# Studies on Elliptic Functions
# (By N. H. Abel)

Translated from the French* by Marcus Emmanuel Barnes

The logarithmic function, the exponential and circular functions were for a long time the only transcendental functions which attracted the attention of geometers. Only recently have others been considered. Among these, we must distinguish certain functions named elliptic, as they have nice analytical properties for application in diverse branches of mathematics. The idea for these functions was given by the immortal Euler, who demonstrated that the separable equation

$$\frac{\partial x}{\sqrt{\alpha+\beta x+\gamma x^2+\delta x^3+\epsilon x^4}}+\frac{\partial y}{\sqrt{\alpha+\beta y+\gamma y^2+\delta y^3+\epsilon y^4}}=0$$

has an algebraic integral. [1] After Euler, Lagrange added something, by giving his elegant theory of the transformation of the integral $\int \frac{R.\partial x}{\sqrt{[(1-p^2x^2)(1-q^2x^2)]}}$, where $R$ is a rational function in $x$. But the first, and if I am not mistaken, the only one who deepened the nature of these functions, was M. Legendre, who, first in his memoir on elliptic functions[2], and then in his excellent mathematical exercises, developed a number of elegant properties of these functions and showed their application. At the time of the publication of this work, nothing has been added to M. Legendre's theory. I believe that one will see with pleasure here the subsequent study of these functions.

In general, by the term elliptic function we understand any function included in the integral

$$\int \frac{R\partial x}{\sqrt{(\alpha+\beta x+\gamma x^2+\delta x^3+\epsilon x^4)}},$$

*N. H. Abel, Recherches sur les fonctions elliptiques, *Journal fr die reine und angewandte Mathematik*, Vol. 2, 1827. pp. 101-181.

[1] Presumably, Abel is referring to "Integratio aequationis $dx/\sqrt{(A+Bx+Cx^2+Dx^3+Ex^4)} = dy/\sqrt{(A+By+Cy^2+Dy^3+Ey^4)}$ which was originally published in *Novi Commentarii academiae sceintiarum Petropolitanea* 12, 1768, pp.3-16; the paper has been republished in Euler's *Opera Omnia*: Series 1, Vol. 20, pp. 302-317.

[2] Here Abel is presumably referring to Legendre's *Exercises de calcul intégral*, 1811.

where $R$ is a rational function and $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$ are real constants. M. Legendre demonstrated that with appropriate substitutions one can always reduce this integral to the form

$$\int \frac{P\partial y}{\sqrt{(a+by^2+cy^4)}},$$

where $P$ is a rational function in $y^2$. With appropriate reductions this integral can always be brought to the form

$$\int \frac{A+By^2}{C+Dy^2}\cdot\frac{\partial y}{\sqrt{(a+by^2+cy^4)}},$$

and this to:

$$\int \frac{A+B\sin^2\theta}{C+D\sin^2\theta}\cdot\frac{\partial\theta}{\sqrt{(1-c^2\sin^2\theta)}},$$

where $c$ is real and less than one.

It follows that all elliptic functions can be reduced to one of three forms:

$$\int \frac{\partial\theta}{\sqrt{(1-c^2\sin^2\theta)}};\int \partial\theta\sqrt{(1-c^2\sin^2\theta)};\int \frac{\partial\theta}{(1+n\sin^2\theta)\sqrt{(1-c^2\sin^2\theta)}},$$

which M. Legendre named elliptic functions of the first, second, and third kind. It was these three functions which M. Legendre considered above all, the first which is the simplest and has the most remarkable properties.

I propose in this memoir to consider the inverse functions, that is, the function $\varphi\alpha$, determined by the equations

$$\alpha = \int \frac{\partial\theta}{\sqrt{1-c^2\sin^2\theta}} \text{ and}$$

$$\sin\theta = \varphi(\alpha) = x.$$

The last equation gives

$$\partial\theta.\sqrt{(1-\sin^2\theta)} = \partial.\varphi\alpha = \partial x,$$

so

$$\alpha = \int \frac{\partial x}{\sqrt{[(1-x^2)(1-c^2x^2)]}}.$$

M. Legendre assumes $c^2$ was positive, but I noticed that the previous formulas are simpler if we assume $c^2$ to be negative, $= -e^2$. Similarly, I write for more symmetry $1 - c^2x^2$ in place of $1 - x^2$. Thus the function $\varphi\alpha = x$ will be given by the equation

$$\alpha = \int \frac{\partial x}{\sqrt{[(1 - c^2x^2)(1 + e^2x^2)}},$$

or

$$\varphi'\alpha = \partial\alpha\sqrt{[(1 - c^2\varphi^2\alpha)(1 + e^2\varphi^2\alpha)]}.$$

For short, I introduce two other functions of $\alpha$, namely:

$$f\alpha = \sqrt{(1 - c^2\varphi^2\alpha)};\ F\alpha = \sqrt{(1 + e^2\varphi^2\alpha)}.$$

Several properties of these functions allow one to immediately deduce well known properties of the elliptic function of the first kind, but the others are more hidden. For example, one can show that the equations $\varphi\alpha = 0$, $f\alpha = 0$, $F\alpha = 0$ have a infinite number of roots, all of which one can find. One of the remarkable properties is that one can express $\varphi(m\alpha)$. $f(m\alpha)$, $F(m\alpha)$ ($m$ being a whole number) as rational in $\varphi\alpha$, $f\alpha$,$F\alpha$. Also it is not hard to find $\varphi(m\alpha)$, $f(m\alpha)$, $F(m\alpha)$ when $\varphi\alpha$, $f\alpha$, $F\alpha$ are known; but the inverse problem, to determine $\varphi\alpha$, $f\alpha$, $F\alpha$ in $\varphi(m\alpha)$, $f(m\alpha)$, $F(m\alpha)$ is very difficult because it depends on an equation of higher degree (namely of degree $m^2$).

The solution of this equation is the principle object of this memoir. First we will show how to find all the roots using the functions $\varphi$, $f$, $F$. Later we will treat the algebraic solution of the equation in question and reach a remarkable result, that $\varphi(\frac{\alpha}{m})$; $f(\frac{\alpha}{m})$; $F(\frac{\alpha}{m})$ may be expressed in $\varphi\alpha$, $f\alpha$, $F\alpha$ by means of one function, which in relation to $\alpha$, contains no irrationalities other than radicals. This produces a class of very general equations which can be solved algebraically. We remark that the expressions for the roots contain constant quantities which in general are not expressible as algebraic quantities. These constant quantities depend on an equation of degree $m^2 - 1$. We will show how, by means of algebraic functions, one can reduce its solution to the solution of an equation of degree $m + 1$. We will give several expressions for the functions $\varphi(2n + 1)\alpha$, $f(2n + 1)\alpha$, $F(2n + 1)\alpha$ as functions of $\varphi\alpha$, $f\alpha$, $F\alpha$. We will subsequently deduce the values of $\varphi\alpha$, $f\alpha$, $F\alpha$ as functions of $\alpha$. We will demonstrate that these functions can be decomposed into infinitely many factors and even into infinitely many partial fractions.

## §. I.
## Fundamental Properties of $\varphi\alpha$; $f\alpha$; $F\alpha$.

### 1.

Suppose that $\varphi\alpha = x$ (1.), we have by the preceding:

$$2. \quad \alpha = \int_0^x \frac{\partial x}{\sqrt{[(1-c^2x^2)(1+e^2x^2)]}}.$$

From that one sees that $\alpha$, considered as a function of $x$, is positive from $x = 0$, to $x = \frac{1}{c}$. If we let

$$3. \quad \frac{\omega}{2} = \int_0^{\frac{1}{c}} \frac{\partial x}{\sqrt{[(1-c^2x^2)(1+e^2x^2)]}},$$

it is clear that $\varphi\alpha$ is positive and is increasing from $\alpha = 0$ to $\alpha = \frac{\omega}{2}$. One will have

$$4. \quad \varphi(0) = 0, \varphi\left(\frac{\omega}{2}\right) = \frac{1}{c}.$$

Because $\alpha$ changes sign when one writes $-x$ in place of $x$, the same holds for the function $\varphi\alpha$ in relation to $\alpha$, and as a consequence we have

$$5. \quad \varphi(-\alpha) = -\varphi(\alpha).$$

Putting $xi$ in (1.) instead of $x$ (where for short, $i$ represents the imaginary quantity $\sqrt{-1}$) and designating the value of $\alpha$ as $\beta i$, it becomes

$$6. \quad xi = \varphi(\beta i) \text{ and } \beta = \int_0^x \frac{\partial x}{\sqrt{[(1+c^2x^2)(1-e^2x^2)]}}.$$

where $\beta$ is real and positive from $x = 0$ to $x = \frac{1}{e}$. Thus if we let

$$7. \quad \frac{\varpi}{2} = \int_0^{\frac{1}{e}} \frac{\partial x}{\sqrt{[(1-e^2x^2)(1+c^2x^2)]}},$$

$x$ will be positive from $\beta = 0$ to $\beta = \frac{\varpi}{2}$; that is, the function $\frac{1}{i}\varphi(\beta i)$ is positive between these same limits. Given $\beta = \alpha$ and $y = \frac{\varphi(\alpha i)}{i}$, one has

$$\alpha = \int_0^y \frac{\partial y}{\sqrt{[(1-e^2y^2)(1+c^2y^2)]}}.$$

Thus one sees, supposing $c$ instead of $e$ and $e$ instead of $c$,

$$\frac{\varphi(\alpha i)}{i} \text{ changes to } \varphi\alpha.$$

And since $\begin{aligned} f\alpha &= \sqrt{(1-c^2\varphi^2\alpha)}, \\ F\alpha &= \sqrt{1+e^2\varphi^2\alpha)}, \end{aligned}$

we see that changing $c$ to $e$ and $e$ to $c$, $f(\alpha i)$ and $F(\alpha i)$ turn respectively into $F\alpha$ and $f\alpha$. Finally, the equations (2.) and (7.) show, that by the same transformation $\omega$ and $\varpi$

turn respectively into $\varpi$ and $\omega$. According to (7.) one will have $x = \frac{1}{e}$ for $\beta = \frac{\varpi}{2}$, thus in virtue of the equation $xi = \varphi(\beta i)$ we have

$$8. \quad \varphi\left(\frac{\varpi i}{2}\right) = i.\frac{1}{e}.$$

2.

In virtue of the above, we have the values of $\varphi\alpha$ for all real values of $\alpha$ between $-\frac{\omega}{2}$ and $+\frac{\omega}{2}$, and for all imaginary values of the from $\beta i$ for this value, if $\beta$ is a quantity contained in the limits $-\frac{\varpi}{2}$ and $+\frac{\varpi}{2}$. Now we will find the value of these functions for an arbitrary real or imaginary value of the variable. To that end, we initially establish the fundamental properties of the functions $\varphi$, $f$ and $F$. Because

$$\begin{aligned} f^2\alpha &= 1 - c^2\varphi^2\alpha, \\ F^2\alpha &= 1 + e^2\varphi^2\alpha, \end{aligned}$$

by differentiating we have:

$$\begin{aligned} f\alpha.f'\alpha &= -c^2\varphi\alpha.\varphi'\alpha, \\ F\alpha.F'\alpha &= e^2\varphi\alpha\varphi'\alpha, \end{aligned}$$

Now, according to (2.) we have

$$\varphi'\alpha = \sqrt{[(1 - c^2\varphi^2\alpha)(1 + e^2\varphi^2\alpha)]} = f\alpha.F\alpha.$$

Thus by substituting this value of $\varphi'$ into the two preceding equations, we find that the functions $\varphi\alpha$, $f\alpha$, $F\alpha$ are related by the equations

$$9. \quad \left\{ \begin{aligned} \varphi'\alpha &= f\alpha.F\alpha, \\ f'\alpha &= -c^2\varphi\alpha.F\alpha, \\ F'\alpha &= e^2\varphi\alpha.f\alpha. \end{aligned} \right.$$

That established, denote two indeterminates by $\alpha$ and $\beta$. We have

$$10. \quad \left\{ \begin{aligned} \varphi(\alpha+\beta) &= \frac{\varphi\alpha.f\beta.F\beta + \varphi\beta.f\alpha.F\alpha}{1 + e^2c^2.\varphi^2\alpha.\varphi^2\beta}, \\ f(\alpha+\beta) &= \frac{f\alpha.f\beta - c^2.\varphi\alpha.\varphi\beta.F\alpha.F\beta}{1 + e^2c^2\varphi^2\alpha.\varphi^2\beta}, \\ F(\alpha+\beta) &= \frac{F\alpha.F\beta + e^2.\varphi\alpha.\varphi\beta.f\alpha.f\beta}{1 + e^2c^2\varphi^2\alpha.\varphi^2\beta}. \end{aligned} \right.$$

These formulas can be deduced at once from the known properties of elliptic functions *)[3]; but one can also verify them easily in the following manner.

Letting r be the right side of the first equation in (10.), we have by differentiating with respect to $\alpha$:

$$\begin{aligned}\left(\frac{\partial r}{\partial \alpha}\right) &= \frac{\{\varphi'\alpha.f\beta.F\beta + \varphi\beta.F\alpha.f'\alpha + \varphi\beta.f\alpha.F'\alpha\}}{1 + e^2c^2\varphi^2\alpha.\varphi^2\beta} \\ &- \frac{(\varphi\alpha.f\alpha.F\beta + \varphi\beta.f\alpha.F\alpha).2e^2c^2\varphi\alpha.\varphi^2\beta.\varphi'\alpha}{(1 + e^2c^2\varphi^2\alpha.\varphi^2\beta)^2}.\end{aligned}$$

Substituting the values given in equations (9.) for $\varphi'\alpha$, $f'\alpha$, $F'\alpha$, it becomes

$$\left(\frac{\partial r}{\partial \alpha}\right) = \frac{f\alpha.F\alpha.f\beta.F\beta}{1 + e^2c^2\varphi^2\alpha.\varphi^2\beta} - \frac{2e^2c^2\varphi^2\alpha.\varphi^2\beta.f\alpha.f\beta.F\alpha.F\beta}{(1 + e^2c^2\varphi^2\alpha\varphi^2\beta)^2}$$
$$+\frac{\varphi\alpha.\varphi\beta.\{1 + e^2c^2\varphi^2\alpha\varphi^2\beta\}\{-c^2F^2\alpha + e^2f^2\alpha\} - 2e^2c^2\varphi\alpha\varphi\beta.\varphi^2\beta.f^2\alpha.F^2\alpha}{(1 + e^2c^2\varphi^2\alpha\varphi^2\beta)^2},$$

Whence, substituting for $f^2\alpha$ and $F^2\alpha$ their values: $1-c^2\varphi^2\alpha$, $1+e^2\varphi^2\alpha$ and simplifying, one concludes

$$\left(\frac{\partial r}{\partial \alpha}\right) = \frac{(1 - e^2c^2\varphi^2\alpha.\varphi^2\beta)\{(e^2 - c^2) - \varphi\alpha.\varphi\beta + f\alpha.f\beta.F\alpha.F\beta\} - 2e^2c^2\varphi\alpha\varphi\beta(\varphi^2\alpha + \varphi^2\beta)}{(1 + e^2c^2\varphi^2\alpha.\varphi^2\beta)^2}.$$

Now $\alpha$ and $\beta$ enter symmetrically into the expression for $r$; thus one will have the value of $\left(\frac{\partial r}{\partial \beta}\right)$ by permuting $\alpha$ and $\beta$ in the expression for $\left(\frac{\partial r}{\partial \alpha}\right)$. However, since the expression for $\left(\frac{\partial r}{\partial \alpha}\right)$ does not change value, we thus have $\left(\frac{\partial r}{\partial \alpha}\right) = \left(\frac{\partial r}{\partial \beta}\right)$.

