Usefulness? The reader may believe that an event such as the one described above is unlikely to occur. However, a situation in which a student actually did something equivalent on a test is related in [**4**], and is addressed in slightly greater generality in [**1**].

The standard sum, product, and quotient rules from differential calculus also lend themselves to a similar interpretation: To differentiate $f(x)g(x)$ (or $f(x) + g(x)$ or $f(x)/g(x)$), first differentiate as if $g(x)$ was constant, then as if $f(x)$ was constant; add the results. Indeed, a function $f(x)$ which is comprised of an algebraic combination of an arbitrary number of simpler functions $f_i(x)$ can be differentiated in this way, differentiating as if all the f_i are constant except for f_1; repeat for all f_j; add the results.

This point of view on the standard "differentiation rules" might be a useful unifying idea. Students typically see these rules as a disparate collection of formulas to memorize. At the least, pointing out this connection between the sum, product, quotient, and exponentiation rules raises the question of "why?" The answer—the proof of the multivariable chain rule—is a semester or two away, however. When it does arrive, these rules are nice examples to have ready.

A possible approach would be to teach the differentiation of functions of the form $y = f(x)^{g(x)}$ using this point of view. An advantage might be that students would not have to learn yet another technique (logarithmic differentiation), and could instead simply combine two formulas that they have already learned. However, the technique of logarithmic differentiation is probably more important to most instructors than simply the ability to differentiate $f(x)^{g(x)}$.

This observation about differentiation rules does raise an interesting grading question: If a student writes "the derivative of x^x is xx^{x-1}," how much partial credit will you give? Shouldn't you give the answer half credit, since it is half right?

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Placing the Natural Logarithm and the Exponential Function on an Equal Footing

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In calculus, the ln function is usually introduced by an integral and its derivative is then found by means of the Fundamental Theorem of Calculus (FTC). Once this is done, we can show that $\ln : (0, \infty) \to \Re$ is one-to-one and onto. The exponential function is then defined as the inverse of ln. Another approach is to first define $\exp : \Re \to (0, \infty)$ as the solution on the real line of the initial value problem $y' = y$, $y(0) = 1$, and then prove that exp is one-to-one and onto. Afterwards ln is defined as the inverse of exp and all the usual properties of both functions are proven. In this note we will review both methods and then suggest a new approach in which the natural logarithm and the exponential are defined through two basic properties, and thereafter are shown to be inverses. Let us begin by succinctly developing the well-known first approach.

Define ln through a Riemann integral, namely $\ln x = \int_1^x \frac{dt}{t}$, $x > 0$. Then FTC implies $\ln' x = \frac{1}{x}$ and right away we can prove that $\ln ab = \ln a + \ln b$ ([**5**, p. 459]). The proof that $\ln : (0, \infty) \to \Re$ is one-to-one is straightforward because $\ln' x > 0$ implies that ln is strictly increasing. With regard to the property of being onto, assume $y > 0$; then choose a natural number n such that $n > y/\ln 2$. So $\ln 1 = 0 < y < \ln 2^n$, which in turn implies, thanks to the intermediate value theorem, that $\ln x = y$ for some real number x. On the other hand, if $y < 0$ then $-y > 0$ and we can use the preceding argument.

The exponential function $\exp : \Re \to (0, \infty)$ is then defined as the inverse of ln. In particular $\exp(0) = 1$ since $\ln 1 = 0$, and in general $\ln(\exp(x)) = x$ for all $x \in \Re$, $\exp(\ln(x)) = x$ for all positive x. Right away we can prove that $\exp(x + y) = \exp(x)\exp(y)$. It seems plausible that the inverse of a differentiable function is differentiable too. Thus the chain rule allows us to conclude that $\ln'(\exp(x))\exp'(x) = 1$ for all $x \in \Re$. Hence $\exp'(x) = \exp(x)$. Thereafter all the usual properties of exp and ln can be developed. It should be noted that circa 1650 it was known that if $A_{a,b}$ denotes the area under the curve $y = 1/t$ between a and b and we define $L(x) = A_{1,x}$ $(x > 1)$, $L(x) = -A_{x,1}$ $(x < 1)$, then $L(xy) = L(x) + L(y)$ and $L'(x) = 1/x$; this last result was found without recourse to FTC, which had not yet been discovered at that time ([**3**, pp. 154–158]). Due to the geometrical interpretation of the Riemann integral we realize that $L(x) = \ln x$, so the first approach to the ln function had its origins in the first half of the 17th century!

