Example 1

For the left nested radical

\[
\sqrt{6 + \sqrt{6 - \sqrt{6 + \sqrt{6 - \sqrt{\ldots}}}}},
\]

the corresponding sequence of nested radicals is

\[
\{y_n\} = \sqrt{6} \cdot \sqrt{6 - \sqrt{6 + \sqrt{6 - \sqrt{\ldots}}}},
\]

The numerical approximation of this sequence is given below.

\[
\{1.8171205928, 2.0452088908, 2.0037603331, 1.9990596957, 1.9999216382, 2.0000195903, 2.0000016325, 1.9999995919, 1.999999966, 2.0000000085, 2.0000000007, 1.9999999998, 2., 2., 2., 2.\}
\]

The left nested radical converges to 2.

For the right nested radical

\[
\sqrt{6 - \sqrt{6 + \sqrt{6 - \sqrt{6 + \sqrt{\ldots}}}}},
\]

the corresponding sequence of nested radicals is

\[
\{z_n\} = \sqrt{6} \cdot \sqrt{6 - \sqrt{6 + \sqrt{6 - \sqrt{\ldots}}}},
\]

The numerical approximation of this sequence is given below.

\[
\{1.8171205928, 1.5255633869, 1.4742399368, 1.4822688551, 1.4836891137, 1.4834667904, 1.4834274685, 1.4834336237, 1.4834347124, 1.483434542, 1.483434518, 1.4834345165, 1.4834345174, 1.4834345172, 1.4834345172\}
\]

The right nested radical converges to approximately 1.48343452
Example 2

For the right nested radical

$$\sqrt{22 + \sqrt{22 + \sqrt{22 + \sqrt{22 + \ldots}}}}$$

the corresponding sequence of nested radicals is

$$\{z_0\} = \sqrt{22} , \sqrt{22 + \sqrt{22}} , \sqrt{22 + \sqrt{22 + \sqrt{22}}} , \ldots$$

The numerical approximation of this sequence is given below.

{ 4.6904157598, 4.9801645887, 4.9988488487, 4.999926517, 4.9999957364, 4.9999997278, 4.9999999842, 4.999999999, 5., 5., 5., 5. }

The right nested radical converges to 5.

For the left nested radical

$$\sqrt{22 + \sqrt{22 + \sqrt{22 + \sqrt{22 + \ldots}}}}$$

the corresponding sequence of nested radicals is

$$\{y_0\} = \sqrt{22} , \sqrt{22 + \sqrt{22}} , \sqrt{22 + \sqrt{22 + \sqrt{22}}} , \ldots$$

The numerical approximation of this sequence is given below.

{ 4.6904157598, 2.988489812, 4.9988488487, 2.9999573642, 4.9999957364, 2.9999998421, 4.999999999, 2.9999999994, 4.9999999999, 3., 5., 3., 5., 3. }

The sequence of left nested radicals asymptotically converges to the periodic sequence whose terms alternate between 5 and 3.
Example 3a

For the left nested radical

\[ \ldots - \sqrt[3]{6 + \sqrt{6} + \sqrt{6 + \sqrt{6}}} \]

the corresponding sequence of nested radicals is

\[ \{y_n\} = \sqrt[3]{6}, \sqrt[3]{12 - \sqrt{6}}, \sqrt[3]{6 + \sqrt{12 - \sqrt{6}}}, \sqrt[3]{12 - \sqrt{6 + \sqrt{12 - \sqrt{6}}}}, \ldots \]

The numerical approximation of this sequence is given below.

\{ 2.4494897428, 3.0903899846, 3.0150273605, 2.9974943936, 2.995823699, 3.0000696042, 3.000116007, 2.9999996778, 3.000000537, 3.00000009, 2.999999985, 2.999999998 \}

The sequence of left nested radicals converges to 3.

Example 3b

For the left nested radical

\[ \ldots + \sqrt[3]{6 + \sqrt{24} + \sqrt{6 + \sqrt{24}}} \]

the corresponding sequence of nested radicals is

\[ \{y_n\} = \sqrt[3]{6}, \sqrt[3]{24 + \sqrt{6}}, \sqrt[3]{6 + \sqrt{24 + \sqrt{6}}}, \sqrt[3]{24 + \sqrt{6 + \sqrt{24 + \sqrt{6}}}}, \ldots \]

The numerical approximation of this sequence is given below.

\{ 2.4494897428, 2.9794705658, 2.9965764742, 2.9998731974, 2.9999999992, 3., 3., 3., 3. \}

The sequence of left nested radicals converges to 3.
For the left nested radical

\[ \sqrt{11 - \sqrt{11 - \sqrt{11 - \sqrt{11 - \sqrt{11}...}}} } , \]

the corresponding sequence of nested radicals is

\[ \{y_n\} = \sqrt{11}, \sqrt{11 - \sqrt{11}}, \sqrt{11 - \sqrt{11 - \sqrt{11}}}, \]
\[ \sqrt{11 - \sqrt{11 - \sqrt{11 - \sqrt{11}}}}, \sqrt{11 - \sqrt{11 - \sqrt{11 - \sqrt{11}}}}, \ldots . \]

The numerical approximation of this sequence is given below.