This partial differential equation shows $r$ is a function of $\alpha + \beta$; thus

$$r = \psi(\alpha + \beta).$$ [4]

The form of the function $\psi$ is found by giving $\beta$ a particular value. Suppose, for example, that $\beta = 0$ with the condition that $\varphi(0) = 0$, $f(0) = 1$, $F(0) = 1$. The two values of $r$ will become

$$r = \varphi(\alpha) \quad \text{and} \quad r = \psi(\alpha).$$

[3]*) Legendre Exercices de calcul intégral.

[4]Abel himself proved this in "Untersuchung der Functionen zweier unabhängig veränderlicher Größen $x$ und $y$, wie $f(x, y)$, welche die Eigenschaft haben, daß $f(z, f(x, y))$ eine symmetrische Function von $z$, $x$ und $y$ ist," *Journal Fúr die reine und angewandte Mathematik*, **1**(1), 11-15.

Thus

$$\psi(\alpha) = \varphi(\alpha),$$

and hence,

$$r = \psi(\alpha+\beta) = \varphi(\alpha+\beta).$$

The first of the formulas in (10.) thus indeed takes place.
In the same manner, we verify the other two formulas.

3.

From the formulas in (10.) we can deduce many others. I will show some of the most remarkable: For short, I let

$$11. \quad 1 + e^2c^2\varphi^2\alpha.\varphi^2\beta = R.$$

By first changing the sign of $\beta$, we obtain

$$12. \quad \begin{cases} \varphi(\alpha+\beta) + \varphi(\alpha-\beta) &= \dfrac{2\varphi\alpha.f\beta.F\beta}{R}, \\ \varphi(\alpha+\beta) - \varphi(\alpha-\beta) &= \dfrac{2\varphi\beta.f\alpha.F\alpha}{R}, \\ f(\alpha+\beta) + f(\alpha-\beta) &= \dfrac{2f\alpha.f\beta}{R}, \\ f(\alpha+\beta) - f(\alpha-\beta) &= \dfrac{-2c^2.\varphi\alpha.\varphi\beta.F\alpha.F\beta}{R}, \\ F(\alpha+\beta) + F(\alpha-\beta) &= \dfrac{2F\alpha.F\beta}{R}, \\ F(\alpha+\beta) - F(\alpha-\beta) &= \dfrac{2e^2.\varphi\alpha.\varphi\beta.f\alpha.f\alpha}{R}. \end{cases}$$

By taking the product of $\varphi(\alpha+\beta)$ and $\varphi(\alpha-\beta)$, we find that

$$\begin{aligned} \varphi(\alpha+\beta).\varphi(\alpha-\beta) &= \frac{\varphi\alpha.f\beta.F\beta + \varphi\beta.f\alpha.F\alpha}{R} \cdot \frac{\varphi\alpha.f\beta.F\beta - \varphi\beta.f\alpha.F\alpha}{R} \\ &= \frac{\varphi^2\alpha.f^2\beta.F^2\beta - \varphi^2\beta.f^2\alpha.F^2\alpha}{R^2}, \end{aligned}$$

or by substituting the values of $f^2\beta$, $F^2\beta$, $f^2\alpha$, $F^2\alpha$ in $\varphi\beta$ and $\varphi\alpha$:

$$\begin{aligned} \varphi(\alpha+\beta).\varphi(\alpha-\beta) &= \frac{\varphi^2\alpha - \varphi^2\beta - e^2c^2\varphi^2\alpha.\varphi^4\beta + e^2c^2\varphi^2\beta.\varphi^4\alpha}{R^2} \\ &= \frac{(\varphi^2\alpha - \varphi^2\beta)(1 + e^2c^2\varphi^2\alpha.\varphi^2\beta)}{R^2}; \end{aligned}$$

now $R = 1 + e^2c^2\varphi^2\alpha.\varphi^2\beta$, thus

$$13.\qquad \varphi(\alpha+\beta).\varphi(\alpha-\beta) = \frac{\varphi^2\alpha - \varphi^2\beta}{R}.$$

Similarly one finds

$$14.\qquad \left\{\begin{array}{l} f(\alpha+\beta).f(\alpha-\beta) = \dfrac{f^2\alpha - c^2\varphi^2\beta.F^2\alpha}{R} = \dfrac{f^2\beta - c^2\varphi^2\alpha.F^2\beta}{R} \\ = \dfrac{1 - c^2\varphi^2\alpha - c^2\varphi^2\beta - c^2e^2\varphi^2\alpha.\varphi^2\beta}{R} = \dfrac{f^2\alpha.f^2\beta - c^2(c^2+e^2)\varphi^2\alpha.\varphi^2\beta}{R}, \\ F(\alpha+\beta).F(\alpha-\beta) = \dfrac{F^2\alpha + e^2\varphi^2\beta.f^2\alpha}{R} = \dfrac{F^2\beta + e^2\varphi^2\alpha.f^2\beta}{R} \\ = \dfrac{1 + e^2\varphi^2\alpha + e^2\varphi^2\beta - e^2c^2\varphi^2\alpha.\varphi^2\beta}{R} = \dfrac{F^2\alpha.F^2\beta - e^2(c^2+e^2)\varphi^2\alpha.\varphi^2\beta}{R}. \end{array}\right.$$

4.

Letting $\beta = \pm\frac{\omega}{2}$, $\beta = \pm\frac{\varpi}{2}i$ and noticing that $f(\pm\frac{\varpi}{2}) = 0$, $F(\pm\frac{\varpi}{2}i) = 0$ we get

$$15.\qquad \left\{\begin{array}{rl} \varphi(\alpha \pm \dfrac{\omega}{2}) = & \pm\varphi\dfrac{\omega}{2}.\dfrac{f\alpha}{F\alpha};\quad f(\alpha\pm\dfrac{\omega}{2}) = \mp\dfrac{F\frac{\omega}{2}}{\varphi\frac{\omega}{2}}.\dfrac{\varphi\alpha}{F\alpha}; \\ & F(\alpha\pm\frac{\omega}{2}) = \frac{F\frac{\omega}{2}}{F\alpha}; \\ \varphi(\alpha\pm\dfrac{\varpi}{2}i) = & \pm\varphi\frac{\varpi}{2}i.\frac{F\alpha}{f\alpha};\quad F(\alpha\pm\frac{\varpi}{2}i) = \mp\frac{f\frac{\varpi}{2}i}{\varphi\frac{\varpi}{2}i}.\frac{\varphi\alpha}{f\alpha}; \\ & f(\alpha\pm\dfrac{\varpi}{2}i) = \dfrac{f\frac{\varpi}{2}i}{f\alpha}; \end{array}\right.$$

or:

$$16.\qquad \left\{\begin{array}{rl} \varphi(\alpha\pm\dfrac{\omega}{2}) = & \pm\dfrac{1}{c}\dfrac{f\alpha}{F\alpha};\quad f(\alpha\pm\dfrac{\omega}{2}) = \mp\sqrt{(e^2+c^2)}.\dfrac{\varphi\alpha}{F\alpha}; \\ & F(\alpha\pm\dfrac{\omega}{2}) = \dfrac{\sqrt{(e^2+c^2)}}{c}.\dfrac{1}{F\alpha}; \\ \varphi(\alpha\pm\dfrac{\varpi}{2}i) = & \pm\dfrac{i}{e}.\dfrac{F\alpha}{f\alpha};\quad F(\alpha\pm\dfrac{\varpi}{2}i) = \pm i\sqrt{(e^2+c^2)}.\dfrac{\varphi\alpha}{f\alpha}; \\ & f(\alpha\pm\dfrac{\varpi}{2}i) = \dfrac{\sqrt{(e^2+c^2)}}{e}.\dfrac{1}{f\alpha}. \end{array}\right.$$

From which one concludes at once:

$$17.\qquad \left\{\begin{array}{rl} \varphi(\dfrac{\omega}{2}+\alpha) = & \varphi(\dfrac{\omega}{2}-\alpha);\quad f(\dfrac{\omega}{2}+\alpha) = -f(\dfrac{\omega}{2}-\alpha); \\ & F(\dfrac{\omega}{2}+\alpha) = F(\dfrac{\omega}{2}-\alpha); \\ \varphi(\dfrac{\varpi}{2}i+\alpha) = & \varphi(\frac{\varpi}{2}i-\alpha);\quad F(\frac{\varpi}{2}i+\alpha) = -F(\frac{\varpi}{2}i-\alpha); \\ & f(\dfrac{\varpi}{2}i+\alpha) = f(\dfrac{\varpi}{2}i-\alpha); \end{array}\right.$$

18. $\varphi\left(\alpha \pm \frac{\omega}{2}\right).\varphi\left(\alpha + \frac{\varpi}{2}i\right) = \pm\frac{i}{ce}; \quad F\left(\alpha \pm \frac{\omega}{2}\right).F\alpha = \frac{b}{c}; f\left(\alpha \pm \frac{\varpi}{2}i\right).f\alpha = \frac{b}{e}.$

Letting $\alpha = \frac{\omega}{2}$ and $\frac{\varpi}{2}i$, we find that

$$\varphi\left(\frac{\omega}{2} + \frac{\varpi}{2}i\right) = \frac{1}{0}, \quad f\left(\frac{\omega}{2} + \frac{\varpi}{2}i\right) = \frac{1}{0}, F\left(\frac{\omega}{2} + \frac{\varpi}{2}i\right) = \frac{1}{0}.$$

Replacing $\alpha$ with $\alpha + \frac{\omega}{2}$ in the first three equations in (17.) and replacing $\alpha$ with $\alpha + \frac{\varpi}{2}i$ in the last three, gives the following

$$19. \quad \begin{cases} \varphi(\alpha+\omega) = -\varphi\alpha; \quad f(\alpha+\omega) = -f\alpha; \quad F(\alpha+\omega) = F\alpha; \\ \varphi(\alpha+\varpi i) = -\varphi\alpha; \quad f(\alpha+\varpi i) = f\alpha; \quad F(\alpha+\varpi i) = -F\alpha; \end{cases}$$

and by replacing $\alpha$ with $\alpha+\omega$ and $\alpha+\varpi i$:

$$20. \quad \begin{cases} \varphi(2\omega+\alpha) = \varphi\alpha; \quad \varphi(2\varpi i+\alpha) = \varphi\alpha; \quad \varphi(\omega+\varpi i+\alpha) = \varphi\alpha; \\ f(2\omega+\alpha) = f\alpha; \quad f(\varpi i+\alpha) = f\alpha; \\ F(\omega+\alpha) = F\alpha; \quad F(2\varpi i+\alpha) = F\alpha. \end{cases}$$

These equations show that the functions $\varphi\alpha$, $f\alpha$, $F\alpha$ are periodic functions. We deduce without trouble that when $m$ and $n$ are two positive or negative whole numbers:

$$21. \quad \begin{cases} \varphi((m+n)\omega + (m-n)\varpi i + \alpha) = \varphi\alpha; \quad \varphi((m+n)\omega + (m-n+1)\varpi i + \alpha) = -\varphi\alpha; \\ f(2m\omega + n\varpi i + \alpha) = f\alpha; \quad f((2m+1)\omega + n\varpi i + \alpha) = -f\alpha; \\ F(m\omega + 2n\varpi i + \alpha) = +F\alpha; \quad F(m\omega + (2n+1)\varpi i + \alpha) = -F\alpha. \end{cases}$$

These formulas may also be written as:

$$22. \quad \begin{cases} \varphi(m\omega + n\varpi i \pm \alpha) = \pm(-1)^{m+n}.\varphi\alpha, \\ f(m\omega + n\varpi i \pm \alpha) = (-1)^m.f\alpha, \\ F(m\omega + n\varpi i \pm \alpha) = [(-1)^n].F\alpha. \end{cases}$$ [5]

We make note of these particular cases:

$$22.' \quad \begin{cases} \varphi(m\omega \pm \alpha) = \pm(-1)^m.\varphi\alpha; & \varphi(n\varpi i \pm \alpha) = \pm(-1)^n.\varphi\alpha; \\ f(m\omega \pm \alpha) = (-1)^m.f\alpha; & f(n\varpi i \pm \alpha) = f\alpha; \\ F(m\omega \pm \alpha) = F\alpha; & F(n\varpi i \pm \alpha) = (-1)^n.F\alpha. \end{cases}$$

5.

[5] In the original, $[(-1)^n]$ was written as $(-)^n$.

The formulas that we have just established show that we will have the values of the functions $\varphi\alpha$; $f\alpha$; $F\alpha$ for all real and imaginary values of the variable, once we know them for real values between $\frac{\omega}{2}$ and $-\frac{\omega}{2}$ and imaginary values of the form $\beta i$, where $\beta$ is between $\frac{\varpi}{2}$ and $-\frac{\varpi}{2}$.

Indeed, let us suppose that one wants the values of the functions $\varphi(\alpha+\beta i)$, $f(\alpha+\beta i)$, $F(\alpha+\beta i)$, where $\alpha$ and $\beta$ are any real quantities. Putting $\beta i$ in place of $\beta$ in the formulas in (10.), it is clear that the three functions concerned will be expressed by the functions: $\varphi\alpha$; $f\alpha$; $F\alpha$;$\varphi(\beta i)$; $f(\beta i)$; $F(\beta i)$. Thus it only remains to determine the latter. Now, whatever the values of $\alpha$ and $\beta$, we can always find two integers $m$ and $n$, such as $\alpha = m.\omega \pm \alpha'$, $\beta = n\varpi \pm \beta'$, where $\alpha'$ is a quantity between 0 and $+\frac{\omega}{2}$, and $\beta'$ between 0 and $+\frac{\varpi}{2}$. Thus in virtue of equations (22.), substituting the previous values of $\alpha$ and $\beta$ gives:

$$\begin{aligned}
\varphi(\alpha) &= \varphi(m\omega \pm \alpha') = \pm(-1)^m.\varphi\alpha', \\
f(\alpha) &= f(m\omega \pm \alpha') = (-1)^m.f\alpha', \\
F(\alpha) &= F(m\omega \pm \alpha') = F\alpha', \\
\varphi(\beta i) &= \varphi(n\varpi i \pm \beta' i) = \pm(-1)^n.\varphi(\beta' i), \\
f(\beta i) &= f(n\varpi i \pm \beta' i) = f(\beta'), \\
F(\beta i) &= F(n\varpi i \pm \beta' i) = (-1)^n.F(\beta' i).
\end{aligned}$$

Therefore the functions $\varphi\alpha$, $f\alpha$, $F\alpha$, $\varphi(\beta i)$, $f(\beta i)$, $F(\beta i)$, are expressible in the way we just said, as will be the functions $\varphi(\alpha+\beta i)$; $f(\alpha+\beta i)$; $F(\alpha+\beta i)$.

