
Example 1

For the left nested radical

$$\ldots - \sqrt[3]{6 + \sqrt{6 - \sqrt[3]{6 + \sqrt{6 - \sqrt[3]{6}}}}} \ ,$$

the corresponding sequence of nested radicals is

$$\{y_n\} = \sqrt[3]{6}\,,\ \sqrt{6 - \sqrt[3]{6}}\,,\ \sqrt[3]{6 + \sqrt{6 - \sqrt[3]{6}}}\,,$$

$$\sqrt{6 - \sqrt[3]{6 + \sqrt{6 - \sqrt[3]{6}}}}\,,\ \sqrt[3]{6 + \sqrt{6 - \sqrt[3]{6 + \sqrt{6 - \sqrt[3]{6}}}}}\,,\ \sqrt{6 - \sqrt[3]{6 + \sqrt{6 - \sqrt[3]{6 + \sqrt{6 - \sqrt[3]{6}}}}}}\,,\ \ldots\ .$$

The numerical approximation of this sequence is given below.

{ 1.8171205928, 2.0452088908, 2.0037603331, 1.9990596957, 1.9999216382, 2.0000195903,
2.0000016325, 1.9999995919, 1.999999966, 2.0000000085, 2.0000000007,
1.9999999998, 2., 2., 2., 2.}

The left nested radical converges to 2.

For the right nested radical

$$\sqrt[3]{6 - \sqrt{6 + \sqrt[3]{6 - \sqrt{6 + \sqrt[3]{6 - \ldots}}}}} \ ,$$

the corresponding sequence of nested radicals is

$$\{z_n\} = \sqrt[3]{6}\,,\ \sqrt[3]{6 - \sqrt{6}}\,,\ \sqrt[3]{6 - \sqrt{6 + \sqrt[3]{6}}}\,,\ \sqrt[3]{6 - \sqrt{6 + \sqrt[3]{6 - \sqrt{6}}}}\,,$$

$$\sqrt[3]{6 - \sqrt{6 + \sqrt[3]{6 - \sqrt{6 + \sqrt[3]{6}}}}}\,,\ \sqrt[3]{6 - \sqrt{6 + \sqrt[3]{6 - \sqrt{6 + \sqrt[3]{6 - \sqrt{6}}}}}}\,,\ \ldots\ .$$

The numerical approximation of this sequence is given below.

{ 1.8171205928, 1.5255633869, 1.4742399368, 1.4822688551, 1.4836891137,
1.4834667904, 1.4834274685, 1.4834336237, 1.4834347124, 1.483434542,
1.4834345118, 1.4834345165, 1.4834345174, 1.4834345172, 1.4834345172}

The right nested radical converges to approximately 1.48343452

Example 2

For the right nested radical

$$\sqrt{22+\sqrt[3]{22+\sqrt{22+\sqrt[3]{22+\sqrt{22+\ldots}}}}}\;,$$

the corresponding sequence of nested radicals is

$$\{z_n\} = \sqrt{22}\,,\ \sqrt{22+\sqrt[3]{22}}\,,\ \sqrt{22+\sqrt[3]{22+\sqrt{22}}}\,,$$

$$\sqrt{22+\sqrt[3]{22+\sqrt{22+\sqrt[3]{22}}}}\,,\ \sqrt{22+\sqrt[3]{22+\sqrt{22+\sqrt[3]{22+\sqrt{22}}}}}\,,\ \ldots\ .$$

The numerical approximation of this sequence is given below.

{ 4.6904157598, 4.9801645887, 4.9988488487, 4.999926517,
4.9999957364, 4.9999997278, 4.9999999842, 4.999999999,
4.9999999999, 5., 5., 5., 5.}

The right nested radical converges to 5.

