The Role of Paradoxes in the Evolution of Mathematics

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A paradox has been described as a truth standing on its head to attract attention. Undoubtedly, paradoxes captivate. They also cajole, provoke, amuse, exasperate, and seduce. More importantly, they arouse curiosity, stimulate, and motivate.

In this paper we present examples of paradoxes from the history of mathematics which have inspired the clarification of basic concepts and the introduction of major results. Our examples will deal with numbers, logarithms, functions, continuity, tangents, infinite series, sets, curves, and decomposition of geometric objects.

We will use the term "paradox" in a broad sense to mean an inconsistency, a counterexample to widely held notions, a misconception, a true statement that seems to be false, or a false statement that seems to be true. It is in these various senses that paradoxes have played an important role in the evolution of mathematics. Indeed, as Bell and Davis, respectively, put it:

> The mistakes and unresolved difficulties of the past in mathematics have always been the opportunities of its future ([**1**], p. 283).

> One of the endlessly alluring aspects of mathematics is that its thorniest paradoxes have a way of blooming into beautiful theories ([**6**], p. 55).

Paradoxes can also serve a useful role in the classroom. The temporary confusion and insecurity which they may engender in students can be put to good use. Conflict and predicament are useful pedagogical devices (provided, of course, that they are dealt with). They may foster a positive attitude to "getting stuck," provide the opportunity to participate in debate and controversy over mathematical issues, and promote the realization that mathematics often develops in this very way. Teachers may gain a better appreciation of students' difficulties in coming to grips with concepts and results with which some of the greatest mathematicians of all time struggled. Such concepts and results, while paradoxical and challenging at the time, became commonplaces in subsequent generations. In the words of Kasner and Newman ([**12**], p. 193):

> The testament of science is so continually in a flux that the heresy of yesterday is the gospel of today and the fundamentalism of tomorrow.

PARADOXES INVOLVING NUMBERS. The evolution of the concept of number has been beset by paradoxes almost every step of the way. As P. J. Davis put it ([**7**], p. 305):

> It is paradoxical that while mathematics has the reputation of being the one subject that brooks no contradictions, in reality it has a long history of successful living with contradictions. This

is best seen in the extensions of the notion of number that have been made over a period of 2500 years. From limited sets of integers, to fractions, negative numbers, irrational numbers, complex numbers, transfinite numbers, each extension, in its way, overcame a contradictory set of demands.

The first sentence in the above quotation may be thought of as a "metaparadox"—a nontechnical, paradoxical statement about technical, paradoxical phenomena. We will point out a variety of such metaparadoxes; they are interesting in their own right as issues for philosophical discussion or contemplation. But now to some paradoxes dealing with the evolution of various number systems.

(a) The Pythagoreans of the 6th century B.C. believed that every line segment can be measured by a positive integer or the ratio of two such integers. This was to them not merely a very plausible fact, but an article of faith, an aspect of their philosophy. Moreover, the idea formed the basis of the pythagorean theory of proportion (see [**23**]). It was thus a great shock (paradox) to them when they discovered that the diagonal of a unit square cannot be measured by a whole number or by a ratio of whole numbers; or, as the Greeks put it, that the diagonal and side of a square are *incommensurable*. Their proof of this result is essentially the one we use today to show that $\sqrt{2}$ is irrational. The paradox was arrived at by using the Pythagorean Theorem. Thus the

Metaparadox: The Pythagorean Theorem was the undoing of the pythagorean philosophy and the pythagorean theory of proportion.

The discovery of the incommensurability of the diagonal and side of a square had far-reaching consequences for Greek mathematics. On the positive side, it inspired Eudoxus to found a sophisticated theory of proportion which applied to both commensurable and incommensurable magnitudes. This, in turn, motivated Dedekind more than two millennia later to define the real numbers via Dedekind cuts. On the debit side, it turned the direction of Greek mathematics (at least in its very productive, classical period) from a harmonious collaboration of number and geometry to an almost exclusive concern with geometry.

(b) The introduction of negative numbers into mathematics and their subsequent use occasioned considerable consternation and difficulties. A major conceptual framework that had to be abandoned was the prohibition of subtracting a greater from a smaller number. As Wallis in the 17th century put it ([**20**], p. 438): "[How can] any magnitude... be less than nothing, or any number fewer than none?"