\{ 2.2239800906, 2.9624347941, 2.0031255468, 2.9994790303, 2.0000434132, 2.9999927645, 
2.000000603, 2.9999998995, 2.0000000084, 2.9999999986, 2.0000000001, 
3., 2., 3., 2., 3. \}

The sequence of left nested radicals asymptotically converges to
the periodic sequence whose terms alternate between 2 and 3.

Example 4

For the right nested radical

\[ \sqrt[5]{5 + \sqrt[7]{5 + \sqrt[5]{5 + \sqrt[7]{5 + \ldots}}}} , \]

the corresponding sequence of nested radicals is

\[ \{z_0\} = \sqrt[5]{5}, \sqrt[7]{5 + \sqrt[5]{5}}, \sqrt[5]{5 + \sqrt[7]{5 + \sqrt[5]{5}}}, \]
\[ \sqrt[7]{5 + \sqrt[5]{5 + \sqrt[7]{5 + \sqrt[5]{5}}}}, \sqrt[7]{5 + \sqrt[5]{5 + \sqrt[7]{5 + \sqrt[5]{5}}}}, \ldots . \]

The numerical approximation of this sequence is given below.

\{ 1.7099759467, 1.9700324929, 1.9959306289, 1.9995833508, 1.999943473, 
1.9999942131, 1.9999992149, 1.9999999196, 1.9999999989, 1.9999999998, 
2., 2., 2., 2., 2. \}

The right nested radical converges to 2.
Example 5

For the left nested radical

$$\sqrt{5 + \sqrt{7 + \sqrt{5 + \sqrt{7 + \sqrt{5}}}}}$$,

the corresponding sequence of nested radicals is

$$\{y_n\} = \sqrt{5}, \sqrt{7 + \sqrt{5}}, \sqrt{5 + \sqrt{7 + \sqrt{5}}}, \sqrt{7 + \sqrt{5 + \sqrt{7 + \sqrt{5}}}}, \ldots$$

The numerical approximation of this sequence is given below.

{ 1.7099759467, 2.9512668376, 1.9959306289, 2.9993216948, 1.999943473, 2.9999905788, 
  1.9999992149, 2.9999998692, 1.9999999891, 2.9999999982, 1.9999999998, 
  3., 2., 3., 2., 3.}

The sequence of left nested radicals asymptotically converges to the periodic sequence whose terms alternate between 2 and 3.

Example 6

For the left nested radical

$$\sqrt{2 + \sqrt{14 + \sqrt{6 + \sqrt{2 + \sqrt{14 + \sqrt{6 + \sqrt{2}}}}}}}$$,
the corresponding sequence of nested radicals is

$$\{y_n\} = \sqrt{2}, \sqrt{6+\sqrt{2}}, \sqrt{14+\sqrt{6+\sqrt{2}}}, \sqrt{2+\sqrt{14+\sqrt{6+\sqrt{2}}}}, \ldots$$

The numerical approximation of this sequence is given below.

{ 1.4142135624, 1.9499420156, 1.9984338493, 1.999608424, 1.9999673681, 1.999989803, 1.999997451, 1.999999748, 1.9999999804, 1.9999999994, 1.9999999998, 2., 2., 2., 2., 2., 2. }

The left nested radical converges to 2.

For the right nested radical

$$\sqrt{2 + \sqrt{6 + \sqrt{14 + \sqrt{2 + \ldots}}}}$$

the corresponding sequence of nested radicals is

$$\{z_n\} = \sqrt{2}, \sqrt{2+\sqrt{6}}, \sqrt{2+\sqrt{6+\sqrt{14}}}, \sqrt{2+\sqrt{6+\sqrt{14+\sqrt{2}}}}, \ldots$$

The numerical approximation of this sequence is given below.

{ 1.4142135624, 1.9537452733, 1.998627778, 1.999612941, 1.9999698515, 1.9999991066, 1.999999748, 1.9999999804, 1.9999999994, 1.9999999998, 2., 2., 2., 2., 2., 2. }

The right nested radical also converges to 2.

Example 7
Let \( \{y_n\}_{n=1}^{\infty} \) be the sequence corresponding to the left nested radical

\[
\begin{align*}
\ldots + (337 + 208 + 503 + 729 + \sqrt{3 - 22 + \sqrt{118 + \sqrt{337 + \sqrt{208 + \sqrt{503}}}^3}^{1/3}})^{1/3}.
\end{align*}
\]

We will present numerical calculations that appear to show \( \{y_n\} \) asymptotically converging to the periodic sequence \( \{L_n\} \), which repeats the seven numbers 8, 6, 7, 5, 3, 0, 9.

Consider the subsequence \( \{y_{7n+1}\}_{n=0}^{\infty} \). This is the subsequence that begins with \( y_1 = \sqrt[3]{503} \), and is generated by iterating the function

\[
f(x) = \sqrt[3]{503 + \sqrt[3]{729 + \sqrt[3]{3 - 22 + \sqrt[3]{118 + \sqrt[3]{337 + \sqrt[3]{208 + \sqrt[3]{x}}}^3}^{1/3}}}^{1/3}.
\]

The numerical approximation of this subsequence is given below.

\[
\{ 7.9528476277, 8.0000000244, 7.9999999998, 8., 8., 8., 8., 8., 8., 8., 8., 8., 8.
\}

The subsequence \( \{y_{7n+1}\}_{n=0}^{\infty} \) appears to converge to 8.