For the left nested radical

$$\ldots+\sqrt{22+\sqrt[3]{22+\sqrt{22+\sqrt[3]{22+\sqrt{22}}}}}\;,$$

the corresponding sequence of nested radicals is

$$\{y_n\} = \sqrt{22}\,,\ \sqrt[3]{22+\sqrt{22}}\,,\ \sqrt{22+\sqrt[3]{22+\sqrt{22}}}\,,\ \sqrt[3]{22+\sqrt{22+\sqrt[3]{22+\sqrt{22}}}}\,,$$

$$\sqrt{22+\sqrt[3]{22+\sqrt{22+\sqrt[3]{22+\sqrt{22}}}}}\,,\ \sqrt[3]{22+\sqrt{22+\sqrt[3]{22+\sqrt{22+\sqrt[3]{22+\sqrt{22}}}}}}\,,\ \ldots\ .$$

The numerical approximation of this sequence is given below.

{ 4.6904157598, 2.988489812, 4.9988488487, 2.9999573642,
4.9999957364, 2.9999998421, 4.9999999842, 2.9999999994, 4.9999999999,
3., 5., 3., 5., 3.}

The sequence of left nested radicals asymptotically converges to

the periodic sequence whose terms alternate between 5 and 3.

Example 3a

For the left nested radical

$$...-\sqrt{6+\sqrt{12-\sqrt{6+\sqrt{12-\sqrt{6}}}}},$$

the corresponding sequence of nested radicals is

$$\{y_n\} = \sqrt{6},\ \sqrt{12-\sqrt{6}},\ \sqrt{6+\sqrt{12-\sqrt{6}}},\ \sqrt{12-\sqrt{6+\sqrt{12-\sqrt{6}}}},$$

$$\sqrt{6+\sqrt{12-\sqrt{6+\sqrt{12-\sqrt{6}}}}},\ \sqrt{12-\sqrt{6+\sqrt{12-\sqrt{6+\sqrt{12-\sqrt{6}}}}}},\ ....$$

The numerical approximation of this sequence is given below.

{ 2.4494897428, 3.0903899846, 3.0150273605, 2.9974943936, 2.9995823699, 3.0000696042,
3.0000116007, 2.9999980666, 2.9999996778, 3.0000000537, 3.000000009, 2.9999999985,
2.9999999998, 3., 3., 3., 3.}

The sequence of left nested radicals converges to 3.

For the left nested radical

$$...+\sqrt{6+\sqrt[3]{24+\sqrt{6+\sqrt[3]{24+\sqrt{6}}}}},$$

the corresponding sequence of nested radicals is

$$\{y_n\} = \sqrt{6},\ \sqrt[3]{24+\sqrt{6}},\ \sqrt{6+\sqrt[3]{24+\sqrt{6}}},\ \sqrt[3]{24+\sqrt{6+\sqrt[3]{24+\sqrt{6}}}},$$

$$\sqrt{6+\sqrt[3]{24+\sqrt{6+\sqrt[3]{24+\sqrt{6}}}}},\ \sqrt[3]{24+\sqrt{6+\sqrt[3]{24+\sqrt{6+\sqrt[3]{24+\sqrt{6}}}}}},\ ....$$

The numerical approximation of this sequence is given below.

{ 2.4494897428, 2.9794705658, 2.9965764742, 2.9998731974,
2.9999788662, 2.9999992173, 2.9999998695, 2.9999999952,
2.9999999992, 3., 3., 3., 3.}

The sequence of left nested radicals converges to 3.

Example 3b

For the left nested radical

$$...-\sqrt[3]{11-\sqrt{11-\sqrt[3]{11-\sqrt{11-\sqrt[3]{11}}}}}\ ,$$

the corresponding sequence of nested radicals is

$$\{y_n\} = \sqrt[3]{11},\ \sqrt{11-\sqrt[3]{11}},\ \sqrt[3]{11-\sqrt{11-\sqrt[3]{11}}},$$

$$\sqrt{11-\sqrt[3]{11-\sqrt{11-\sqrt[3]{11}}}},\ \sqrt[3]{11-\sqrt{11-\sqrt[3]{11-\sqrt{11-\sqrt[3]{11}}}}},$$

$$\sqrt{11-\sqrt[3]{11-\sqrt{11-\sqrt[3]{11-\sqrt{11-\sqrt[3]{11}}}}}},\ ...\ .$$

The numerical approximation of this sequence is given below.