Among other paradoxes having to do with negative numbers are the following two:

(i) Wallis "proved" that negative numbers are greater than infinity. His argument was that since (for positive a) $\frac{a}{0} = \infty$, $a/$a neg. no. $> \infty$; this is so because decreasing the denominator increases the fraction.

(ii) In a letter to Leibniz, Arnauld (a 17th-century mathematician and philosopher) objected to the equality $\frac{1}{-1} = \frac{-1}{1}$ on the grounds that the ratio of a greater to a smaller quantity cannot equal the ratio of a smaller to a greater. Leibniz agreed this was a difficulty, but argued for the tolerance of negative numbers because they are useful and, in general, lead to consistent results. See [**5**], pp. 39–40.

Justification of otherwise inexplicable notions on the grounds that they yield useful results has occurred frequently in the evolution of mathematics. This brings up the following

Metaparadox: How can meaningless (or at best inexplicable) things be so useful?

Of course, out of meaninglessness (or confusion) emerged, in time, clarity and understanding.

(c) The solution by radicals of cubic equations was one of the great achievements of 16th-century mathematics. Cardan's solution of the cubic $x^3 = ax + b$ was given by the formula

$$x = \sqrt[3]{\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{3}\right)^3}} + \sqrt[3]{\frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{3}\right)^3}}.$$

Bombelli applied it to the equation $x^3 = 15x + 4$ to obtain $x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$. Cardan had earlier denied the applicability of his formula to such equations since it introduced square roots of negative numbers, which he rejected. But Bombelli noted (by inspection) that $x = 4$ is a solution of $x^3 = 15x + 4$. (The other two roots, $-2 \pm \sqrt{3}$, are also real.) Here was a paradox: The roots of $x^3 = 15x + 4$ are real, yet the formula yielding the roots involved complex, and at the time meaningless, numbers. "The whole matter seemed to rest on sophistry rather than on truth," noted Bombelli ([**15**], p. 2). And he set himself the task of resolving that sophistry, which resulted in the birth of complex numbers.[1] Birth, however, did not entail legitimacy. It took another two and a half centuries before complex numbers were accepted as bona fide mathematical entities.

PARADOXES INVOLVING LOGARITHMS. The issue of the meaning of logarithms of negative and complex numbers arose in the early 18th century in connection with integration. In analogy with the real case, Johann Bernoulli integrated $1/(x^2 + a^2)$ as follows:

$$\int \frac{dx}{x^2 + a^2} = \int \frac{dx}{(x + ai)(x - ai)} = -\frac{1}{2ai}\int\left(\frac{1}{x + ai} - \frac{1}{x - ai}\right) dx$$

$$= -\frac{1}{2ai}[\log(x + ai) - \log(x - ai)] = -\frac{1}{2ai}\log\frac{x + ai}{x - ai}.$$

In an exchange of letters (begun in 1702 and lasting sixteen months) Bernoulli and Leibniz argued about the meaning of $\log(x + ai/x - ai)$, and, in particular, about the meaning of $\log(-1)$. Bernoulli asserted that $\log(-1)$ is real while Leibniz claimed it is imaginary, each advancing various arguments to support his view. For

[1]Bombelli developed rules for manipulating expressions of the form $a + b\sqrt{-1}$ and was thereby able to show that (one of the values of) $\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$ is indeed 4.

example, Bernoulli argued that since

$$\frac{dx}{x} = \frac{d(-x)}{-x}, \quad \int \frac{dx}{x} = \int \frac{d(-x)}{-x},$$

hence $\log x = \log(-x)$. In particular, $\log(-1) = \log 1 = 0$. Among Leibniz' arguments were the following:

(i) Since the range of $\log a$, for $a > 0$, is all real numbers, it follows that $\log a$, for $a < 0$, must be imaginary, because the real numbers have already been "spoken for".

(ii) If $\log(-1)$ were real, then $\log i$ would also be real, since $\log i = \log(-1)^{1/2} = \frac{1}{2}\log(-1)$. But this is clearly absurd, alleges Leibniz.

(iii) Putting $x = -2$ in the expansion

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

yields $\log(-1) = -2 - \frac{4}{2} - \frac{8}{3} - \ldots$. Since the series on the right diverges, it cannot be real, hence must be imaginary.