We saw earlier that $\varphi\alpha$ is real from $\alpha = -\frac{\omega}{2}$ to $\alpha = +\frac{\omega}{2}$ and that $\frac{\varphi(\alpha i)}{i}$ is real from $\alpha = -\frac{\varpi}{2}$ to $\alpha = +\frac{\varpi}{2}$.

Hence, in virtue of equations (22.) it is clear:

1) that $\varphi(\alpha)$ and $\frac{\varphi(\alpha i)}{i}$ are real for all real values of $\alpha$; $\varphi\alpha$ is contained between $-\frac{1}{c}$ and $+\frac{1}{c}$ and $\frac{\varphi(\alpha i)}{i}$ is contained between $-\frac{1}{e}$ and $+\frac{1}{e}$;

2) that $\varphi(\alpha)$ vanishes at $\alpha = m\omega$ and $\frac{\varphi(\alpha i)}{i}$ at $\alpha = m\varpi$, $m$ being a positive or negative integer; but $\varphi(\alpha)$ does not vanish for any other real values of $\alpha$.

Noticing that $f\alpha = \sqrt{(1-c^2\varphi^2\alpha)}$, $F\alpha = \sqrt{(1+e^2\varphi^2\alpha)}$, it follows from what we have just said:

1) that the functions $f(\alpha)$; $F(\alpha)$; $f(\alpha i)$; $F(\alpha i)$ are real for all values of $\alpha$;

2) that $f(\alpha)$ is contained between the limits $-1$ and $+1$ and $F(\alpha)$ is contained between the limits $+1$ and $+\sqrt{\left(1+\frac{e^2}{c^2}\right)}$, so that $F\alpha$ is positive for all real values of $\alpha$;

3) that $f(\alpha i)$ is positive between the limits $+1$ and $\sqrt{\left(1+\frac{c^2}{e^2}\right)}$ and $F(\alpha i)$ between the limits $-1$ and $+1$ for all real values of $\alpha$;

4) that $f(\alpha)$ vanishes at $\alpha = \left(m+\frac{1}{2}\right)\omega$ and $F(i\alpha)$ at $\alpha = \left(m+\frac{1}{2}\right)\varpi$; but for no other values of $\alpha$.

Take note of the following corollaries that can be deduced from formulas (22.):

1) Let $\alpha = 0$. In this case notice that $\varphi(0) = 0$, $f(0) = 1$, $F(0) = 1$. This gives

$$23. \quad \begin{cases} \varphi(m\omega + n\varpi i) = 0, \\ f(m\omega + n\varpi i) = (-1)^m, \\ F(m\omega + n\varpi i) = (-1)^n. \end{cases}$$

2) Let $\alpha = \frac{\omega}{2}$. In virtue of the equations:

$$\varphi\left(\frac{\omega}{2}\right) = \frac{1}{c}; \quad f\left(\frac{\omega}{2}\right) = 0; \quad F\left(\frac{\omega}{2}\right) = \frac{b}{c};$$

one will have

$$24. \quad \begin{cases} \varphi\left(\left(m+\frac{1}{2}\right)\omega + n\varpi i\right) = (-1)^{m+n}.\frac{1}{c}, \\ f\left(\left(m+\frac{1}{2}\right)\omega + n\varpi i\right) = 0, \\ F\left(\left(m+\frac{1}{2}\right)\omega + n\varpi i\right) = (-1)^{n}.\frac{b}{c}. \end{cases}$$

3.) Let $\alpha = \frac{\varpi}{2}i$. In virtue of the equations

$$\varphi\left(\frac{\varpi}{2}i\right) = \frac{i}{e}; \quad f\left(\frac{\varpi}{2}i\right) = \frac{b}{e}; \quad F\left(\frac{\varpi}{2}i\right) = 0;$$

we have:

$$25. \quad \begin{cases} \varphi\left(m\omega + \left(n+\frac{1}{2}\right)\varpi i\right) = (-1)^{m+n}\frac{i}{e}, \\ f\left(m\omega + \left(n+\frac{1}{2}\right)\varpi i\right) = (-1)^{m}.\frac{b}{e}, \\ F\left(m\omega + \left(n+\frac{1}{2}\right)\varpi i\right) = 0. \end{cases}$$

4.) Let $\alpha = \frac{\omega}{2} + \frac{\varpi}{2}i$. In virtue of the equations above,

$$26. \quad \begin{cases} \varphi\left(\left(m+\frac{1}{2}\right)\omega + \left(n+\frac{1}{2}\right)\varpi i\right) = \frac{1}{0}, \\ f\left(\left(m+\frac{1}{2}\right)\omega + \left(n+\frac{1}{2}\right)\varpi i\right) = \frac{1}{0}, \\ F\left(\left(m+\frac{1}{2}\right)\omega + \left(n+\frac{1}{2}\right)\varpi i\right) = \frac{1}{0}. \end{cases}$$

6.

Equations (23.), (24.), (25.) show that the function $\varphi(\alpha)$ always vanishes when $\alpha$ is of the form $\alpha = m\omega + n\varpi i$; that $f\alpha$ always vanishes when $\alpha$ is of the form $\alpha = \left(m+\frac{1}{2}\right)\omega + n\varpi i$, and $F\alpha$ always vanishes when $\alpha$ is of the form $\alpha = m\omega + \left(n+\frac{1}{2}\right)\varpi i$.

However, I claim that for all other values of $\alpha$, the functions $\varphi\alpha$, $f\alpha$, $F\alpha$ will necessarily have a nonzero value.

Indeed, let us suppose that

$$\varphi(\alpha+\beta i) = 0,$$

$\alpha$ and $\beta$ being real quantities. By virtue of the first formula in (10.), this equation can be written as follows:

$$\frac{\varphi\alpha.f(\beta i).F(\beta i) + \varphi(\beta i).f\alpha.F\alpha}{1+e^2c^2\varphi^2\alpha\varphi^2(\beta i)} = 0.$$

Now the quantities $\varphi\alpha$, $f(\beta i)$, $F(\beta i)$ are real and $\varphi(\beta i)$ is of the form $i.A$, where $A$ is real; thus this equation cannot hold unless both:

$$\varphi(\alpha).f(\beta i).F(\beta i) = 0; \quad \varphi(\beta i).f\alpha.F\alpha = 0.$$

These equations can only be satisfied in two ways, namely, by letting

$$\varphi(\alpha) = 0, \quad \varphi(\beta i) = 0,$$

or

$$f(\beta i).F(\beta i) = 0, \quad f\alpha.F\alpha = 0.$$

The first two equations give $\alpha = m\omega$; $\beta = n\varpi$. Noting that $F\alpha$ and $f(\beta i)$ can never vanish, the latter two equations give

$$f\alpha = 0, \quad F(\beta i) = 0,$$

where

$$\alpha = \left(m+\frac{1}{2}\right)\omega, \quad \beta = \left(n+\frac{1}{2}\right)\varpi.$$

However for these values of $\alpha$ and $\beta$ the value of $\varphi(\alpha+\beta i)$ becomes infinite; thus the only values of $\alpha$ and $\beta$ are $\alpha = m\omega$ and $\beta = n\varpi$, and consequently, all the roots of the equation

$$\varphi(x) = 0,$$

can be represented by

$$27. \qquad x = m\omega + n\varpi i.$$

In the same manner, we find that all the roots of the equation

$$f(x) = 0$$

can be represented by

$$28. \quad x = \left(m + \frac{1}{2}\right)\omega + n\varpi i,$$

and those of the equation

$$F(x) = 0,$$

by

$$29. \quad x = m\omega + (n + \frac{1}{2})\varpi i.$$

7.

The formulas in (26.) show that the three equations

$$\varphi(x) = \frac{1}{0}, f(x) = \frac{1}{0}, F(x) = \frac{1}{0},$$

will be satisfied if $x$ is given a value of the form

$$30. \quad x = \left(m + \frac{1}{2}\right)\omega + \left(n + \frac{1}{2}\right)\varpi i.$$

Now one can demonstrate that the equations in question have no other roots. Indeed, if

$$F(x) = \frac{b}{c}.\frac{1}{F\left(x - \frac{\omega}{2}\right)}; \quad \varphi(x) = \frac{-i}{ec}.\frac{1}{\varphi\left(x - \frac{\omega}{2} - \frac{\varpi}{2}i\right)}; \quad f(x) = \frac{b}{e}.\frac{1}{f\left(x - \frac{\varpi}{2}i\right)};$$

the equations in question entail the following:

$$\varphi\left(x - \frac{\omega}{2} - \frac{\varpi}{2}i\right) = 0; \quad f\left(x - \frac{\varpi}{2}i\right) = 0; \quad F\left(x - \frac{\omega}{2}\right) = 0,$$

but in virtue of what we just saw, the equations give respectively:

$$x - \tfrac{\omega}{2} - \tfrac{\varpi}{2}i = m\omega + n\varpi i; \quad x - \tfrac{\varpi}{2}i = \left(m + \tfrac{1}{2}\right)\omega + n\varpi i;$$
$$x - \tfrac{\omega}{2} = m\omega + \left(n + \tfrac{1}{2}\right)\varpi i;$$

that is to say: for the three equations one will have:

$$x = \left(m + \frac{1}{2}\right)\omega + \left(n + \frac{1}{2}\right)\varpi i,$$

Q.E.D.

8.

Having found as above all the roots of the equations

$$\begin{gathered} \varphi(x) = 0; \quad f(x) = 0; \quad F(x) = 0; \\ \varphi(x) = \frac{1}{0}; \quad f(x) = \frac{1}{0}; \quad F(x) = \frac{1}{0}; \end{gathered}$$

I now want to seek the roots of the more general equations

$$\varphi(x) = \varphi a, \quad f(x) = fa, \quad F(x) = Fa,$$

where $a$ is an arbitrary real or imaginary quantity.
First consider the equation

$$\varphi(x) - \varphi a = 0.$$

In the second equation of formulas (12.), by setting

$$\alpha = \frac{x+a}{2}, \quad \beta = \frac{x-a}{2},$$

we find that

$$\varphi x - \varphi a = \frac{2\varphi\left(\frac{x-a}{2}\right).f\left(\frac{x+a}{2}\right).F\left(\frac{x+a}{2}\right)}{1 + e^2c^2.\varphi^2\left(\frac{x+a}{2}\right)\varphi^2\left(\frac{x-a}{2}\right)} = 0.$$

This equation can only hold in one of the five following cases:

1. If $\varphi\left(\frac{x-a}{2}\right) = 0$, then $x = a + 2m\omega + 2n\varpi i$,

2. If $f\left(\frac{x+a}{2}\right) = 0$, then $x = -a + (2m+1)\omega + 2n\varpi i$,

3. If $F\left(\frac{x+a}{2}\right) = 0$, then $x = -a + 2m\omega + (2n+1)\varpi i$,

4. if $\varphi\left(\frac{x-a}{2}\right) = \frac{1}{0}$, then $x = a + (2m+1)\omega + (2n+1)\varpi i$,

5. If $\varphi\left(\frac{x+a}{2}\right) = \frac{1}{0}$, then $x = -a + (2m+1)\omega + (2n+1)\varpi i$.

The solution of these five equations are given in formulas (27.), (28.), (29.).

Having found those values of $x$, we should discard those given by the formula

$$x = -a + (2m+1)\omega + (2n+1)\varpi i,$$

because such a value of $x$ gives, by virtue of (22.):

$$\varphi x = -\varphi a,$$

while one must have $\varphi x = \varphi a$; however the other values of $x$ expressed by the first four formulas are allowed. As we see, they are given by the single formula:

$$31. \quad x = (-1)^{m+n}.a + m\omega + n\varpi i.$$

This is therefore the general expression for all the roots of the equation

$$\varphi x = \varphi a.$$

In the same manner, one finds that all the roots of the equation

$$fx = fa$$

can be represented by the formula

$$32. \quad x = \pm a + 2m\omega + n\varpi i,$$

and all those of the equation

$$Fx = Fa,$$

by the formula

$$33. \quad x = \pm a + m\omega + 2n\varpi i.$$

## §. II.
## Formulas that give the values of $\varphi(n\alpha)$, $f(n\alpha)$, $F(n\alpha)$ expressed as rational functions of $\varphi\alpha$, $f\alpha$, $F\alpha$.

### 9.

Let us start again with the formulas in (12.). Setting $\alpha = n\beta$ in the 1st, 3rd, and 5th, they become:

$$34. \quad \begin{cases} \varphi(n+1)\beta = -\varphi(n-1)\beta + \frac{2\varphi(n\beta).f\beta.F\beta}{R}, \\ f(n+1)\beta = -f(n-1)\beta + \frac{2f(n\beta).f\beta}{R}, \\ F(n+1)\beta = -F(n-1)\beta + \frac{2F(n\beta).F\beta}{R}, \end{cases}$$

where $R = 1 + c^2e^2.\varphi^2(n\beta).\varphi^2\beta$.

