) = 5.$$

We see that this student failed to find an anti-derivative before plugging in the limits of integration. The correct computation is given by

$$\int_0^1 3x^2 + 2x + 3\,dx = \big((1)^3 + (1)^2 + 3(1)\big) - \big((0)^3 + (0)^2 + 3(0)\big) = 5.$$

By happy accident this student found a correct answer, but we know this trick won't always work or we wouldn't devote so much time learning to integrate! Let's call polynomials like the one in the example *self-integrating*. A question naturally arises; how many self-integrating polynomials are there over the interval [0, 1]? We will answer that question in this capsule.

A quick check of the definition of a self-integrating polynomial indicates that the only constant polynomial satisfing the conditon is $p_0(x) \equiv 0$. We next check to see if there are any self-integrating linear polynomials. Solving the linear equation

$$\int_0^1 a_1 x + a_0\,dx = ax + b|_0^1$$

gives $a_1 = 2a_0$, so we define $p_1(x) = x + 1/2$. A similar computation with a quadratic polynomial

$$\int_0^1 a_2 x^2 + a_1 x + a_0\,dx = a_2 x^2 + a_1 x + a_0|_0^1$$

leads to $p_2(x) = x^2 + 2/3$. At this point, we see a pattern and define

$$p_k(x) = x^k + \frac{k}{k+1}.$$

It is easily verified that all polynomials of this form are self-integrating on the interval [0, 1]. We now have a set of self-integrating polynomials

$$S_n = \left\{ p_k(x) = x^k + \frac{k}{k+1} \right\}$$

for $k = 1, 2 \ldots, n$. The linearity property of the integral ensures that any linear combination of polynomials in S_n is self-integrating. In fact, the set of self-integrating polynomials of degree at most n is an n-dimensional subspace of the vector space of all polynomials of degree at most n. An exercise in elementary linear algebra demonstrates that the set S_n is a basis for this subspace, and this description completely characterizes the self-integrating polynomials up to any finite degree n.

For finite dimensions, this is about all we can say. However, if we move into infinite dimensional vector spaces, the situation gets more interesting. Let P be the set of all polynomials and let K be the set of all self-integrating polynomials on the interval [0, 1]. Clearly K is an infinite-dimensional subspace of P. If we equip P with the inner product,

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx,$$

we can measure distance using the norm

$$\|f - g\|_2 = \langle f - g, f - g \rangle^{1/2}. \tag{1}$$

Recall that $L^2[0, 1]$ is the set of all functions f such that $\|f\|_2 < \infty$ and that P and K are subspaces of $L^2[0, 1]$. It is known (see [**1**]) that P is dense in $L^2[0, 1]$. Is K dense too? If the polynomial $p(x) \equiv 1$ can be approximated to any accuracy by elements in K, so can any positive integer power of x. It follows that K is dense in P and hence also in $L^2[0, 1]$.

Taking our cue from the basis S_N, we define the sequence

$$S = \left\{ x^n + \frac{n}{n+1} \right\}_{n=1}^{\infty}.$$

It is easy to verify that S is contained in K. In Figure 1, we see the graph of some of the terms of this sequence. These pictures suggest that this sequence converges to 1, but we must be careful about the *type* of convergence. For all $0 < x < 1$, we have pointwise convergence of

$$x^n + \frac{n}{n+1}$$

to 1 since $x^n \to 0$ for these values of x. At $x = 1$, we see that

$$x^n + \frac{n}{n+1} = 1 + \frac{n}{n+1} \to 2,$$

so S does not converge in the pointwise sense to 1 in the interval [0, 1]. This sequence does converge in $L^2[0, 1]$ to 1 as $n \to \infty$ since this type of convergence only requires that square of the distance between

$$x^n + \frac{n}{n+1}$$

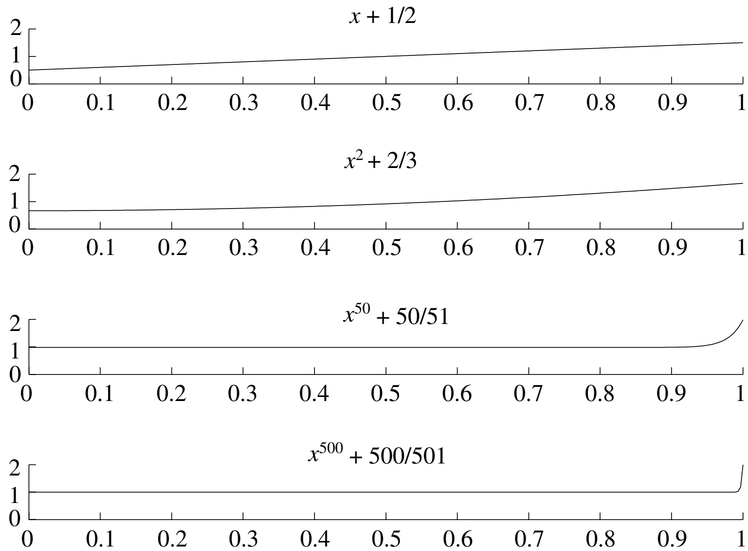

Figure 1. Some terms in the sequence S

and 1 as measured in $L^2[0, 1]$ tends to zero. Using the definition of distance given in (1), the calculation below demonstrates this fact.

$$\left\| 1 - \left(x^n + \frac{n}{n+1} \right) \right\|_2^2 = \int_0^1 \left(1 - \left(x^n + \frac{n}{n+1} \right) \right)^2 dx,$$

which can be shown to equal

$$\frac{n^2}{(2n+1)(n+1)^2},$$

and this expression vanishes as $n \to \infty$.

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Reference

1. W. E. Cheney, *Approximation Theory*, Chelsea, 1982.

A Variant of the Partition Function

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The topic that we consider in this note is the number of ways $c(n)$ that one can write a given positive integer n as a sum of primes. This is in contrast to the classic partition function $p(n)$ (sometimes called the Hardy-Ramanujan partition function), which is the number of ways an integer n can be written as a sum of arbitrary positive