{ 2.2239800906, 2.9624347941, 2.0031255468, 2.9994790303, 2.0000434132, 2.9999927645,
2.000000603, 2.9999998995, 2.0000000084, 2.9999999986, 2.0000000001,
3., 2., 3., 2., 3.}

The sequence of left nested radicals asymptotically converges to

the periodic sequence whose terms alternate between 2 and 3.

Example 4

For the right nested radical

$$\sqrt[3.]{5+\sqrt{7+\sqrt[3.]{5+\sqrt{7+\sqrt[3.]{5+...}}}}}\ ,$$

the corresponding sequence of nested radicals is

$$\{z_n\} = \sqrt[3.]{5},\ \sqrt[3.]{5+\sqrt{7}},\ \sqrt[3.]{5+\sqrt{7+\sqrt[3.]{5}}},$$

$$\sqrt[3.]{5+\sqrt{7+\sqrt[3.]{5+\sqrt{7}}}},\ \sqrt[3.]{5+\sqrt{7+\sqrt[3.]{5+\sqrt{7+\sqrt[3.]{5}}}}},\ \sqrt[3.]{5+\sqrt{7+\sqrt[3.]{5+\sqrt{7+\sqrt[3.]{5+\sqrt{7}}}}}},\ ...\ .$$

The numerical approximation of this sequence is given below.

{ 1.7099759467, 1.9700324929, 1.9959306289, 1.9995833508, 1.999943473,
1.9999942131, 1.9999992149, 1.9999999196, 1.9999999891, 1.9999999989,
1.9999999998, 2., 2., 2., 2.}

The right nested radical converges to 2.

Example 5

For the left nested radical

$$\ldots+\sqrt[3]{5+\sqrt{7+\sqrt[3]{5+\sqrt{7+\sqrt[3]{5}}}}}\quad ,$$

the corresponding sequence of nested radicals is

$$\{y_n\} = \sqrt[3]{5},\ \sqrt{7+\sqrt[3]{5}},\ \sqrt[3]{5+\sqrt{7+\sqrt[3]{5}}},\ \sqrt{7+\sqrt[3]{5+\sqrt{7+\sqrt[3]{5}}}},$$

$$\sqrt[3]{5+\sqrt{7+\sqrt[3]{5+\sqrt{7+\sqrt[3]{5}}}}},\ \sqrt{7+\sqrt[3]{5+\sqrt{7+\sqrt[3]{5+\sqrt{7+\sqrt[3]{5}}}}}},\ \ldots .$$

The numerical approximation of this sequence is given below.

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{ 1.7099759467,  2.9512668376,  1.9959306289,  2.9993216948,   1.999943473,  2.9999905788,
  1.9999992149,  2.9999998692,  1.9999999891,  2.9999999982,  1.9999999998,
            3.,            2.,            3.,            2.,            3.}
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The sequence of left nested radicals asymptotically converges to

the periodic sequence whose terms alternate between 2 and 3.

Example 6

For the left nested radical

$$\ldots+\sqrt{2+\sqrt[4]{14+\sqrt[3]{6+\sqrt{2+\sqrt[4]{14+\sqrt[3]{6+\sqrt{2}}}}}}}\quad ,$$

the corresponding sequence of nested radicals is

$$\{y_n\} = \sqrt{2},\ \sqrt[3]{6+\sqrt{2}},\ \sqrt[4]{14+\sqrt[3]{6+\sqrt{2}}},\ \sqrt{2+\sqrt[4]{14+\sqrt[3]{6+\sqrt{2}}}},$$

$$\sqrt[3]{6+\sqrt{2+\sqrt[4]{14+\sqrt[3]{6+\sqrt{2}}}}},\ \sqrt[4]{14+\sqrt[3]{6+\sqrt{2+\sqrt[4]{14+\sqrt[3]{6+\sqrt{2}}}}}},\ \ldots .$$

The numerical approximation of this sequence is given below.