These formulas give the value of $\varphi(n+1)\beta$ in $\varphi(n-1)\beta$ and $\varphi(n\beta)$; that of $f(n+1)\beta$ in $f(n-1)\beta$ and $f(n\beta)$, and that of $F(n+1)\beta$ in $F(n-1)\beta$ and $F(n\beta)$. Thus by successively setting $n = 1, 2, 3, \ldots$, we find the successive values of the functions:

$$\begin{aligned} &\varphi(2\beta);\ \varphi(3\beta);\ \varphi(4\beta)\ldots\varphi(n\beta), \\ &f(2\beta);\ f(3\beta);\ f(4\beta)\ldots f(n\beta), \\ &F(2\beta);\ F(3\beta);\ F(4\beta)\ldots F(n\beta) \end{aligned}$$

expressed as rational functions of the three quantities

$$\varphi\beta;\quad f\beta;\quad F\beta.$$

Letting, e.g., n=1, we have:

$$35. \quad \begin{cases} \varphi(2\beta) = \frac{2.\varphi\beta.f\beta.F\beta}{1+e^2c^2\varphi^4\beta}, \\ f(2\beta) = -1 + \frac{2f^2\beta}{1+e^2c^2\varphi^4\beta}, \\ F(2\beta) = -1 + \frac{2F^2\beta}{1+e^2c^2\varphi^4\beta}. \end{cases}$$

The functions $\varphi(n\beta)$, $f(n\beta)$, $F(n\beta)$ are rational functions of $\varphi\beta$, $f\beta$, $F\beta$, and they can always be reduced to the form $\frac{P}{Q}$, where $P$ and $Q$ are polynomials in $\varphi\beta$, $f\beta$, $F\beta$. It is likewise clear that for the three functions we consider, the denominator will have the same value. Thus let

$$35.' \quad \varphi(n\beta) = \frac{P_n}{Q_n},\ f(n\beta) = \frac{P'_n}{Q_n},\ F(n\beta) = \frac{P''_n}{Q_n},$$

We also have

$$\begin{aligned} \varphi(n+1)\beta = \frac{P_{n+1}}{Q_{n+1}},\ f(n+1)\beta = \frac{P'_{n+1}}{Q_{n+1}},\ F(n+1)\beta = \frac{P''_{n+1}}{Q_{n+1}}, \\ \varphi(n-1)\beta = \frac{P_{n-1}}{Q_{n-1}},\ f(n-1)\beta = \frac{P'_{n-1}}{Q_{n-1}},\ F(n-1)\beta = \frac{P''_{n-1}}{Q_{n-1}}. \end{aligned}$$

Substituting in these values, the first formula of (34.) becomes:

$$\frac{P_{n+1}}{Q_{n+1}} = -\frac{P_{n-1}}{Q_{n-1}} + \frac{2f\beta.F\beta.\frac{P_n}{Q_n}}{1+c^2e^2\varphi^2\beta.\frac{P_n^2}{Q_n^2}},$$

or

$$\frac{P_{n+1}}{Q_{n+1}} = \frac{-P_{n-1}(Q_n^2+c^2e^2\varphi^2.P_n^2)+2P_n.Q_n.Q_{n-1}.f\beta.F\beta}{Q_{n-1}.(Q_n^2+e^2c^2\varphi^2\beta.P_n^2)}.$$

Equating the numerators and the denominators of these two fractions, we have

$$\begin{array}{lrcl} 36. & P_{n+1} & = & -P_{n-1}(Q_n^2+c^2e^2\varphi^2\beta.P_n^2)+2f\beta.F\beta.P_n.Q_n.Q_{n-1}, \\ 37. & Q_{n+1} & = & Q_{n-1}(Q_n^2+e^2c^2\varphi^2\beta.P_n^2). \end{array}$$

The second and third equations from (34.) will give in the same manner:

$$\begin{array}{lrcl} 38. & P'_{n+1} & = & -P'_{n-1}.(Q_n^2+c^2e^2\varphi^2\beta.P_n^2)+2f\beta P'_n.Q_n.Q_{n-1}, \\ 39. & P''_{n+1} & = & -P''.(Q_n^2+c^2e^2\varphi^2\beta.P_n^2)+2F\beta.P''_n.Q_n.Q_{n-1}. \end{array}$$

By setting $n = 1, 2, 3\ldots$, in the four formulas and noting that:

$$\begin{array}{llll} Q_0 = 1; & Q_1 = 1; & P_0 = 0; & P_1 = \varphi\beta; \\ P'_0 = 1; & P'_1 = f\beta & P''_0 = 1; & P''_1 = F\beta; \end{array}$$

one can successively find the polynomials $Q_n$, $P_n$, $P'_n$, $P''_n$ for all values of $n$. For brevity, let

$$40.\quad \varphi\beta = x,\ f\beta = y,\ F\beta = z \text{ and}$$
$$41.\quad R_n = Q_n^2 + e^2c^2x^2P_n^2.$$

The previous formulas give:

$$42.\quad \begin{cases} Q_{n+1} = Q_{n-1}.R_n, \\ P_{n+1} = -P_{n-1}.R_n + 2yz.P_n.Q_n.Q_{n-1}, \\ P'_{n+1} = -P'_{n-1}R_n + 2y.P'_n.Q_n.Q_{n-1}, \\ P''_{n+1} = -P''_{n-1}.R_n + 2z.P''_n.Q_n.Q_{n-1}. \end{cases}$$

Letting $n = 1, 2,$, we have:

$$43.\quad \begin{cases} R_1 = Q_{n-1}.R_n, \\ Q_2 = Q_0R_1 = 1 + e^2c^2x^4, \\ P_2 = -P_0R_1 + 2yzP_1.Q_1.Q_0 = 2xyz, \\ P'_2 = -P'_0R_1 + 2yP'_1.Q_1.Q_0 = -1 - e^2c^2x^4 + 2y^2, \\ P''_2 = -P''_0R_1 + 2zP''_1.Q_1Q_0 = -1 - e^2c^2x^4 + 2z^2. \end{cases}$$

$$
44. \quad \begin{cases} R_2 = Q_2^2 + e^2c^2x^2.P_2^2 = (1+e^2c^2x^4)^2 + e^2c^2x^2.4x^2y^2z^2, \\ Q_3 = Q_1R_2 = R_2, \\ P_3 = -P_1R_2 + 2yzP_2Q_2Q_1 = -xR_2 + 2y^2z^2x.Q_2 \\ = x\{2y^2z^2.Q_2 - R_2\}, \\ P_3' = -P_1'.R_2 + 2y.P_2'.Q_2Q_1 = -y.R_2 + 2yP_2'Q_2 \\ = y\{2Q_2P_2' - R_2\}, \\ P_3'' = z\{2Q_2P_2'' - R_2\}. \end{cases}
$$

Continuing in this way and noting that $y^2 = 1 - c^2x^2$, $z^2 = 1 + e^2x^2$, we easily see that the quantities:

$$
Q_n, \quad \tfrac{P_{2n}}{xyz}, \quad \tfrac{P_{2n+1}}{x}, \quad P_{2n}', \quad \tfrac{P_{2n+1}'}{y}, \quad P_{2n}'', \quad \tfrac{P_{2n+1}''}{z}
$$

are polynomials in the three quantities $x^2$, $y^2$, $z^2$, and consequently, also one of these for an arbitrary integer $n$.

This shows that the expressions for $\varphi(n\beta)$, $f(n\beta)$, $F(n\beta)$ are of the following form:

$$
45. \quad \begin{cases} \varphi(2n\beta) = \varphi\beta.f\beta.F\beta.T, & \varphi(2n+1)\beta = \varphi\beta.T', \\ f(2n\beta) = T_1, & f(2n+1)\beta = f\beta.T'', \\ F(2n\beta) = T_2, & F(2n+1)\beta = F\beta.T''', \end{cases}
$$

where $T$, etc., represent rational functions of the quantities $(\varphi\beta)^2$, $(f\beta)^2$, $(F\beta)^2$.

## §. III.
## Solving the equations
## $\varphi(n\beta) = \frac{P_n}{Q_n}, \quad f(n\beta) = \frac{P_n'}{Q_n}, \quad F(n\beta) = \frac{P_n''}{Q_n}.$

### 10.

According to what we have seen, the functions $\varphi(n\beta)$, $f(n\beta)$, $F(n\beta)$ may be expressed rationally in $x$, $y$, $z$. The contrary case does not happen, because the equations in (35.) are in general of very high degree. They have for this reason a certain number of roots. We will see how one can easily express all the roots using the functions $\varphi$, $f$, $F$.

A. Consider first the equation $\varphi(n\beta) = \frac{P_n}{Q_n}$, or $Q_n.\varphi(n\beta) = P_n$, and let us seek all the values of $x$.

It is necessary to distinguish two cases, according to whether $n$ is even or odd.:

1) If $n$ is an even number.

According to what we saw in the preceding paragraph (45.), we have in this case

$$
\varphi(2n\beta) = \psi(x^2).x.y.z,
$$

that is to say, in virtue of the formulas

$$y = \sqrt{(1 - c^2x^2)}, \quad z = \sqrt{(1 + e^2x^2)},$$
$$\varphi(2n\beta) = x\psi(x^2).\sqrt{[(1 - c^2x^2)(1 + e^2x^2)]}.$$

Thus the equation in $x$ will become,

$$\varphi^2(2n\beta) = x^2(\psi(x^2))^2.(1 - c^2x^2)(1 + e^2x^2).$$

Designating the right side as $\theta(x^2)$, we have

$$\varphi^2(2n\beta) = \theta(x^2).$$

One of the values of $x$ is $\varphi\beta$, so we have

$$46. \quad \varphi^2(2n\beta) = \theta(\varphi^2\beta),$$

an equation that holds for any value of $\beta$. To find the other values of $x$, that is, an arbitrary root $x = \varphi\alpha$, we must have

$$\varphi^2(2n\beta) = \theta(\varphi^2\alpha).$$

Now substituting $\alpha$ into (46.) instead of $\beta$, we get

$$\varphi^2(2n\alpha) = \theta(\varphi^2\alpha), \text{ thus :}$$
$$47. \qquad \varphi^2(2n\beta) = \varphi^2(2n\alpha);$$

an equation that reduces to these two:

$$\varphi(2n\alpha) = \varphi(2n\beta) \text{ and } \varphi(2n\alpha) = -\varphi(2n\beta).$$

The first gives, in virtue of (31.),

$$2n\alpha = (-1)^{m+\mu}2n\beta + m\omega + \mu\varpi i,$$

where $m$ and $\mu$ are two arbitrary positive or negative integers (including zero).

The second gives the same values for $2m\alpha$, but of opposite sign, as it is easy to see by writing as follows:

$$\varphi(-2n\alpha) = \varphi(2n\beta).$$

All values of $2n\alpha$ that satisfy equation (47.) can thus be represented by

$$2n\alpha = \pm\left[(-1)^{m+\mu}2n\beta + m\omega + \mu\varpi i\right].$$

From there one deduces the value of $\alpha$. Dividing by $2n$, we get

$$\alpha = \pm\left[(-1)^{m+\mu}.\beta + \frac{m}{2n}\omega + \frac{\mu}{2n}\varpi i\right].$$

With the value for $\alpha$, we get

$$48. \quad \varphi\alpha = \pm\varphi\left[(-1)^{m+\mu}.\beta + \frac{m}{2n}\omega + \frac{\mu}{2n}\varpi i\right] = x.$$

The values of $x$ are described by this expression. Thus they will be given explicitly by letting the numbers $m$ and $\mu$ be integers between $-\infty$ and $\infty$. However, to insure they are distinct, it is enough to let $m$ and $\mu$ be integers less than $2n$. In fact, whatever these number are, we may assume they can always be reduced to the following form:

$$m = 2n.k + m', \;\; \mu = 2n.k' + \mu',$$

where $k$, $k'$ are whole numbers and $m'$, $\mu'$ are integers less than $2n$. Substituting these values into the expression for $x$, we get:

$$x = \pm\varphi\left\{(-1)^{m'+\mu'}\beta + \frac{m'}{2n}\omega + \frac{\mu'}{2n}\varpi i + k\omega + k'\varpi i\right\},$$

Now in virtue of (22.), this expression can be simplified to

$$49. \quad x = \pm\varphi\left\{(-1)^{m'+\mu'}\beta + \frac{m'}{2n}\omega + \frac{\mu'}{2n}\varpi i\right\}.$$

This value of $x$ is of the same form as the preceding in (48.), only $m$ and $\mu$ are replaced by $m'$ and $\mu'$, which are both positive and less than $2n$; thus one obtains all the distinct values of $x$ by merely letting $m$ and $\mu$ be all the integers between zero and $2n$ inclusive. All the values are necessarily distinct. Indeed, suppose for example that we have

$$\begin{aligned} &\pm\varphi\left\{(-1)^{m'+\mu'}\beta + \tfrac{m'}{2n}\omega + \tfrac{\mu'}{2n}\varpi i\right\} \\ = \quad &\pm\varphi\left\{(-1)^{m+\mu}\beta + \tfrac{m}{2n}\omega + \tfrac{\mu}{2n}\varpi i\right\}, \end{aligned}$$

from this it follows, according to (31.);

$$(-1)^{m'+\mu'}\beta + \frac{m'}{2n}\omega + \frac{\mu'}{2n}\varpi i = \pm\left\{(-1)^{m+\mu}.\beta + \frac{m}{2n}\omega + \frac{\mu}{2n}\varpi i\right\} + k\omega + k'\varpi i,$$

$k$ and $k'$ integers.

This equation gives:

$$\mu' = k'.2n \pm \mu, \quad m' = k.2n \pm m, \quad (-1)^{m'+\mu'} = \pm(-1)^{m+\mu}.$$

The first two equations cannot hold unless $k' = 1$, $k = 1$, $\mu' = 2n - \mu$, $m' = 2n - m$. The latter becomes:

$$(-1)^{4n-m-\mu} = -(-1)^{m+\mu},$$

from which one concludes:

$$(-1)^{nm+2\mu} = -1,$$

an absurd result.

Thus all the values of $x$ given in formula (48.) are distinct if $m$ and $\mu$ are positive and less than $[2m]$.[6]

The total number of values of $x$ is, as it is easy to see, equal to $2(2n)^2 = 8n^2$. However, the equation $\varphi^2(2n\beta) = \theta(x^2)$ cannot have equal roots, because in this case, one has $\frac{\partial.\theta(x^2)}{\partial x} = 0$, which gives $x$ a value independent of $\beta$. Thus the degree of the equation $\varphi^2(2n\beta) = \theta(x^2)$ is equal to the number of roots, that is, to $8n^2$. If for example $n = 1$, one will have the equation

$$\varphi^2(2\beta) = \theta(x^2) = \frac{4x^2(1-c^2x^2)(1+e^2x^2)}{(1+e^2c^2x^4)^2},$$

or

$$(1+e^2c^2x^4)^2.\varphi^2(2\beta) = 4x^2(1-c^2x^2)(1+e^2x^2),$$

and according to formula (48.) the eight roots of this equation will be

$$\begin{aligned}&x = \pm\varphi\beta,\ x = \pm\varphi(-\beta + \tfrac{\omega}{2}),\\ &x = \pm\varphi(-\beta + \tfrac{\varpi}{2}i),\ x = \pm\varphi(\beta + \tfrac{\omega}{2} + \tfrac{\varpi}{2}i).\end{aligned}$$

2) If $n$ is an odd number $= 2n + 1$.

In this case $\frac{P_{2n+1}}{Q_{2n+1}}$ is, as we saw, a rational function of $x$ and as a consequence the equation for $x$ will be:

$$50. \quad \varphi(2n+1)\beta = \frac{P_{2n+1}}{Q_{2n+1}}.$$

Precisely like in the preceding case, we find that all the roots of this equation can be represented by

$$51. \quad x = \varphi\left((-1)^{m+\mu}.\beta + \frac{m}{2n+1}\omega + \frac{\mu}{2n+1}\varpi i\right),$$

[6] The original read "... and less than $m$."

where it is necessary to let $m$ and $\mu$ be all the integers from $-n$ to $+n$ inclusive. Thus the number of distinct roots is $(2n+1)^2$. This is also the degree of the equation in question. We can also express the roots by

$$x = (-1)^{m+\mu}.\varphi\left(\beta + \frac{m}{2n+1}\omega + \frac{\mu}{2n+1}\varpi i\right).$$

If for example $n = 1$, one will have an equation of degree $3^2 = 9$.