The above are indeed interesting examples of the art (not to say "science") of symbolic manipulation practiced by some of the greatest mathematicians of the 17th and 18th centuries. The resulting paradoxes had "for a long time... tormented me," noted Euler ([**17**], p. 72). He resolved them in a 1749 paper. We quote from its interesting introduction ([**14**], p. 4):

> Since logarithms are clearly part of pure mathematics it may well be surprising to learn that they have been until now the subject of an embarrassing controversy in which whatever side is taken contradictions appear that seem completely impossible to resolve. Meanwhile if truth is to be universal there can be no doubt that these contradictions,..., however unresolved they seem can only be apparent. ...I will bring out fully all the contradictions involved so that it may be seen how difficult it is to discover truth and to guard against inconsistency even when two great men are working on the problem.

The crux of Euler's solution was the Euler-Cotes formula $e^{i\theta} = \cos\theta + i\sin\theta$. It implies that $e^{i(\pi+2n\pi)} = \cos(\pi + 2n\pi) + i\sin(\pi + 2n\pi) = \cos\pi + i\sin\pi = -1$, so that $\log(-1) = i(\pi + 2n\pi)$, where $n = 0, \pm 1, \pm 2, \ldots$ .Thus $\log(-1)$ is *multivalued* (in fact, infinite-valued) and all its values are complex. Both Bernoulli and Leibniz were wrong, the former "more so" than the latter.

PARADOXES INVOLVING FUNCTIONS. The concept of function originated in the early 18th century. Newton and Leibniz invented the calculus in the latter part of the 17th century. Here, then, is a

Metaparadox: Calculus without functions.

The calculus of Newton and Leibniz was a calculus of curves (given by equations) rather than of functions.

A function was viewed at different times as a formula, a curve, or an arbitrary correspondence. Paradoxes turned up to dethrone one or another of these views of functionality. Even the very meaning of a formula, as well as its scope (i.e., the functions that are representable by formulas), changed over time, and were often subjects of considerable controversy. For example:

(a) To Euler and his contemporaries of the mid-18th century a function meant a formula, where the latter concept, though not rigorously defined, was broadly construed to allow (among other things) infinite sums and products in its formation. There were several implicit assumptions:

(i) The function (formula) had to be given by a *single* expression. For example,

$$f(x) = \begin{cases} x, & x > 0 \\ -x, & x \leq 0 \end{cases} \quad \text{was not considered a function.}$$

(ii) The independent variable had to range over *all* real numbers (except possibly for isolated points, as in $f(x) = \frac{1}{x}$). For example, $f(x) = x, 0 \leq x \leq 1$, was not considered a function

(iii) Two functions which agreed on an interval were assumed to agree everywhere on the line.

The significance of these assumptions was the fact that the algorithms of the calculus applied at that time only to such functions.

Many of the 18th-century (mis)conceptions about functions were overturned by Fourier's work on heat conduction in the early decades of the 19th century. As a result of this work Fourier claimed to have shown that *any* function defined on some interval can be represented on that interval as an infinite series of sines and cosines—a *Fourier series*.[2] For example, if

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 0, & x = 0 \\ 1, & 0 < x < \pi \end{cases},$$

then

$$f(x) = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right) \quad \text{for all} \quad x \in (-\pi, \pi).^3$$

Several fundamental departures concerning functions resulted from Fourier's work:

1. It became legitimate, and important, to consider functions whose domain is an *interval* rather than the entire real line.
2. Two functions could agree on an interval but differ outside the interval.
3. A function given by two or more distinct expressions could equal a function given by a single expression.

(b) In an 1829 paper on Fourier series Dirichlet introduced the so-called Dirichlet function

$$D(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

This function was neither a formula nor a curve. It was a new type of function, described by a correspondence. It was the first of many functions which came to be called "pathological"—but not for very long (see [**25**]).

[2]Although this result is, of course, incorrect (given *our* conception of functions) in the generality which Fourier claimed for it, a large class of functions *can* be represented by Fourier series. In fact, Fourier's contemporaries would have been hard put to find an exception.

[3]In the latter part of the 18th century, following debates surrounding the famous vibrating-string problem, it became legitimate (at least in some quarters) to consider functions defined by several expressions. See [**16**].