{ 1.4142135624, 1.9499420156, 1.9984338493, 1.999608424, 1.9999673681, 1.9999989803, 1.9999997451, 1.9999999788, 1.9999999993, 1.9999999998, 2., 2., 2., 2., 2., 2.}

The left nested radical converges to 2.

For the right nested radical

$$\sqrt{2+\sqrt[3]{6+\sqrt[4]{14+\sqrt{2+\sqrt[3]{6+\sqrt[4]{14+\sqrt{2+\ldots}}}}}}},$$

the corresponding sequence of nested radicals is

$$\{z_n\} = \sqrt{2},\ \sqrt{2+\sqrt[3]{6}},\ \sqrt{2+\sqrt[3]{6+\sqrt[4]{14}}},$$

$$\sqrt{2+\sqrt[3]{6+\sqrt[4]{14+\sqrt{2}}}},\ \sqrt{2+\sqrt[3]{6+\sqrt[4]{14+\sqrt{2+\sqrt[3]{6}}}}},$$

$$\sqrt{2+\sqrt[3]{6+\sqrt[4]{14+\sqrt{2+\sqrt[3]{6+\sqrt[4]{14}}}}}},\ \ldots .$$

The numerical approximation of this sequence is given below.

{ 1.4142135624, 1.9537452733, 1.998627778, 1.999612941, 1.9999698515, 1.9999991066, 1.999999748, 1.9999999804, 1.9999999994, 1.9999999998, 2., 2., 2., 2., 2., 2.}

The right nested radical also converges to 2.

Example 7

Let $\{y_n\}_{n=1}^{\infty}$ be the sequence corresponding to the left nested radical

$$\ldots+\left(337+\left(208+\left(503+\sqrt[3]{729+\sqrt[3]{3-\sqrt[3]{22+\sqrt[3]{118+\sqrt[3]{337+\sqrt[3]{208+\sqrt[3]{503}}}}}}}\right)^{\wedge}(1/3)\right)^{\wedge}(1/3)\right)^{\wedge}(1/3).$$

We will present numerical calcuations that appear to show $\{y_n\}$ asymptotically converging to the periodic sequence $\{L_n\}$, which repeats the seven numbers 8, 6, 7, 5, 3, 0, 9.

Consider the subsequence $\{y_{7n+1}\}_{n=0}^{\infty}$. This is the subsequence that begins with $y_1=\sqrt[3]{503}$ and is generated by iterating the function

$$f(x)=\sqrt[3]{503+\sqrt[3]{729+\sqrt[3]{3-\sqrt[3]{22+\sqrt[3]{118+\sqrt[3]{337+\sqrt[3]{208+x}}}}}}}.$$

The numerical approximation of this subsequence is given below.

{ 7.9528476277, 8.0000000244, 7.9999999998, 8., 8., 8., 8., 8., 8., 8.}

The subsequence $\{y_{7n+1}\}_{n=0}^{\infty}$ appears to converge to 8.

Consider the subsequence $\{y_{7n+2}\}_{n=0}^{\infty}$. This is the subsequence that begins with $y_2=\sqrt[3]{208+\sqrt[3]{503}}$ and is generated by iterating the function

$$g(x)=\sqrt[3]{208+\sqrt[3]{503+\sqrt[3]{729+\sqrt[3]{3-\sqrt[3]{22+\sqrt[3]{118+\sqrt[3]{337+x}}}}}}}.$$

The numerical approximation of this subsequence is given below.

{ 5.9945895531, 6.0000000005, 6., 6., 6., 6., 6., 6., 6., 6.}

The subsequence $\{y_{7n+2}\}_{n=0}^{\infty}$ appears to converge to 6.