Formula (51.) gives the following 9 values for $x$:

$$\begin{array}{l}
\varphi(\beta); \\
\varphi(-\beta - \frac{\omega}{3}); \\
\varphi(-\beta + \frac{\omega}{3}); \\
\varphi(-\beta - \frac{\varpi}{3}i); \\
\varphi(-\beta + \frac{\varpi}{3}i); \\
\varphi(\beta - \frac{\omega}{3} - \frac{\varpi}{3}i); \\
\varphi(\beta - \frac{\omega}{3} + \frac{\varpi}{3}i); \\
\varphi(\beta + \frac{\omega}{3} - \frac{\varpi}{3}i); \\
\varphi(\beta + \frac{\omega}{3} + \frac{\varpi}{3}i);
\end{array}$$

B. Now consider the equation

$$52. \qquad f(n\beta) = \frac{P'_n}{Q_n}$$

and let us find the values of $y$ that satisfy this equation. The function $\frac{P'_n}{Q_n}$ is, as we saw above, rational in $y$: setting $\frac{P'_n}{Q_n} = \psi(y)$, the equation in $y$ is

$$f(n\beta) = \psi(y).$$

One of the roots of this equation is $y = f\beta$, thus for arbitrary $\beta$:

$$53. \quad f(n\beta) = \psi(f\beta).$$

To find the other values of $y$, let $\alpha$ be a new unknown such that $y = f\alpha$. We have

$$f(n\beta) = \psi(f\alpha);$$

now in virtue of (53.), the right hand side is equal to $f(n\alpha)$. Thus to find $\alpha$, we have the equation:

$$f(n\alpha) = f(n\beta).$$

In virtue of (32.), this equation gives as the general expression of $n\alpha$:

$$n\alpha = \pm n\beta + 2m\omega + \mu\varpi i,$$

$m$ and $\mu$ being two positive or negative integers, zero included.

From here we conclude

$$\alpha = \pm\beta + \frac{2m}{n}\omega + \frac{\mu}{n}\varpi i,$$

and as a consequence:

$$f\alpha = f\left(\pm\beta + \frac{2m}{n}\omega + \frac{\mu}{n}\varpi i\right) = y.$$

This is the general value of $y$. Now to get the distinct values for $y$, I claim that it is enough to take $\beta$ with the $+$ sign, and to let $m$ and $\mu$ take on all the integers less than $n$. Indeed, since we have $f(+\alpha) = f(-\alpha)$, initially we get:

$$f\left(-\beta + \frac{2m}{n}\omega + \frac{\mu}{n}\varpi i\right) = f\left(\beta - \frac{2m}{n}\omega - \frac{\mu}{n}\varpi i\right).$$

Thus, in the expression for $y$, one can always take $\beta$ with the $+$ sign. Hence all the values of $y$ are given by the expression

$$54. \quad y = f\left(\beta + \frac{2m}{n}\omega + \frac{\mu}{n}\varpi i\right).$$

Now whatever the numbers $m$ and $\mu$, we may suppose that

$$m = k.n + m',\ \mu' = k'.n + \mu',$$

where $k$, $k'$, $m'$, $\mu'$ are integers, the two latter being both positive and less than $n$.

Substituting we get

$$y = f\left(\beta + \frac{2m'}{n}\omega + \frac{\mu'}{n}\varpi i + 2k\omega + k'\varpi i\right).$$

Now in virtue of (22.), the right side of this equation is equal to

$$55. \quad f\left(\beta + \frac{2m'}{n}\omega + \frac{\mu'}{n}\varpi i\right) = y,$$

a quantity of the same form as the right side of (54.); only $m'$ and $\mu'$ are positive and less than $n$. Thus etc.

Letting $m$ and $\mu$ take on all the possible values less than $n$, one finds $n^2$ values for $y$. However in general, all these quantities are distinct. Indeed, suppose for example,

$$f\left(\beta + \frac{2m}{n}\omega + \frac{\mu}{n}\varpi i\right) = f\left(\beta + \frac{2\mu'}{n} + \frac{\mu'}{n}\varpi i\right),$$

in virtue of (32.), by letting $k$, $k'$ be two integers:

$$\beta + \frac{2m}{n}\omega + \frac{\mu}{n}\varpi i = \frac{2m'}{n}\omega + \frac{\mu'}{n}\varpi i + 2k\omega + k'\varpi i,$$

Since $\beta$ can have any irrational value, it is clear that this equation can not hold unless one chooses the upper sign $(+)$ on the right side. This then gives

$$\frac{2m}{n}\omega + \frac{\mu}{n}\varpi i = \frac{2m'}{n}\omega + \frac{\mu'}{n}\varpi i + 2k\omega + k'\varpi i,$$

from which we deduce by equating the real and imaginary parts:

$$m = m' + kn, \;\; \mu = \mu' + k'n,$$

absurd equations, given that the numbers $m$, $m'$, $\mu$, and $\mu'$ are all positive and smaller than $n$. So in general the equation

$$f(n\beta) = \psi(y)$$

has $n^2$ distinct roots and no more.

Now generally all the roots of this equation are different. Indeed, if two were equal, we would have both:

$$f(n\beta) = \psi(y) \text{ and } 0 = \psi'(y),$$

and this is impossible, if we note that the coefficients of $y$ in $\psi(y)$ do not contain $\beta$. The general equation (52.) is necessarily of degree $n^2$.

C. The equation

$$56. \quad F(n\beta) = \frac{P''_n}{Q_n},$$

can be treated in absolutely the same manner with respect to $z$ as the equation $f(n\beta) = \frac{P'}{Q_n}$ was with respect to $y$, giving the general expression for the values of $z$:

$$57. \quad z = F\left(\beta + \frac{m}{n}\omega + \frac{2\mu}{n}\varpi i\right),$$

where $m$ and $\mu$ are positive integers less than $n$. There are $n^2$ values of $z$, and in general they are distinct.

Therefore the general equation (56.) is of degree $n^2$.

11.

Above we found all the roots of the equations

$$\varphi(n\beta)=\frac{P_n}{Q_n},\ f(n\beta)=\frac{P'_n}{Q_n},\ F(n\beta)=\frac{P''_n}{Q_n},$$

roots that are expressed by the formulas in (48.), (51.), (54.), (57.). All these roots are different, except for the case of particular values of $\beta$; but for these values, the distinct roots are given by same formulas. – In the last case, a certain number of the values of the quantities $x$, $y$, $z$ will be equal; but it is clear that all the equal or unequal values will be nevertheless the roots of the equations concerned. This can be seen by making $\beta$ converge towards a particular value, which gives equal values for $x$, or $y$, or $z$ .

Letting $\beta=\frac{\alpha}{2n}$ in formula (48.), the equation 58. $\varphi^2\alpha=\frac{P_{2n}^2}{Q_{2n}^2}$, has roots $x=\pm\varphi\left((-1)^{m+\mu}\frac{\alpha}{2n}+\frac{m}{2n}\omega+\frac{\mu}{2n}\varpi i\right)$ where $m$ and $\mu$ take on all the positive integers less than $2n$.

The same for formula (50.), letting $\beta=\frac{\alpha}{2n+1}$ we have $\varphi\alpha=\frac{P_{2n+1}}{Q_{2n+1}}$, which has roots

$$59.\qquad x=(-1)^{m+\mu}.\varphi\left(\frac{\alpha}{2n+1}+\frac{m\omega+\mu\varpi i}{2n+1}\right),$$

$m$ and $\mu$ taking on all the integers from $-n$ to $+n$.

Finally in (52.), (56.), letting $\beta=\frac{\alpha}{n}$, we have the equation $f\alpha=\frac{P'_n}{Q_n}$, which has roots

$$60.\qquad y=f\left(\frac{\alpha}{n}+\frac{2m}{n}\omega+\frac{\mu}{n}\varpi i\right),$$

and equation $F\alpha=\frac{P''_n}{Q_n}$ has roots

$$61.\qquad z=F\left(\frac{\alpha}{n}+\frac{m}{n}\omega+\frac{2\mu}{n}\varpi i\right),$$

where $m$ and $\mu$ are contained between the limits 0 and $n-1$ inclusive. If $n$ is odd $=2n+1$, we may suppose that

$$\begin{aligned} y&=(-1)^m.f\left(\tfrac{\alpha}{2n+1}+\tfrac{m}{2n+1}\omega+\tfrac{\mu}{2n+1}\varpi i\right),\\ z&=(-1)^\mu.F\left(\tfrac{\alpha}{2n+1}+\tfrac{m}{2n+1}\omega+\tfrac{\mu}{2n+1}\varpi i\right),\end{aligned}$$

where $m$ and $\mu$ take on integer values from $-n$ to $+n$.

In all these equations the quantity $\alpha$ can have an arbitrary value.

Like the specific case, we must notice the following:

1) Letting $\alpha=0$ in (58.) and (59.), we get the equations

$$62.\qquad \begin{cases} P_{2n}^2=0, & \text{the roots are } x=\pm\varphi\left(\frac{m}{2n}\omega+\frac{\mu}{2n}\varpi i\right)\\ & \text{(the limits of } m \text{ and } \mu \text{ being 0 and } 2n-1),\\ P_{2n+1}=0, & \text{the roots are } x=\varphi\left(\frac{m}{2n+1}\omega+\frac{\mu}{2n+1}\varpi i\right)\\ & \text{(the limits of } m \text{ and } \mu \text{ being } -n \text{ and } +n).\end{cases}$$

2) Letting $\alpha = \frac{\omega}{2}$ in (60.) and letting $\alpha = \frac{\varpi}{2}i$ in (61.), noting that $f(\frac{\omega}{2}) = 0$, $F(\frac{\varpi}{2}i) = 0$, we obtain the two equations:

$$\left.\begin{array}{ll} 63. & P'_n = 0, \text{the roots are } y = f\left((2m+\frac{1}{2})\frac{\omega}{n} + \frac{\mu}{n}\varpi i\right) \\ 64. & P''_n = 0, \text{the roots are } z = F\left(\frac{m}{n}\omega + (2\mu+\frac{1}{2})\frac{\varpi i}{n}\right) \end{array}\right\} \begin{array}{l}\text{(the limits of} \\ m \text{ and } \mu \text{ are} \\ 0 \text{ and } n-1\text{ )}\end{array}$$

3) Letting $\alpha = \frac{\omega}{2} + \frac{\varpi}{2}i$ in (58.), noting that $\varphi(\frac{\omega}{2} + \frac{\varpi}{2}i) = \frac{1}{0}$, we get the equation

$$Q_{2n}^2 = 0,$$

whose roots are:

$$x = \pm\varphi\left[\left(m + \frac{1}{2}(-1)^{m+\mu}\right)\frac{\omega}{2n} + \left(\mu + \frac{1}{2}(-1)^{m+\mu}\right).\frac{\varpi i}{2n}\right].$$

The values of $x$ must be equal in pairs and one will easily see that the distinct values can be represented by

$$65. \quad x = \varphi\left[\left(m + \frac{1}{2}\right)\frac{\omega}{2n} + \left(\mu + \frac{1}{2}\right)\frac{\varpi i}{2n}\right],$$

by letting $m$ and $\mu$ take on all the integer values from 0 to $2n-1$. These are thus the roots of the equation

$$Q_{2n} = 0, \text{ with respect to } x.$$

Likewise, letting $\alpha = \frac{\omega}{2} + \frac{\varpi}{2}i$ in (58.) gives the equation

$$Q_{2n+1} = 0,$$

whose roots are:

$$66. \quad \left\{\begin{array}{l} x = (-1)^{m+\mu}.\varphi\left[(m+\frac{1}{2})\frac{\omega}{2n+1} + (\mu+\frac{1}{2})\frac{\varpi i}{2n+1}\right], \\ y = (-1)^{m}.f\left[(m+\frac{1}{2})\frac{\omega}{2n+1} + (\mu+\frac{1}{2})\frac{\varpi i}{2n+1}\right], \\ z = (-1)^{\mu}.F\left[(m+\frac{1}{2})\frac{\omega}{2n+1} + (\mu+\frac{1}{2})\frac{\varpi i}{2n+1}\right], \end{array}\right.$$

$m$ and $\mu$ taking all the integers from $-n$ to $+n$.

Among the values of $x$, $y$, $z$, we must note those that correspond to $m = n$, $\mu = n$.

Then we have

$$\begin{array}{l} x = (-1)^n.\varphi(\frac{\omega}{2} + \frac{\varpi}{2}i) = \frac{1}{0}, \\ y = (-1)^n.f(\frac{\omega}{2} + \frac{\varpi}{2}i) = \frac{1}{0}, \\ z = (-1)^n.F(\frac{\omega}{2} + \frac{\varpi}{2}i) = \frac{1}{0}. \end{array}$$

These infinite values show that the equation $Q_{2n+1} = 0$ is of degree one less than the equation it came from. By discarding these values, the $(2n+1)^2 - 1$ remaining will be the roots of the equation $Q_{2n+1} = 0$.

§. IV.

The algebraic solution of the equations

$\varphi\alpha = \frac{P_{2n+1}}{Q_{2n+1}}$, $f\alpha = \frac{P'_{2n+1}}{Q_{2n+1}}$, $F\alpha = \frac{P''_{2n+1}}{Q_{2n+1}}$.

12.

We saw in the preceding §, how we can easily express the roots of the equations in question using the functions $\varphi$, $f$, $F$. We will now deduce the solutions to these same equations, or determine the functions $\varphi(\frac{\alpha}{n})$, $f(\frac{\alpha}{n})$, $F(\frac{\alpha}{n})$, as functions of $\varphi\alpha$, $f\alpha$, $F\alpha$.

As we have

$$\varphi\left(\frac{\alpha}{m\mu}\right) = \varphi\left(\frac{1}{m}\left(\frac{\alpha}{\mu}\right)\right),$$

we may assume that $n$ is a prime number. Let us first consider the case when $n = 2$, and then when $n$ is an odd number.

A. Expressions of the Functions $\varphi(\frac{\alpha}{2})$, $f(\frac{\alpha}{2})$, $F(\frac{\alpha}{2})$.

13.