At the end of the 19th century Baire extended (again) the notion of formula. To him it meant an expression obtained from a variable and constants by a (possibly countable) iteration of additions, multiplications, and the taking of limits. He called such a function *analytically representable* and showed that the Dirichlet function is of this type: $D(x) = \lim_{m\to\infty}\lim_{n\to\infty}\cos(m!\pi x)^n$. Thus the "pathological" Dirichlet function became a "tame" analytically representable function.

Is analytic representability a universal mode of representability of functions? That is, are there functions which are not analytically representable? Yes and no. If you are a formalist, you can show by a counting argument that the set of analytically representable functions has cardinality c, while the set of all functions (clearly) has cardinality 2^c. Thus there are uncountably many functions which are not analytically representable. But no one has given a *constructive* example of even one.

PARADOXES INVOLVING CONTINUITY. Although the concept of continuity is nowadays fundamental in mathematics, its modern definition was not formulated until the 19th century, about 150 years after the invention of the calculus by Newton and Leibniz. In the 18th century, Euler did define a notion of continuity in response to the famous vibrating-string controversy ([**8**], p. 301). Thus a continuous function was one given by a single expression (formula), while a function given by several expressions was considered *discontinuous*. For example, to Euler the function

$$f(x) = \begin{cases} x, & x > 0 \\ -x, & x \le 0 \end{cases}$$

was discontinuous, while the function comprising the two branches of a hyperbola was considered continuous, since it is given by the single expression $f(x) = \frac{1}{x}$ (see [**16**], p. 301).

The work on Fourier series showed the untenability of the 18th-century notion of continuity. For example, the function

$$g(x) = \begin{cases} -1, & -\pi < x < 0 \\ 0, & x = 0 \\ 1, & 0 < x < \pi \end{cases}$$

could (as we have seen) be represented by a single expression, namely its Fourier series, hence it was both continuous and discontinuous in the 18th-century sense of that concept.

In an 1821 work Cauchy initiated a reappraisal and reorganization of the foundations of 18th-century calculus. In this work Cauchy defined continuity essentially as we understand the concept today, although he used the then-prevailing language of infinitesimals rather than the now-accepted $\varepsilon - \delta$ formulation given by Weierstrass in the 1850s. The shift in point of view from Euler's to Cauchy's conceptions of continuity was fundamental. In the former case continuity was a global property while in the latter case it was a local property. But the concept of continuity proved to be very subtle, and was not completely understood even by Cauchy and his contemporaries of the early to mid-19th century. For example:

(a) Cauchy "proved" that an infinite sum (a convergent series) of continuous functions is a continuous function ([**4**], p. 110). This, of course, is incorrect. A

counterexample was given by Abel in the 1820s—it is essentially the series

$$\sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1}$$

we encountered earlier, which is discontinuous at $x = k\pi, k = 0, \pm 1, \pm 2, \ldots$. The error in Cauchy's proof resulted from his failure to distinguish between convergence and uniform convergence of a series of functions. In fact, "the realization of the central role of the concept of uniform convergence in analysis came about slowly in the last [19th] century " ([**21**], p. 97).

(b) Euler's continuous functions were, in practice, differentiable (except possibly at isolated points). So were Cauchy's—at least this is what Cauchy and his contemporaries believed, and what some of them "proved" (see [**26**]). It was therefore astonishing when Weierstrass in the 1860s gave an example of a continuous function which is *nowhere* differentiable, namely $f(x) = \sum_{n=1}^{\infty} b^n \cos(a^n \pi x)$, a an odd integer, b a real number in $(0, 1)$, and $ab > 1 + (3\pi/2)$. This and similar examples showed for the first time that the concept of continuity is considerably broader than that of differentiability, and thus established continuity as an important concept of investigation in its own right. The examples also showed the limitations of intuitive geometric reasoning in analysis, and thus the need for careful, analytic formulations of basic notions.