Consider the subsequence \( \{y_{7n+2}\}_{n=0}^{\infty} \). This is the subsequence that begins with \( y_2 = \sqrt[3]{208 + \sqrt[3]{503}} \), and is generated by iterating the function

\[
g(x) = \sqrt[3]{208 + \sqrt[3]{503 + \sqrt[3]{729 + \sqrt[3]{3 - 22 + \sqrt[3]{118 + \sqrt[3]{337 + \sqrt[3]{x}}}^3}^{1/3}}}^{1/3}.
\]

The numerical approximation of this subsequence is given below.

\[
\}
The subsequence \( \{y_{7n+2}\}_{n=0}^{\infty} \) appears to converge to 6.

Consider the subsequence \( \{y_{7n+3}\}_{n=0}^{\infty} \). This is the subsequence that begins with
\[
y_3 = \sqrt[3]{337 + \sqrt{208 + \sqrt{503}}} \quad \text{and is generated by iterating the function}
\]
\[
h(x) = \sqrt[3]{337 + \sqrt{208 + \sqrt{503}}} \sqrt{729 + \sqrt[3]{3 - \sqrt{22 + \sqrt{118 + x}}}}.
\]

The numerical approximation of this subsequence is given below.
\[
\{ 6.9999970297, \quad \quad 7., \quad \quad 7., \quad \quad 7., \quad \quad 7., \quad \quad 7., \quad \quad \}
\]

The subsequence \( \{y_{7n+3}\}_{n=0}^{\infty} \) appears to converge to 7.

Consider the subsequence \( \{y_{7n+4}\}_{n=0}^{\infty} \). This is the subsequence that begins with
\[
y_4 = \sqrt[3]{118 + \sqrt[3]{337 + \sqrt{208 + \sqrt{503}}}} \quad \text{and is generated by iterating the function}
\]
\[
i(x) = \sqrt[3]{118 + \sqrt[3]{337 + \sqrt{208 + \sqrt{503}}} \sqrt{729 + \sqrt[3]{3 - \sqrt{22 + \sqrt{118 + x}}}}).
\]

The numerical approximation of this subsequence is given below.
\[
\{ 4.999999604, \quad \quad 5., \quad \quad 5., \quad \quad 5., \quad \quad 5., \quad \quad 5., \quad \quad \}
\]

The subsequence \( \{y_{7n+4}\}_{n=0}^{\infty} \) appears to converge to 5.
Consider the subsequence \( \{y_{n+5}\}_{n=0}^{\infty} \). This is the subsequence that begins with

\[
y_5 = 22 + \sqrt{118 + \sqrt{337 + \sqrt{208 + \sqrt{503}}}}
\]

and is generated by iterating the function

\[
j(x) = 22 + \sqrt{118 + \sqrt{337 + \sqrt{208 + \sqrt{503 + \sqrt{729 - x}}}}}
\]

The numerical approximation of this subsequence is given below.

\[
\{ 2.9999999985, 3., 3., 3., 3., 3., 3., 3., 3., 3. \}
\]

The subsequence \( \{y_{n+5}\}_{n=0}^{\infty} \) appears to converge to 3.

Consider the subsequence \( \{y_{n+6}\}_{n=0}^{\infty} \). This is the subsequence that begins with

\[
y_6 = 3 - 22 + \sqrt{118 + \sqrt{337 + \sqrt{208 + \sqrt{503}}}}
\]

and is generated by iterating the function

\[
k(x) = 3 - 22 + \sqrt{118 + \sqrt{337 + \sqrt{208 + \sqrt{503 + \sqrt{729 + x}}}}}
\]

The numerical approximation of this subsequence is given below.

\[
\{ 0.0011362043368, -7.6293945313 \times 10^{-6}, 0., 0., 0., 0., 0., 0., 0., 0. \}
\]

The subsequence \( \{y_{n+6}\}_{n=0}^{\infty} \) appears to converge to 0.
Consider the subsequence \( \{y_{n,7}\}_{n=0}^\infty \). This is the subsequence that begins with
\[
y_7 = \sqrt[7]{729 + \sqrt[7]{3 - \sqrt[7]{22 + \sqrt[7]{118 + \sqrt[7]{337 + \sqrt[7]{208 + \sqrt[7]{503}}}}}}}.
\]
and is generated by iterating the function
\[
1(x) = \sqrt[7]{729 + \sqrt[7]{3 - \sqrt[7]{22 + \sqrt[7]{118 + \sqrt[7]{337 + \sqrt[7]{208 + \sqrt[7]{503 + x}}}}}}}
\].

The numerical approximation of this subsequence is given below.
\[
\}

The subsequence \( \{y_{n,7}\}_{n=0}^\infty \) appears to converge to 9.