The values of $\varphi(\frac{\alpha}{2})$, $f(\frac{\alpha}{2})$, $F(\frac{\alpha}{2})$ can be found very easily in the following manner. In formulas (35.), suppose $\beta = \frac{\alpha}{2}$, set

$$x = \varphi\left(\frac{\alpha}{2}\right), \; y = f\left(\frac{\alpha}{2}\right), \; z = F\left(\frac{\alpha}{2}\right),$$

this gives:

$$f(\alpha) = \frac{y^2 - c^2x^2z^2}{1 + e^2c^2x^4}, \; F(\alpha) = \frac{z^2 + e^2y^2x^2}{1 + e^2c^2x^4},$$

or, by substituting the values of $y^2$ and $x^2$ and $x^2$:

$$f(\alpha) = \frac{1 - 2c^2x^2 - c^2e^2x^4}{1 + e^2c^2x^4}, \; F(\alpha) = \frac{1 + 2e^2x^2 - e^2c^2x^4}{1 + e^2c^2x^4},$$

These equations give

$$1 + f\alpha = \frac{2(1 - 2c^2x^2)}{1 + e^2c^2x^4}, \; 1 - f\alpha = \frac{2c^2x^2(1 + e^2x^2)}{1 + e^2c^2x^4},$$
$$F\alpha - 1 = \frac{2e^2x^2(1 - c^2x^2)}{1 + e^2c^2x^4}, \; F\alpha + 1 = \frac{2(1 + e^2x^2)}{1 + e^2c^2x^4},$$

which gives

$$\frac{F\alpha-1}{1+f\alpha}=e^2.x^2,\ \frac{1-f\alpha}{F\alpha+1}=c^2x^2,$$

and from this, noting that $y^2=1-c^2x^2$, $z^2=1+e^2x^2$:

$$z^2=\frac{F\alpha+f\alpha}{1+f\alpha},\ y^2=\frac{F\alpha+f\alpha}{1+F\alpha}.$$

From these equations one concludes that, extracting the square roots and replacing $x$, $y$, $x$ with their values $\varphi(\frac{\alpha}{2})$, $f(\frac{\alpha}{2})$, $F(\frac{\alpha}{2})$:

$$67.\quad \begin{cases} \varphi\left(\frac{\alpha}{2}\right)=\frac{1}{c}\sqrt{\left(\frac{1-f\alpha}{1+F\alpha}\right)}=\frac{1}{e}\sqrt{\left(\frac{F\alpha-1}{f\alpha+1}\right)}, \\ f\left(\frac{\alpha}{2}\right)=\sqrt{\left(\frac{F\alpha+f\alpha}{1+F\alpha}\right)},\ F\left(\frac{\alpha}{2}\right)=\sqrt{\left(\frac{F\alpha+f\alpha}{1+f\alpha}\right)}. \end{cases}$$

Such are the simplest forms that one can give for the values of the functions $\varphi(\frac{\alpha}{2})$, $f(\frac{\alpha}{2})$, $F(\frac{\alpha}{2})$. In this manner we can express $\varphi(\frac{\alpha}{2})$, $f(\frac{\alpha}{2})$, $F(\frac{\alpha}{2})$ in terms of $f\alpha$, $F\alpha$ algebraically. In the same way, $\varphi(\frac{\alpha}{4})$, $f(\frac{\alpha}{4})$, $F(\frac{\alpha}{4})$ are expressible in $f(\frac{\alpha}{2})$, $F(\frac{\alpha}{2})$, and so on. Thus in general, the functions $\varphi(\frac{\alpha}{2^n})$, $f(\frac{\alpha}{2^n})$, $F(\frac{\alpha}{2^n})$ can be expressed by means of the extraction of square roots of a function of three quantities $\varphi\alpha$, $f\alpha$, $F\alpha$.

Applying the above bisection formulas to an example, suppose $\alpha=\frac{\omega}{2}$.

So we have $f(\frac{\omega}{2})=0$, $F(\frac{\omega}{2})=\frac{\sqrt{(e^2+c^2)}}{c}$, and thus by substituting:

$$\begin{aligned} &\varphi(\tfrac{\omega}{4})=\tfrac{1}{c}\sqrt{\left(\tfrac{1}{1+\frac{1}{c}\sqrt{(e^2+c^2)}}\right)}=\tfrac{1}{e}.\sqrt{\left(\tfrac{1}{c}\sqrt{(e^2+c^2)}-1\right)}, \\ &f(\tfrac{\omega}{4})=\sqrt{\tfrac{\frac{1}{c}\sqrt{(e^2+c^2)}}{1+\frac{1}{c}\sqrt{(e^2+c^2)}}}, \\ &F(\tfrac{\omega}{4})=\sqrt{(\tfrac{1}{c}\sqrt{(e^2+c^2)})} \end{aligned}$$

or

$$\begin{aligned} &\varphi(\tfrac{\omega}{4})=\frac{1}{\sqrt{\left(c^2+c\sqrt{(e^2+c^2)}\right)}}=\frac{\sqrt{\left(\sqrt{(e^2+c^2)}-c\right)}}{ec}, \\ &f(\tfrac{\omega}{4})=\frac{\sqrt[4]{(e^2+c^2)}}{\sqrt{\left(c+\sqrt{(e^2+c^2)}\right)}}=\tfrac{1}{e}.\sqrt{\left(e^2+c^2-c\sqrt{(e^2+c^2)}\right)} \\ &F(\tfrac{\omega}{4})=\sqrt[4]{\left(1+\tfrac{e^2}{c^2}\right)}=\sqrt{[F(\tfrac{\omega}{2})]}. \end{aligned}$$

B. Expressions for the Functions $\varphi(\frac{\alpha}{2n+1})$, $f(\frac{\alpha}{2n+1})$, $F(\frac{\alpha}{2n+1})$ in algebraic functions of the quantities $\varphi\alpha$, $f\alpha$, $F\alpha$.

To find the values of $\varphi(\frac{\alpha}{2n+1})$, $f(\frac{\alpha}{2n+1})$, $F(\frac{\alpha}{2n+1})$ in $\varphi\alpha$, $f\alpha$, $F\alpha$ it is necessary to solve the equations

$$\varphi\alpha = \frac{P_{2n+1}}{Q_{2n+1}},\ f\alpha = \frac{P'_{2n+1}}{Q_{n+1}},\ F\alpha = \frac{P''_{2n+1}}{Q_{2n+1}},$$

that are all of degree $(2n+1)^2$. We will see that it is always possible solve these algebraically.

Let

$$68. \quad \varphi_1\beta = \sum_{m=-n}^{+n} \varphi\left(\beta + \frac{2m\omega}{2n+1}\right)$$

and

$$69. \quad \psi\beta = \sum_{\mu=-n}^{+n} \theta^\mu.\varphi_1\left(\beta + \frac{2\mu\varpi i}{2n+1}\right),\ \psi_1\beta = \sum_{\mu=-n}^{+n} \theta^\mu\varphi_1\left(\beta - \frac{2\mu\varpi i}{2n+1}\right),$$

where $\theta$ is an arbitrary imaginary root of the equation $\theta^{2n+1} - 1 = 0$.

That stated, I claim that the two quantities

$$\psi\beta.\psi_1\beta \text{ and } (\psi\beta)^{2n+1} + (\psi_1\beta)^{2n+1},$$

can be expressed rationally in $\varphi(2n+1)\beta$.

First write $\varphi_1\beta$ as follows:

$$\begin{aligned}
\varphi_1\beta &= \varphi\beta.\sum_{m=1}^{n}.\left\{\varphi\left(\beta + \frac{2m\omega}{2n+1}\right) + \varphi\left(\beta - \frac{2m\omega}{2n+1}\right)\right\} \\
&= \varphi\beta.\sum_{m=1}^{n}(-1)^m.\frac{2\varphi\beta.f(\frac{2m\omega}{2n+1}).F(\frac{2m\omega}{2n+1})}{1 + e^2c^2\varphi^2(\frac{2m\omega}{2n+1}).\varphi^2\beta},
\end{aligned}$$

we see that $\varphi_1\beta$ can be expressed rationally in $\varphi\beta$. Thus let $\varphi_1\beta = \chi(\varphi\beta)$. In similar manner:

$$\begin{aligned}
\varphi_1\left(\beta \pm \frac{2\mu\varpi i}{2n+1}\right) &= \chi\left(\varphi\left(\beta \pm \frac{2\mu\varpi i}{2n+1}\right)\right) \\
&= \chi\left(\frac{\varphi\beta.f(\frac{2\mu\varpi i}{2n+1}).F(\frac{2\mu\varpi i}{2n+1}) \pm \varphi(\frac{2\mu\varpi i}{2n+1}).f\beta.F\beta}{1 + e^2c^2.\varphi^2(\frac{2\mu\varpi i}{2n+1}).\varphi^2\beta}\right),
\end{aligned}$$

or by setting $\varphi\beta = x$:

$$f\left(\frac{2\mu\varpi i}{2n+1}\right).F\left(\frac{2\mu\varpi i}{2n+1}\right) = a,\ \varphi\left(\frac{2\mu\varpi i}{2n+1}\right) = b,$$

and by substituting the values of $\sqrt{(1-c^2x^2)}$ and $\sqrt{(1+e^2x^2)}$ for $f\beta$ and $F\beta$:

$$\varphi_1\left(\beta \pm \frac{2\mu\varpi i}{2n+1}\right) = \chi\left(\frac{ax \pm b.\sqrt{[(1-c^2x^2)(1+e^2x^2)]}}{1+e^2c^2b^2.x^2}\right);$$

where $\chi$ designates a rational function. The right side of this equation can be put into the form

$$R_\mu \pm R'_\mu.\sqrt{[(1-c^2x^2)(1+e^2x^2)]},$$

where $R_\mu$ and $R'_\mu$ are rational functions of $x$.
Thus

$$\varphi_1\left(\beta \pm \frac{2\mu\varpi i}{2n+1}\right) = R_\mu \pm R'_\mu.\sqrt{[(1-c^2x^2)(1+e^2x^2)]}$$

Substituting this into the expressions for $\psi\beta$ and $\psi_1\beta$, we get:

$$70.\qquad \begin{cases} \psi\beta = \sum\limits_{\mu=-n}^{+n}(+\theta)^\mu.R_\mu + \sqrt{[(1-c^2x^2)(1+e^2x^2)]}.\sum\limits_{\mu=-n}^{+n}(+\theta)^\mu.R'_\mu, \\ \psi_1\beta = \sum\limits_{\mu=-n}^{+n}(+\theta)^\mu.R_\mu - \sqrt{[(1-c^2x^2)(1+e^2x^2)]}.\sum\limits_{\mu=-n}^{+n}(+\theta)^\mu.R'_\mu. \end{cases}$$

Now $R_\mu$ and $R'_\mu$ are rational functions of $x$ as well as the quantities $\sum\limits_{\mu=-n}^{+n}(+\theta)^\mu.R_\mu$ and $\sum\limits_{\mu=-n}^{+n}(+\theta)^\mu.R'_\mu$.

Therefore by raising $\psi\beta$ and $\psi_1\beta$ to the $(2n+1)$st power, the two quantities $(\psi\beta)^{2n+1}$ and $(\psi_1\beta)^{2n+1}$ can be put into the form:

$$(\psi\beta)^{2n+1} = t + t'.\sqrt{[(1-c^2x^2)(1+e^2x^2)]},$$
$$(\psi_1\beta)^{2n+1} = t - t'.\sqrt{[(1-c^2x^2)(1+e^2x^2)]},$$

$t$ and $t'$ being rational functions of $x$. By taking the sum of the values of $(\psi\beta)^{2n+1}$ and $(\psi_1\beta)^{2n+1}$ we have

$$(\psi\beta)^{2n+1}+(\psi_1\beta)^{2n+1}=2t.$$

Thus the quantity $(\psi\beta)^{2n+1}+(\psi_1\beta)^{2n+1}$ can be expressed rationally in $x$. Similarly for the product $\psi\beta.\psi_1\beta$, as we see from the equations in (70.).

Thus we can make

$$71. \quad \begin{cases} \psi\beta.\psi_1\beta=\lambda(x), \\ (\psi\beta)^{2n+1}+(\psi_1\beta)^{2n+1}=\lambda_1(x), \end{cases}$$

$\lambda(x)$ and $\lambda_1(x)$ designating rational functions of $x$. Now these functions have the property of not changing value when we substitute forf $x$ another arbitrary root of the equation

$$\varphi(2n+1)\beta=\frac{P_{2n+1}}{Q_{2n+1}}.$$

First consider the function $\lambda(x)$. Setting the value of $x=\varphi\beta$ we have

$$\psi\beta.\psi_1\beta=\lambda(\varphi\beta),$$

from which we conclude, substituting $\beta+\frac{2k\omega}{2n+1}+\frac{2k'\varpi i}{2n+1}$ in place of $\beta$:

$$\lambda\left[\varphi\left(\beta+\frac{2k\omega}{2n+1}+\frac{2k'\varpi i}{2n+1}\right)\right]=\psi\left(\beta+\frac{2k\omega}{2n+1}+\frac{2k'\varpi i}{2n+1}\right).\psi_1\left(\beta+\frac{2k\omega}{2n+1}+\frac{2k'\varpi i}{2n+1}\right).$$

That given, note that:

$$72. \quad \sum_{m=-n}^{+n}\psi(m+k)=\sum_{m=-n}^{+n}\psi(m)+\sum_{m=1}^{k}[\psi(m+n)-\psi(m-n-1)],$$

setting $\beta=\beta+\frac{2k\omega}{2n+1}$ in the expression for $\psi_1\beta$, we have :

$$\begin{aligned}\varphi_1\left(\beta+\frac{2k\omega}{2n+1}\right) &= \sum_{m=-n}^{n}\varphi\left(\beta+\frac{2(k+m)}{2n+1}\omega\right) \\ &= \varphi_1\beta+\sum_{m=1}^{k}\left\{\varphi\left(\beta+\frac{2(n+m)\omega}{2n+1}\right)-\varphi\left(\beta+\frac{2(m-n-1)\omega}{2n+1}\right)\right\},\end{aligned}$$

now: $\varphi\left(\beta+\frac{2(m-n-1)}{2n+1}\omega\right)=\varphi\left(\beta+\frac{2(m+n)}{2n+1}\omega-2\omega\right)=\varphi\left(\beta+\frac{2(m+n)}{2n+1}\omega\right)$,

thus:

$$73. \quad \varphi_1\left(\beta+\frac{2k\omega}{2n+1}\right)=\varphi_1\beta.$$

Substituting $\beta + \frac{2k'\varpi i}{2n+1} + \frac{2k\omega}{2n+1}$ in place of $\beta$ in the expression for $\psi\beta$ we find

$$\psi\left(\beta + \frac{2k'\varpi i}{2n+1} + \frac{2k\omega}{2n+1}\right) = \sum_{\mu=-n}^{+n} \theta^{\mu}.\varphi_1\left(\beta + \frac{2(k'+\mu)\varpi i}{2n+1} + \frac{2k\omega}{2n+1}\right),$$

now in virtue of (73.):

$$\varphi_1\left(\beta + \frac{2(k'+\mu)\varpi i}{2n+1} + \frac{2k\omega}{2n+1}\right) = \varphi_1\left(\beta + \frac{2(k'+\mu)\varpi i}{2n+1}\right),$$

thus

$$\psi\left(\beta + \frac{2k'\varpi i}{2n+1} + \frac{2k\omega}{2n+1}\right) = \sum_{\mu=-n}^{+n} \theta^{\mu}.\varphi_1\left(\beta + \frac{2(k'+\mu)\varpi i}{2n+1}\right).$$