The second approach consists of defining exp as the solution of the initial value problem (IVP) $y'(x) = y(x)$, $y(0) = 1$, $x \in \Re$ ([**7**, p. 281], [**4**, p. 228]). Uniqueness can be proven using elementary tools, and will be shown in detail later on. A proof of existence of solution on the real line for the IVP under consideration is more difficult and for this reason it belongs to the realm of real analysis ([**1**]). We could present a plausible argument for existence using power series, but the theory of series is usually taught at the end of first-year calculus long after the ln and exp functions are needed.

Next we can prove that $\exp(a + b) = \exp(a)\exp(b)$ ([**7**, p. 285]). One property is particularly important, namely the fact that $\exp(x) > 0$. Indeed

$$\exp(x)\exp(-x) = \exp\big(x + (-x)\big) = \exp(0) = 1,$$

which implies that $\exp(x) \neq 0$ for all x. If there were to exist a real number c such that $\exp(c) < 0$ then the inequalities $\exp(c) < 0 < \exp(0)$ and the intermediate value theorem would imply that $\exp(x) = 0$ for some x, which is impossible. Thus exp goes from $\Re$ into $(0, \infty)$, i.e. $\exp : \Re \to (0, \infty)$. Since $\exp'(x) = \exp(x) > 0$ it follows that exp is strictly increasing and a fortiori one-to-one.

The proof of the fact that exp is an onto function can be carried out by proving first the inequality $\exp(x) \geq 1 + x$ for all x. Then assuming $y > 1$ we get $\exp(y - 1) \geq 1 + (y - 1) = y$. If $\exp(y - 1) = y$ we are done. Otherwise $\exp(y - 1) > y > 1 = \exp(0)$ implies, through the use of the intermediate value theorem, that there exists

x with $\exp(x) = y$. What happens if $0 < y < 1$? We note that $1/y > 1$, so there exists x such that $\exp(x) = 1/y$. Thus $y = 1/\exp(x) = \exp(-x)$. Having proven that $\exp : \Re \to (0, \infty)$ is one-to-one and onto we can define its inverse $\ln : (0, \infty) \to \Re$ and show all the properties of the natural logarithm. For instance, since $\exp(\ln x) = x$ we get $\exp'(\ln x) \ln' x = 1$. So $\ln' x = 1/x$.

It is time to compare both approaches to ln and exp. The first one defines the natural logarithm as an integral and then uses FTC to derive all its properties, as well as developing the exponential function in a straightforward fashion. No wonder that this has been the preferred presentation in most textbooks, especially since Richard Courant's classic book appeared almost seventy years ago ([**2**, p. 168]). The second approach avoids using FTC, so in principle could be presented quite early in a calculus course, but it is more sophisticated due to the need to justify the existence of a solution of the IVP $y' = y$, $y(0) = 1$, $x \in \Re$. For pedagogical reasons we should circumvent any concerns with regard to existence and proceed without further ado; thus, the second approach is a viable option in the calculus classroom setting.

Yet another approach. Napier thought of the natural logarithm as being defined by the two properties $\ln' x = 1/x$ and $\ln 1 = 0$ ([**6**, p. 108]). Thus let us define ln as the solution of the IVP $y' = 1/x$, $y(1) = 0$, on $(0, \infty)$. Such an IVP has a unique solution, namely $y(x) = \int_1^x dt/t$, $x > 0$. With regard to the definition of the exponential function we will follow the second path mentioned above. That is to say, let us define the function exp as the solution on the real line of the IVP $y' = y$, $y(0) = 1$. It should come as no surprise that the initial condition in this IVP is $y(0) = 1$ while in the former it is $y(1) = 0$; after all, we are thinking of showing that their solutions are inverse functions of one another.