In a modern development of a different kind, Schwartz and Sobolev showed in the 1940s that every continuous function is, indeed, "differentiable". But the derivative is now a "generalized function" (a "distribution"). For example, if

$$f(x) = \begin{cases} 1, & x > 0 \\ \frac{1}{2}, & x = 0 \\ 0, & x < 0, \end{cases}$$

then

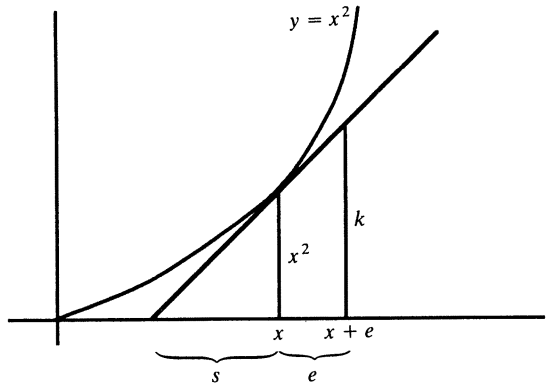
$$f'(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0, \end{cases}$$

which is the Dirac delta "function" $\delta(x)$. As this example shows, there are even discontinuous functions which are differentiable (in the Schwartz/Sobolev sense) —a shocking realization (it would have been) for mathematicians of the second half of the 19th century.

PARADOXES INVOLVING ASPECTS OF THE CALCULUS (OTHER THAN CONTINUITY). (a) The calculus was invented (independently) by Newton and Leibniz in the last third of the 17th century. But many of its important ideas were foreshadowed in early-17th century works of prominent mathematicians, notably Fermat. In the late 1630s he devised a method for dealing with problems on tangents and on maxima and minima. The following example illustrates Fermat's approach (see [**8**], p. 122):

Suppose we wish to find the tangent to the parabola $y = x^2$ at some point (x, x^2). Let $x + e$ be a nearby point on the x-axis and let s denote the *subtangent* of the curve at the point (x, x^2) (see accompanying diagram). Similarity of triangles yields $x^2/s = k/(s + e)$. Fermat notes that k is "approximately equal" to $(x + e)^2$; writing this as $k \simeq (x + e)^2$ we get

$$\frac{x^2}{s} \simeq \frac{(x+e)^2}{s+e}.$$



Solving for s we have

$$s \simeq \frac{ex^2}{(x+e)^2 - x^2} = \frac{ex^2}{x^2 + 2ex + e^2 - x^2} = \frac{ex^2}{e(2x+e)} = \frac{x^2}{2x+e},$$

hence $x^2/s \simeq 2x + e$. Note that x^2/s is the slope of the tangent to the parabola at (x, x^2). Fermat now "deletes" e and claims that the slope of the tangent is $2x$.

Fermat's method was severely criticized by some of his contemporaries. They objected to his introduction and subsequent suppression of the mysterious e. Dividing by e meant regarding it as not zero. Discarding e implied treating it as zero. This is inadmissible, they rightly claimed. In a somewhat different context, but with equal justification, Bishop Berkeley in the 18th century would refer to such e's as "the ghosts of departed quantities," arguing that "by virtue of a twofold mistake... [one] arrive[d], though not at a science yet at the truth" ([**13**], p. 428).

The justification of 17th- and 18th-century algorithms of the calculus was that they yielded correct results—another important example of the utility of "meaningless" procedures (cf. p. 965). The end seemed to have justified the means. *Rigorous* justification of the calculus—of one kind—came with the 1821 introduction of limits by Cauchy, and—of another kind—with the 1960 introduction of infinitesimals by Robinson.

Metaparadox: How can the calculus be founded on two distinct, and in some ways incompatible, theories: limits, based on the real numbers, and infinitesimals, based on the hyperreal numbers? Or, as Steen put it: "The epistemological foundation of mathematical analysis is far from settled" ([**22**], p. 92).

(b) Power series were a potent tool in 17th- and especially 18th-century calculus. They were manipulated as polynomials, with little if any attention paid to questions of convergence. In fact, Euler and others consciously used *divergent* series to great advantage. The results thus obtained were impressive and important, but errors and paradoxes became unavoidable. Here are two:

(i) There is undoubtedly a touch of the metaphysical in the mathematical infinite. The following example, due to Euler, confirms it ([**13**], p. 447): Letting $x = -1$ in $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$, he gets

$$\infty = 1 + 2 + 3 + 4 + \cdots (*).$$

Letting $x = 2$ in $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots$, one has

$$-1 = 1 + 2 + 4 + 8 + \cdots (**).$$

Since each term on the right side of $(**)$ is greater than or equal to the corresponding term on the right side of $(*)$, $-1 > \infty$. But clearly $\infty > 1$. Hence $-1 > \infty > 1$. Euler infers that ∞ must be a type of limit between the positive and negative numbers, and in this sense resembles 0.