In virtue of (72.) we have

$$\begin{aligned}
&\sum_{\mu=-n}^{+n} \theta^{\mu}.\varphi_1\left(\beta + \frac{2(k'+\mu)\varpi i}{2n+1}\right) \\
&= \theta^{-k'}.\sum_{\mu=-n}^{+n} \theta^{\mu}.\varphi_1\left(\beta + \frac{2\mu\varpi i}{2n+1}\right) + \sum_{\mu=1}^{k'}.\theta^{n+\mu-k'}.\varphi_1\left(\beta + \frac{2(\mu+n)\varpi i}{2n+1}\right) \\
&-\sum_{\mu=1}^{k'} \theta^{\mu-n-1-k'}.\varphi_1\left(\beta + \frac{2(\mu-n-1)\varpi i}{2n+1}\right),
\end{aligned}$$

thus, noting that $\theta^{n+\mu-k'} = \theta^{\mu-n-1-k'}$ and

$$\varphi_1\left(\beta + \frac{2(\mu-n-1)\varpi i}{2n+1}\right) = \varphi_1\left(\beta + \frac{2(\mu+n)\varpi i}{2n+1} - 2\varpi i\right) = \varphi_1\left(\beta + \frac{2(\mu+n)\varpi i}{2n+1}\right),$$

we get

$$74. \quad \psi\left(\beta + \frac{2k'\varpi i}{2n+1} + \frac{2k\omega}{2n+1}\right) = \theta^{-k'}.\psi\beta.$$

In the same manner one can also find

$$\psi_1\left(\beta + \frac{2k'\varpi i}{2n+1} + \frac{2k\omega}{2n+1}\right) = \theta^{k'}.\psi_1\beta.$$

These two equations give

$$\psi\left(\beta+\frac{2k\omega+2k'\varpi i}{2n+1}\right).\psi_1\left(\beta+\frac{2k\omega+2k'\varpi i}{2n+1}\right)=\psi\beta.\psi_1\beta,$$
$$\left\{\psi\left(\beta+\frac{2k\omega+2k'\varpi i}{2n+1}\right)\right\}^{2n+1}+\left\{\psi_1\left(\beta+\frac{2k\omega+2k'\varpi i}{2n+1}\right)\right\}^{2n+1}=(\psi\beta)^{2n+1}+(\psi_1\beta)^{2n+1}.$$

In virtue of these equations we obtained, substituting in the values of $\lambda(\varphi\beta)$ and $\lambda_1(\varphi\beta)$, and $\beta+\frac{2k\omega+2k'\varpi i}{2n+1}$ in place of $\beta$:

$$\lambda(\varphi\beta)=\lambda\left[\varphi\left(\beta+\frac{2k\omega+2k'\varpi i}{2n+1}\right)\right],$$
$$\lambda_1(\varphi\beta)=\lambda_1\left[\varphi\left(\beta+\frac{2k\omega+2k'\varpi i}{2n+1}\right)\right].$$

Now $\varphi\left(\beta+\frac{2k\omega+2k'\varpi i}{2n+1}\right)$ expresses an arbitrary root of the equation

$$\varphi(2n+1)\beta=\frac{P_{2n+1}}{Q_{2n+1}}.$$

Thus as we claimed, the functions $\lambda(x)$ and $\lambda_1(x)$ have the same values whatever the root we put in place $x$.

Hence, let $x_0, x_1, x_2, \ldots, x_{2n+1}$ be these roots. We have

$$\lambda(x)=\frac{1}{2n+1}\{\lambda(x_0)+\lambda(x_1)+\cdots+\lambda(x_{2n})\},$$
$$\lambda_1(x)=\frac{1}{2n+1}\{\lambda_1(x_0)+\lambda_1(x_1)+\cdots+\lambda_1(x_{2n})\}.$$

Now the right hand side of these equations are symmetric rational functions of the roots of the equation $\varphi(2n+1)\beta=\frac{P_{2n+1}}{Q_{2n+1}}$. So $\lambda(x)$ and $\lambda_1(x)$ can be expressed rationally in $\varphi(2n+1)\beta$. In equations (71), making

$$\lambda(x)=B,\ \lambda_1=2A,$$

gives

$$(\psi\beta)^{2n+1}.(\psi_1\beta)^{2n+1}=B^{2n+1};\ (\psi\beta)^{2n+1}+(\psi_1\beta)^{2n+1}=2A,$$

from which we conclude

$$75.\quad \psi\beta=\sqrt[2n+1]{[A+\sqrt{(A^2-B^{2n+1})}]}=\sum_{\mu=-n}^{+n}\theta^\mu.\varphi_1\left(\beta+\frac{2\mu\varpi i}{2n+1}\right).$$

Having found the value of $\psi\beta$, we easily deduce those of $\varphi_1\beta$.

Indeed, by successively taking all the imaginary roots of the equation $\theta^{2n+1}-1=0$ for $\theta$, indicating the corresponding values of $A$ and $B$ for $A_1$, $B_1$, $A_2$, $B_2$ etc., we obtain:

$$\begin{aligned}
\sqrt[2n+1]{[A_1+\sqrt{(A_1^2-B_1^{2n+1})}]} &= \sum_{\mu=-n}^{+n}\theta_1^{\mu}.\varphi_1\left(\beta+\frac{2\mu\varpi i}{2n+1}\right),\\
\sqrt[2n+1]{[A_2+\sqrt{(A_2^2-B_2^{2n+1})}]} &= \sum_{\mu=-n}^{+n}\theta_2^{\mu}.\varphi_1\left(\beta+\frac{2\mu\varpi i}{2n+1}\right),\\
\dots\\
\sqrt[2n+1]{[A_{2n}+\sqrt{(A_{2n}^2-B_{2n}^{2n+1})}]} &= \sum_{\mu=-n}^{+n}\theta_{2n}^{\mu}.\varphi_1\left(\beta+\frac{2\mu\varpi i}{2n+1}\right),
\end{aligned}$$

Similarly knowing the sum of the roots:

$$\sum_{m=-n}^{+n}\sum_{-n}^{+n}\varphi\left(\beta+\frac{2m\omega}{2n+1}+\frac{2\mu\varpi i}{2n+1}\right)=\sum_{\mu=-n}^{+n}\varphi_1\left(\beta+\frac{2\mu\varpi i}{2n+1}\right),$$

which equals $(2n+1)\varphi(2n+1)\beta$ (as we will see later). Adding these equations term by term, then multiplying the first by $\theta_0^{-k}$, the second by $\theta_1^{-k}$, the third by $\theta_2^{-k}\dots$ and the $(2n)$th by $\theta_{2n}^{-k}$, we get:

$$\begin{aligned}
&\sum_{\mu=-n}^{+n}(\theta_0^{\mu-k}+\theta_1^{\mu-k}+\theta_2^{\mu-k}+\cdots+\theta_{2n}^{\mu-k}+1).\varphi_1\left(\beta+\frac{2\mu\varpi i}{2n+1}\right)\\
= \quad &(2n+1).\varphi(2n+1)\beta+\sum_{\mu=1}^{2n}\theta_\mu^{-k}.\sqrt[2n+1]{[A_\mu+\sqrt{(A_\mu^2-B_\mu^{2n+1})}]};
\end{aligned}$$

Now the sum

$$1+\theta_1^{\mu-k}+\theta_2^{\mu-k}+\cdots+\theta_{2n}^{\mu-k}$$

reduces to zero for all values of $k$ except for $k=\mu$. In the latter case, it equals $2n+1$. So the first term in the preceding equation becomes

$$(2n+1).\varphi_1\left(\beta+\frac{2k\varpi i}{2n+1}\right),$$

substituting and dividing by $(2n+1)$, we thus have:

76. $\varphi_1\left(\beta + \dfrac{2k\varpi i}{2n+1}\right) =$

$$\varphi(2n+1)\beta + \frac{1}{2n+1}.\Big\{\theta_1^{-k}\sqrt[2n+1]{[A_1 + \sqrt{(A_1^2 - B_1^{2n+1})}]} + \theta_2^{-k}\sqrt[2n+1]{[A_2 + \sqrt{(A_2^2 - B_2^{2n+1})}]} + \dots$$
$$\dots + \theta_{2n}^{-k}\sqrt[2n+1]{[A_{2n} + \sqrt{(A_{2n}^2 - B_{2n}^{2n+1})}]}\Big\}$$

For $k = 0$, we have:

$$77. \quad \varphi_1\beta = \varphi(2n+1)\beta + \frac{1}{2n+1}\Big\{\sqrt[2n+1]{[A_1 + \sqrt{(A_1^2 - B_1^{2n+1})}]} + \sqrt[2n+1]{[A_2 + \sqrt{(A_2^2 - B_2^{2n+1})}]} + \cdots$$
$$\cdots + \sqrt[2n+1]{[A_{2n} + \sqrt{(A_{2n}^2 - B_{2n}^{2n+1})}]}.$$

16.

Having thus found the value of $\varphi_1\beta$, we must find $\varphi\beta$. Now this can be done easily as follows:

Let

$$78. \quad \psi_2\beta = \sum_{m=-n}^{+n}\theta^m.\varphi\left(\beta + \frac{2m\omega}{2n+1}\right);\ \psi_3\beta = \sum_{m=-n}^{+n}\theta^m.\varphi\left(\beta - \frac{2m\omega}{2n+1}\right),$$

we have

$$\varphi\left(\beta \pm \frac{2m\omega}{2n+1}\right) = \frac{\varphi\beta.f(\frac{2m\omega}{2n+1}).F(\frac{2m\omega}{2n+1}) \pm f\beta.F\beta.\varphi(\frac{2m\omega}{2n+1})}{1 + e^2c^2.\varphi^2(\frac{2m\omega}{2n+1}).\varphi^2\beta}.$$

From there it follows that one can put

$$\varphi_2\beta = r + f\beta.F\beta.s;\ \varphi_3\beta = r - f\beta.F\beta.s,$$

where $r$ and $s$ are rational functions of $\varphi\beta$.

From there we conclude

$$79. \quad \left\{\begin{array}{rl} \psi_2\beta.\psi_3\beta = & \chi(\varphi\beta), \\ (\psi_2\beta)^{2n+1} + (\psi_3\beta)^{2n+1} = & \chi_1(\varphi\beta), \end{array}\right.$$

$\chi(\varphi\beta)$ and $\chi_1(\varphi\beta)$ being two rational functions of $\varphi\beta$.

That stated, I claim that $\chi(\varphi\beta)$ and $\chi_1(\varphi\beta)$ can be expressed rationally in $\varphi_1\beta$.

We saw that

$$80. \qquad \varphi_1\beta = \varphi\beta + \sum_{m=1}^{n} \left\{ \frac{2\varphi\beta.F(\frac{2m\omega}{2n+1}).F(\frac{2m\omega}{2n+1})}{1+e^2c^2.\varphi^2(\frac{2m\omega}{2n+1})\varphi^2\beta} \right\}.$$

Setting $\varphi\beta = x$, we have an equation in $x$ of degree $(2n+1)$. A root of this equation is $x = \varphi\beta$; now substituting $\beta + \frac{2k\omega}{2n+1}$ for $\beta$, $\varphi_1\beta$ does not change the value. Thus $x = \varphi\left(\beta + \frac{2k\omega}{2n+1}\right)$ will be a root whatever the integer $k$. By now giving $k$ the integer values from $-n$ to $+n$, $\varphi\left(\beta + \frac{2k\omega}{2n+1}\right)$ takes $2n+1$ different values. Hence these $2n+1$ quantities will be precisely the $2n+1$ roots of the equation in $x$.

That stated, substituting $\beta + \frac{2k\omega}{2n+1}$ for $\beta$ in the expressions for $\psi_2\beta$ it becomes, in virtue of (72.):

$$\begin{aligned}
\psi_2\left(\beta + \frac{2k\omega}{2n+1}\right) &= \sum_{m=-n}^{+n} \theta^m \varphi\left(\beta + \frac{2(k+m)\omega}{2n+1}\right) \\
&= \theta^{-k}\psi_2\beta + \sum_{m=1}^{k} \theta^{m+n-k}.\varphi\left(\beta + \frac{2(m+n)\omega}{2n+1}\right) \\
&- \sum_{m=1}^{k} \theta^{m-n-1-k}.\varphi\left(\beta + \frac{2(m-n-1)\omega}{2n+1}\right),
\end{aligned}$$

whence, seeing that $\theta^{m+n-k} = \theta^{m-n-1-k}$ and $\varphi\left(\beta + \frac{2(m-n-1)}{2n+1}\omega\right) = \varphi\left(\beta + \frac{2(m+n)\omega}{2n+1}\right)$, the result is

$$81. \qquad \psi_2\left(\beta + \frac{2k\omega}{2n+1}\right) = \theta^{-k}.\psi_2\beta.$$

In the same way we have

$$\psi_3\left(\beta + \frac{2k\omega}{2n+1}\right) = \theta^{+k}.\psi_3\beta.$$

We see in virtue of these relations that the equations that give the values of of the functions $\chi(\varphi\beta)$ and $\chi_1(\varphi\beta)$ lead to the two equalities:

$$\begin{aligned}
\chi_1\left[\varphi\left(\beta + \frac{2k\omega}{2n+1}\right)\right] &= \chi_1(\varphi\beta), \\
\chi\left[\varphi\left(\beta + \frac{2k\omega}{2n+1}\right)\right] &= \chi(\varphi\beta).
\end{aligned}$$

From which one deduces

$$\begin{aligned}\chi(\varphi\beta) &= \frac{1}{2n+1}.\sum_{k=-n}^{+n}\chi\left[\varphi\left(\beta+\frac{2k\omega}{2n+1}\right)\right],\\ \chi_1(\varphi\beta) &= \frac{1}{2n+1}.\sum_{k=-n}^{+n}\chi_1\left[\varphi\left(\beta+\frac{2k\omega}{2n+1}\right)\right].\end{aligned}$$

Now these values of $\chi(\varphi\beta)$ and $\chi_1(\varphi\beta)$ are rational functions and symmetric for all of the roots of equation (80.). So they can be expressed rationally by the coefficients of the same equation, i.e., rationally in $\varphi_1\beta$.