The time has come to prove uniqueness of solution of $y' = y$, $y(0) = 1$. Assume f and g are solutions, fix any $x \in \Re$ and define $h(t) = f(t)g(x - t)$ for all real numbers t. Then $h'(t) = f'(t)g(x - t) - f(t)g'(x - t) = f(t)g(x - t) - f(t)g(x - t) = 0$, so $h(t)$ has to be constant. In particular $h(0) = h(x)$, that is to say $f(0)g(x - 0) = f(x)g(x - x)$. But $f(0) = g(0) = 1$, consequently $f(x) = g(x)$. The arbitrary nature of x allows us to conclude that $f = g$. As mentioned before, it is best to take existence for granted.

Now our task is to prove that ln and exp are inverse functions of one another. First of all, let us recall that previously we proved that $\exp(x) > 0$ for all real numbers x. Let $h = \ln \circ \exp$. Then $h'(x) = \ln'(\exp(x)) \exp'(x) = \frac{1}{\exp(x)} \exp(x) = 1$, which in turn implies that $h(x) = x + c$ for a certain constant c. But $c = h(0) = \ln(\exp(0)) = \ln(1) = 0$, leading to the equality $h(x) = x$ for every real number x. Next let $g = \exp \circ \ln$. Then $\ln g(x) = \ln(\exp(\ln x)) = \ln x$ because $\ln(\exp(y)) = y$ for any y has already been established. Since $\ln' x > 0$ we can ascertain that ln is increasing, thus it is one-to-one. Therefore $g(x) = x$ for all positive numbers x (it is worth mentioning that, in general, if two functions F and G satisfy $F(G(y)) = y$ and F is one-to-one, then $G(F(x)) = x$.) The fact that $\ln \circ \exp(x) = x$ and $\exp \circ \ln(x) = x$, on their respective domains of definition, implies that exp and ln are one-to-one and onto and one is the inverse of the other. From this point onwards all the properties of exp and ln can be deduced.

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Trigonometric Identities on a Graphing Calculator

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Many calculus texts (for example [**1**], [**2**], and [**3**]) mention the power and utility of graphing calculators along with the various "pitfalls" one can experience. Figure 1 displays a misleading graph of the $\sin(190x)$ which should have 190 periods on a TI-83 graphing calculator's interval of XMIN $= 0$ to XMAX $= 2\pi$, but only appears to display two periods.

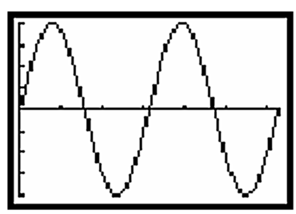

Figure 1. $Y_1 = \sin(190x)$ in the TI-83 window $[0, 2\pi]$ by $[-1, 1]$ in radian mode.

Here's a detailed mathematical explanation for this apparent discrepancy.

Understanding calculator graphics. Graphing calculators simulate the graph of a continuous function. The resolution of the graphics screen on the TI-83 has 5985 pixels, or picture elements, that can be either turned on or off. Figure 2 illustrates the 0 through 62 rows and the 0 through 94 columns of pixels. See [**4**, page 8-6].

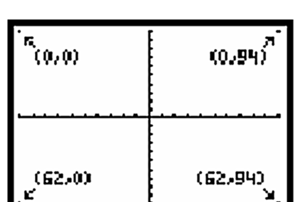

Figure 2. Pixel positions on the TI-83 graphing calculator.

Using the appropriate transformations, the graph of a continuous function is simulated by sampling the functional values for a finite number of points, actually for up to 95 equally spaced horizontal values along the domain window.