(ii) Occasionally 17th- and 18th-century mathematicians revelled in the art of series-manipulation if for no better reason (it would seem) that to demonstrate their prowess. For example, putting $x = 1$ in $\log(1 + x) = x - x^2/2 + x^3/3 - \ldots$ yields $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. So far so good. But now, the argument went, the right side equals

$$\left(1 + \tfrac{1}{3} + \tfrac{1}{5} + \cdots\right) + \left(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \cdots\right) - 2\left(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \cdots\right)$$
$$= \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \cdots\right) - \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \cdots\right) = 0,$$

hence $\log 2 = 0$. It was only in the mid-19th century that Riemann resolved this paradox by proving that the sum of a conditionally convergent series can assume, upon rearrangement, *any* value. "The discovery of this apparent paradox contributed essentially to a re-examination and rigorous founding... of the theory of infinite series" ([**21**], p. 30).

PARADOXES INVOLVING SETS. (a) During nearly the last three decades of the 19th century Cantor developed many important set-theoretic ideas using an intuitive ("naive") notion of set. Eventually his concept proved inadequate and led to paradoxes. Perhaps the best known is Russell's classic paradox of 1902: Let $R = \{x: x \notin x\}$. Then $R \in R$ if and only if $R \notin R$. This paradox had a profound effect on a number of mathematicians (see [**19**]). It devastated the logician Frege, who had just completed a two-volume treatise on the foundations of arithmetic which relied on set-theoretic notions. Learning of Russell's paradox, he lamented ([**13**], p. 1192):

> A scientist can hardly meet with anything more undesirable than to have the foundation give way just as the work is finished. I was put in this position by a letter from Mr. Bertrand Russell when the work was nearly through the press.

On the other hand, the paradoxes of set theory had positive effects. In particular, they provoked mathematicians to give precise meaning to the notion of set by devising various axiomatizations of set theory (e.g., the Zermelo-Fraenkel axioms, the Russell and Whitehead theory of types, the Gödel-Bernays system). Although such axiom-systems avoided the known paradoxes, they did not guarantee that new ones would not emerge. As Poincaré put it picturesquely ([**13**], p. 1186):

> We have put a fence around the herd to protect it from the wolves but we do not know whether some wolves were not already within the fence.

Here are two metaparadoxes resulting from Cantor's work in set theory:

Metaparadox 1: Infinity comes in different sizes; in fact, in *infinitely many* different sizes.

The second metaparadox comes from juxtaposing the following two quotations by Poincaré and Hilbert, respectively ([**13**], p. 1003):

Metaparadox 2: (a) "Later generations will regard *Mengenlehre* [set theory] as a disease from which one has recovered."

(b) "No one shall expel us from the paradise which Cantor created for us."

PARADOXES INVOLVING CURVES. The notion of curve is, of course, fundamental in geometry. To Euclid it meant "breadthless length". The collection of curves known to his contemporaries was small—the conic sections, the conchoid, the cissoid, the spiral, the quadratrix, and a very few others. The situation changed dramatically with the invention of analytic geometry in the 17th century. Now any equation in two variables came to represent a (plane) curve, although "seventeenth-century mathematicians did not have a uniform definition of the concept of curve (nor apparently did they feel the need for such a definition)" ([**3**], p. 296). The study of curves was pursued vigorously for the next three centuries, attracting some of the best mathematicians who attacked it by geometric, analytic, algebraic, arithmetic, and topological means.

"Pathological" functions introduced in the second half of the 19th century raised questions about the nature of curves. For example, in what sense does a continuous nowhere-differentiable function represent a curve? Jordan responded (in 1887) with what came to be the first formal definition of a curve (other than perhaps Euclid's). To him a curve was the path of a continuously moving point. More precisely, it was $\{(f(t), g(t)) | f, g: [0,1] \to \mathbb{R}$ are continuous functions$\}$.[4] In 1890 Peano gave his famous and astounding example of a "space-filling curve;" that is, he exhibited a continuous mapping of the unit interval onto a square (including its interior). But that, according to Jordan's definition, made the square into a curve—a not very desirable state of affairs. "How was it possible that intuition could so deceive us?", wondered Poincaré ([**24**], p. 123). Jordan's definition was too broad and had to be modified.