Consider the subsequence $\{y_{7n+3}\}_{n=0}^{\infty}$. This is the subsequence that begins with

$y_3 = \sqrt[3]{337 + \sqrt[3]{208 + \sqrt[3]{503}}}$ and is generated by iterating the function

$$h(x) = \sqrt[3]{337 + \sqrt[3]{208 + \sqrt[3]{503 \sqrt[3]{729 + \sqrt[3]{3 - \sqrt[3]{22 + \sqrt[3]{118 + x}}}}}}}.$$

The numerical approximation of this subsequence is given below.

{ 6.9999970297, 7., 7., 7., 7.,
7., 7., 7., 7., 7.}

The subsequence $\{y_{7n+3}\}_{n=0}^{\infty}$ appears to converge to 7.

Consider the subsequence $\{y_{7n+4}\}_{n=0}^{\infty}$. This is the subsequence that begins with

$y_4 = \sqrt[3]{118 + \sqrt[3]{337 + \sqrt[3]{208 + \sqrt[3]{503}}}}$ and is generated by iterating the function

$$i(x) = \sqrt[3]{118 + \sqrt[3]{337 + \sqrt[3]{208 + \sqrt[3]{503 + \sqrt[3]{729 + \sqrt[3]{3 - \sqrt[3]{22 + x}}}}}}}.$$

The numerical approximation of this subsequence is given below.

{ 4.9999999604, 5., 5., 5., 5.,
5., 5., 5., 5., 5.}

The subsequence $\{y_{7n+4}\}_{n=0}^{\infty}$ appears to converge to 5.

Consider the subsequence $\{y_{7n+5}\}_{n=0}^{\infty}$. This is the subsequence that begins with

$y_5 = \sqrt[3]{22 + \sqrt[3]{118 + \sqrt[3]{337 + \sqrt[3]{208 + \sqrt[3]{503}}}}}$ and is generated by iterating the function

$$j(x) = \sqrt[3]{22 + \sqrt[3]{118 + \sqrt[3]{337 + \sqrt[3]{208 + \sqrt[3]{503 + \sqrt[3]{729 + \sqrt[3]{3 - x}}}}}}}.$$

The numerical approximation of this subsequence is given below.

{ 2.9999999985, 3., 3., 3., 3., 3., 3., 3., 3., 3.}

The subsequence $\{y_{7n+5}\}_{n=0}^{\infty}$ appears to converge to 3.

Consider the subsequence $\{y_{7n+6}\}_{n=0}^{\infty}$. This is the subsequence that begins with

$y_6 = \sqrt[3]{3 - \sqrt[3]{22 + \sqrt[3]{118 + \sqrt[3]{337 + \sqrt[3]{208 + \sqrt[3]{503}}}}}}$ and is generated by iterating the function

$$k(x) = \sqrt[3]{3 - \sqrt[3]{22 + \sqrt[3]{118 + \sqrt[3]{337 + \sqrt[3]{208 + \sqrt[3]{503 + \sqrt[3]{729 + x}}}}}}}.$$

The numerical approximation of this subsequence is given below.

$\{$ 0.0011362043368, $-7.6293945313 \times 10^{-6}$, 0., 0., 0., 0., 0., 0., 0., 0.$\}$

The subsequence $\{y_{7n+6}\}_{n=0}^{\infty}$ appears to converge to 0.

Consider the subsequence $\{y_{7n+7}\}_{n=0}^{\infty}$. This is the subsequence that begins with

$$y_7 = \sqrt[3]{729 + \sqrt[3]{3 - \sqrt[3]{22 + \sqrt[3]{118 + \sqrt[3]{337 + \sqrt[3]{208 + \sqrt[3]{503}}}}}}}$$

and is generated by iterating the function

$$l(x) = \sqrt[3]{729 + \sqrt[3]{3 - \sqrt[3]{22 + \sqrt[3]{118 + \sqrt[3]{337 + \sqrt[3]{208 + \sqrt[3]{503 + x}}}}}}}.$$

The numerical approximation of this subsequence is given below.

{ 9.0000046757, 8.9999999686, 9., 9., 9.,
9., 9., 9., 9., 9.}

The subsequence $\{y_{7n+7}\}_{n=0}^{\infty}$ appears to converge to 9.