But Jordan's definition also turned out to be too narrow. For we would surely want the graph of $y = \sin \frac{1}{x}$ and its limit points on the y-axis (i.e., $\{(x, \sin \frac{1}{x}): x \in (-\infty, 0) \cup (0, \infty)\} \cup \{(0, y): -1 \leq y \leq 1\}$) to be called a curve, but it is not the image of a continuously moving point.[5]

Metaparadox: How can a definition be both too broad and too narrow?

A satisfactory resolution of the dilemma was achieved (by Menger and Uryson) only in the 1920s. First one had to clarify the notion of dimension ([**18**]).[6] When this was done, a curve was defined as a one-dimensional continuum (see [**28**]).[7] The definition proved adequate until the 1970s when Mandelbrot introduced curves—his fractals—whose dimensions are fractions. See [**9**].

[4]It was in this context that he stated, and proved (incorrectly, as it later turned out) the celebrated "Jordan-Curve Theorem".

[5]This is intuitively clear, although to prove it we need topological notions. See [**11**], p. 1968.

[6]That notion, too, was challenged by the paradoxical Peano curve which implied that a square is one-dimensional since it is the continuous image of the unit interval. Cantor's proof of the one-one correspondence between an interval and a square also put to question the intuitive notion of dimension.

[7]A continuum is a closed, connected set of points.

PARADOXES INVOLVING DECOMPOSITION OF GEOMETRIC OBJECTS.

(a) In 1924 Banach and Tarski proved that a pea and the sun are equidecomposable. That is, the pea may be cut up into finitely many pieces[8] which can be rearranged to yield the sun (in volume if not in substance). This is the celebrated *Banach-Tarski paradox* (see [**27**]). In fact, Banach and Tarski have shown that *any* two bounded sets in Euclidean space $\mathbb{R}^n$ are equidecomposable if they contain interior points and if $n > 2$ (see [**2**], p. 351).[9]

Of course, the pieces into which the pea is cut in the Banach-Tarski decomposition are *not measurable*; that is, they do not have a volume. They are not the kinds of pieces that can be obtained using scissors or other cutting devices. They are obtained using the axiom of choice.

Metaparadox: How can simple assumptions (e.g. the axiom of choice) have such formidable consequences (e.g. the Banach-Tarski paradox)?

Of course, the axiom of choice may not be such a simple assumption after all (see [**19**]). But it would have been very helpful to the Delians of Greek antiquity ([**27**], p. v):

Delians: "How can we be rid of the plague?"
Delphic Oracle: "Construct a cubic altar double the size of the existing altar."
Banach and Tarksi: "Can we use the axiom of choice?"

(b) At long last, the circle has been squared. This is no hoax. It is the title of an article which appeared recently in the reputable *Notices of the American Mathematical Society* ([**10**]). In 1988 the Hungarian mathematician Laczkovich showed that the circle can be decomposed into finitely many pieces which can be reassembled to give a square of equal area. But the pieces are not measurable (none has an area) and the decomposition is secured using the axiom of choice. See [**10**].

CONCLUDING REMARKS. We have presented a variety of mathematical paradoxes from different historical periods. They resulted from (among other things) debates and controversies among mathematicians, counterexamples to what were thought to be immutable notions, failures to see the need for tightening (broadening) a concept or broadening (tightening) a result, and the application of a "principle of continuity" which suggested the transferability of procedures from a given case to what appeared to be like cases. We saw that such paradoxical phenomena have had a very substantial impact on the development of mathematics through the refinement and reshaping of concepts, the broadening of existing theories and the rise of new ones. Moreover, this process is ongoing.

We have also suggested roles for paradoxes in the teaching and learning of mathematics. They can generate curiosity, increase motivation, create an effective environment for debate, encourage the examination of underlying assumptions, and show that faulty logic and erroneous arguments are not an uncommon feature of the mathematical enterprise.

[8]It was shown in the 1940s that five pieces suffice; in fact, no number less than five will do.

[9]If one allows for *denumerable* decompositions, then this result holds also for $n = 2$ (see [**2**], p. 351).

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