Let

$$\chi(\varphi\beta) = D,\ \chi_1(\varphi\beta) = 2C,$$

the equations in (79.) give

$$\psi_2\beta = \sqrt[2n+1]{[C+\sqrt{(C^2-D^{2n+1})}]},$$

from where, substituting in the value of $\psi_2\beta$:

$$82. \qquad \sqrt[2n+1]{[C+\sqrt{(C^2-D^{2n+1})}]} = \sum_{m=-n}^{+n}\theta^m.\varphi\left(\beta+\frac{2m\omega}{2n+1}\right).$$

From which we conclude, substituting $\theta_\mu$ for $\theta$ and designating the values corresponding to $C$ and $D$ by $C_\mu$ and $D_\mu$:

$$\theta_\mu^{-k}\ \sqrt[2n+1]{[C_\mu+\sqrt{(C_\mu^2-D_\mu^{2n+1})}]} = \sum_{m=-n}^{+n}\theta_\mu^{m-k}.\varphi\left(\beta+\frac{2m\omega}{2n+1}\right).$$

Joining it to the equation

$$\varphi_1\beta = \sum_{m=-n}^{+n}\varphi\left(\beta+\frac{2m\omega}{2n+1}\right),$$

we easily deduce that:

$$83. \qquad (2n+1).\varphi\left(\beta+\frac{2k\omega}{2n+1}\right) = \varphi_1\beta+\sum_{\mu=1}^{2n}\theta_\mu^{-k}.\ \sqrt[2n+1]{[C_\mu+\sqrt{(C_\mu^2-D_\mu^{2n+1})}]}.$$

Setting $k=0$, it becomes:

84. $\varphi\beta = \dfrac{1}{2n+1}\left\{\varphi_1\beta + \sqrt[2n+1]{[C_1 + \sqrt{(C_1^2 - D_1^{2n+1})}]} + \cdots + \sqrt[2n+1]{[C_{2n} + \sqrt{(C_{2n}^2 - D_{2n}^{2n+1})}]}\right\}.$

This equation gives $\varphi\beta$ as an algebraic function of $\varphi_1\beta$. Now previously we found that $\varphi_1\beta$ is an algebraic function of $\varphi(2n+1)\beta$. So in place of $\beta$ putting $\frac{\alpha}{2n+1}$, we have $\varphi(\frac{\alpha}{2n+1})$ as an algebraic function of $\varphi\alpha$.

Using a totally similar analysis, we find $f(\frac{\alpha}{2n+1})$ in $f\alpha$ and $F(\frac{\alpha}{2n+1})$ in $F\alpha$.

17.

The values that we have just found for the quantities $\varphi_1\beta$ and $\varphi\beta$, the first in $\varphi(2n+1)\beta$ and the second in $\varphi_1\beta$, each contain a sum of $2n$ different radicals of degree $(2n+1)$. The result is $(2n+1)^{2n}$ values for $\varphi\beta, \varphi_1\beta,$ . . . ., whereas each of these quantities is the root of an equation of $(2n+1)th$ degree. So we can give the expressions for $\varphi\beta$ and $\varphi_1\beta$ a form so that the number of values for these quantities is precisely equal to $2n+1$.

That is for

$$\theta = \cos\frac{2\pi}{2n+1} + i\sin\frac{2\pi}{2n+1},$$

we can make

$$\theta_1 = \theta, \theta_2 = \theta^2, \theta_3 = \theta^3, \ldots, \theta_{2n} = \theta^{2n}.$$

The same for

$$85. \quad \begin{cases} \psi^k(\beta) = \sum_{\mu=-n}^{+n} \theta^{k\mu}.\varphi_1\left(\beta + \dfrac{2\mu\varpi i}{2n+1}\right), \\ \psi_1^k(\beta) = \sum_{\mu=-n}^{+n} \theta^{k\mu}.\varphi_1\left(\beta - \dfrac{2\mu\varpi i}{2n+1}\right), \end{cases}$$

we have in virtue of (74.)

$$\begin{aligned} \psi^k\left(\beta + \frac{2\nu\varpi i}{2n+1}\right) &= \theta^{-k\nu}.\psi^k(\beta), \\ \psi_1^k\left(\beta + \frac{2\nu\varpi i}{2n+1}\right) &= \theta^{+k\nu}.\psi_1^k(\beta), \\ \psi^1\left(\beta + \frac{2\nu\varpi i}{2n+1}\right) &= \theta^{-\nu}.\psi^1(\beta), \\ \psi_1^1\left(\beta + \frac{2\nu\varpi i}{2n+1}\right) &= \theta^{+\nu}.\psi_1^1(\beta). \end{aligned}$$

Now let

$$86.\quad \begin{cases} \dfrac{\psi^k(\beta)}{(\psi^1\beta)^k} + \dfrac{\psi_1^k(\beta)}{(\psi_1^1\beta)^k} = P(\varphi\beta), \\ \dfrac{\psi^k(\beta)}{(\psi^1\beta)^{k-2n-1}} + \dfrac{\psi_1^k(\beta)}{\psi_1^1(\beta)^{k-2n-1}} = Q(\varphi\beta), \end{cases}$$

$P(\varphi\beta)$ and $Q(\varphi\beta)$ being rational functions of $\varphi\beta$; however, by substituting $\beta+\frac{2m\omega+2\mu\varpi i}{2n+1}$ for $\beta$, it is clear in virtue of the formulas that $P$ and $Q$ do not change values; thus we have

$$P(\varphi\beta) = \frac{1}{(2n+1)^2}.\sum_{m=-n}^{+n}\sum_{\mu=-n}^{+n} P\left[\varphi\left(\beta + \frac{2m\omega+2\mu\varpi i}{2n+1}\right)\right];$$

however the right side is a symmetric rational function of the roots of the equation $\varphi(2n+1)\beta = \frac{P_{2n+1}}{Q_{2n+1}}$, so $P(\varphi\beta)$ can be expressed rationally in $\varphi(2n+1)\beta$. It is the same for $Q(\varphi\beta)$. Knowing these two quantities, the equations in (86.) give

$$\frac{\psi^k(\beta)}{(\psi^1\beta)^k}\left\{1 - \left(\frac{\psi_1^1\beta}{\psi^1\beta}\right)^{2n+1}\right\} = P(\varphi\beta) - \frac{Q(\varphi\beta)}{(\psi_1^1\beta)^{2n+1}},$$

now

$$\begin{aligned} \psi_1^1\beta &= A_1 + \sqrt{(A_1^2 - B_1^{2n+1})}, \\ \psi_1^1\beta &= A_1 - \sqrt{(A_1^2 - B_1^{2n+1})} \end{aligned}$$

hence

$$\frac{\psi^k(\beta)}{(\psi^1\beta)^k}.2\sqrt{(A_1^2 - B_1^{2n+1})} = Q(\varphi\beta) - \left[A_1 - \sqrt{(A_1^2 - B_1^{2n+1})}\right].P(\varphi\beta).$$

So we have

$$\psi^k\beta = (\psi^1\beta)^k.\left\{F_k + H_k.\sqrt{(A_1^2 - B_1^{2n+1})}\right\},$$

where $F_k$ and $H_k$ are rational functions of $\varphi(2n+1)\beta$. Replacing $A_1$ and $B_1$ and substituting in the values of $\psi^k\beta$ and $(\psi^1\beta)^k$, it becomes:

$$\sqrt[2n+1]{\left[A_k + \sqrt{\left(A_k^2 - B_k^{2n+1}\right)}\right]} = \left[A + \sqrt{(A^2 - B^{2n+1})}\right]^{\frac{k}{2n+1}}\left[F_k + H_k.\sqrt{(A^2 - B^{2n+1})}\right],$$

therefore the value of $\varphi_1\beta$ becomes:

$$87.\ \varphi_1\beta = \varphi(2n+1)\beta + \frac{1}{2n+1}.\Big\{ \Big[A+\sqrt{(A^2-B^{2n+1})}\Big]^{\frac{1}{2n+1}}$$
$$+ \Big[F_2 + H_2\sqrt{(A^2-B_{2n}^{2n+1})}\Big]\Big[A+\sqrt{(A^2-B^{2n+1})}\Big]^{\frac{2}{2n+1}}$$
$$+ \cdots + \Big[F_{2n} + H_{2n}\sqrt{(A^2-B^{2n+1})}\Big]\Big[A+\sqrt{(A^2-B^{2n+1})}\Big]^{\frac{2n}{2n+1}} \Big\}.$$

By a very similar process we find that

$$88.\ \varphi\beta = \frac{1}{2n+1}\Big\{\varphi_1\beta + \Big[C+\sqrt{(C^2-D^{2n+1})}\Big]^{\frac{1}{2n+1}}$$
$$+ \Big[K_2 + L_2\sqrt{(C^2-D^{2n+1})}\Big]\Big[C+\sqrt{(C^2-D^{2n+1})}\Big]^{\frac{2}{2n+1}}$$
$$+ \cdots + \Big[K_{2n} + L_{2n}\sqrt{(C^2-D^{2n+1})}\Big]\Big[C+\sqrt{(C^2-D^{2n+1})}\Big]^{\frac{2n}{2n+1}} \Big\},$$

where $K_2, L_2, K_3, L_3 \ldots . K_{2n}, L_{2n}$ are rational functions of $\varphi_1\beta$.

These expressions for $\varphi_1\beta$ and $\varphi\beta$ only have $2n+1$ different values that one obtains by assigning the $2n+1$ values of the radicals. — It follows from our analysis that we can take $\sqrt{(A^2-B^{2n+1})}$ and $\sqrt{(C^2-D^{2n+1})}$ with whatever sign we please.

## 18.

The value that we found for $\varphi(\beta)$ or $\varphi(\frac{\alpha}{2n+1})$ gives, in addition to the function $\varphi\alpha$, the following ones:

$$e,\ c,\ \theta$$
$$\varphi(\tfrac{m\omega}{2n+1}),\ \varphi(\tfrac{m\varpi i}{2n+1}),\ f(\tfrac{m\omega}{2n+1}),$$
$$f(\tfrac{m\varpi i}{2n+1}),\ F(\tfrac{m\omega}{2n+1}), F(\tfrac{m\varpi i}{2n+1}),$$

for arbitrary values of $m$ from 1 to $2n$. Now, whatever the value of $m$, we can always algebraically express $\varphi(\frac{m\omega}{2n+1})$, $f(\frac{m\omega}{2n+1})$, $F(\frac{m\omega}{2n+1})$ and $\varphi(\frac{\omega}{2n+1})$, and $\varphi(\frac{\mu\varpi i}{2n+1})$, $f(\frac{\mu\varpi i}{2n+1})$, $F(\frac{\mu\varpi i}{2n+1})$ and $\varphi(\frac{\varpi i}{2n+1})$. Everything there is known in the expression for $\varphi(\frac{\alpha}{2n+1})$ except the two quantities that are independent of $\alpha$; $\varphi(\frac{\omega}{2n+1})$, $\varphi(\frac{\varpi i}{2n+1})$. — These quantities depend only on $c$ and $e$ and they can be found by solving an equation of degree $(2n+1)^2-1$, knowing the equation $\frac{P_{2n+1}}{x}=0$. — We will see in the following section how we can reduce the solution to that of equations of lower degree.

## §. V.
## C. On the equation $P_{2n+1}=0$.

19.

The expression we just found for $\varphi(\frac{\alpha}{2n+1})$ contains, as we saw, the two constants $\varphi(\frac{\omega}{2n+1})$ and $\varphi(\frac{\varpi i}{2n+1})$. We found these quantities by solving the equation

$$P_{2n+1}=0,$$

for which the roots are represented by

$$x=\varphi\left(\frac{m\omega+\mu\varpi i}{2n+1}\right),$$

where $m$ and $\mu$ can be integers from $-n$ to $+n$. One of these roots, corresponding to $m=0$, $\mu=0$, is equal to zero. So $P_{2n+1}$ is divisible by $x$. Ignoring this factor, we have an equation

$$R=0,\ \text{of degree } (2n+1)^2-1.$$

Setting $x^2=r$, equation (90.) $R=0$ in $r$, is of degree $\frac{(2n+1)^2-1}{2}=2.n(n+1)$, and the roots of the equation are

$$91.\quad r=\varphi^2\left(\frac{m\omega+\mu\varpi i}{2n+1}\right),$$

$\mu$ and $m$ being all the positive values below $n$, disregarding the zero root.

We will now see how one can reduce the solution of the equation $R=0$ to that of of two equations, one of degree $n$ and the other of degree $2n+2$.

To begin with, I claim that the values of $r$ can all be represented by

$$92.\quad \varphi^2\left(\frac{m\omega}{2n+1}\right)\ \text{and}\ \varphi^2\left(m\frac{\mu\omega+\varpi i}{2n+1}\right)\ldots,$$

where we give $\mu$ all the integer values from 0 to $2n$, and $m$ those from 1 to $n$.

Indeed, $\varphi^2\left(\frac{m\omega}{2n+1}\right)$ represents initially $n$ values of $r$; now the other values can be represented by $\varphi^2\left(m\frac{\mu\omega+\varpi i}{2n+1}\right)$. For example, let $m\mu=(2n+1)k+m'$, where $m'$ is an integer between $-n$ and $n$. Substituting we have

$$\begin{aligned}\varphi^2\left(m.\frac{\mu\omega+\varpi i}{2n+1}\right)&=\varphi^2\left(k.\omega+\frac{m'\omega+m\varpi i}{2n+1}\right)\\&=\varphi^2\left(\frac{m'\omega+m\varpi i}{2n+1}\right)=\varphi^2\left(\frac{-m'\omega-m\varpi i}{2n+1}\right).\end{aligned}$$

thus $\varphi^2\left(m.\frac{\mu\omega+\varpi i}{2n+1}\right)$ is a value of $r$; now each value of $m$ gives a different value of $m'$. Since if one has

$$m_1\mu = (2n+1)k_1 + m',$$

it follows that

$$(m_1 - m)\mu = (2n+1).(k - k_1),$$

which is impossible, as $2n+1$ is a prime number.

Thus $\varphi^2\left(m.\frac{\mu\omega+\varpi i}{2n+1}\right)$ combined with $\varphi^2\left(\frac{m\omega}{2n+1}\right)$ represent all the values of $r$.

That stated, let

$$93. \quad \left[r - \varphi^2\left(\frac{\omega}{2n+1}\right)\right]\left[r - \varphi^2\left(\frac{2\omega}{2n+1}\right)\right]\cdots\left[r - \varphi^2\left(\frac{n\omega}{2n+1}\right)\right]$$
$$= r^n + p_{n-1}.r^{n-1} + p_{n-2}.r^{n-2} + \cdots + p_1.r + p_0.$$

The quantities $p_0, p_1, \ldots, p_{n-1}$ are symmetric rational functions of $\varphi^2\left(\frac{\omega}{2n+1}\right), \varphi^2\left(\frac{2\omega}{2n+1}\right)$, $\ldots, \varphi^2\left(\frac{n\omega}{2n+1}\right)$; now these functions can be found by means of an equation of degree $2n+2$.

Let $p$ be an arbitrary symmetric rational function of $\varphi^2\left(\frac{\omega'}{2n+1}\right), \varphi^2\left(\frac{2\omega'}{2n+1}\right), \ldots, \varphi^2\left(\frac{n\omega'}{2n+1}\right)$, where $\omega'$ designates the quantity $m\omega + \mu\varpi i$.

By the formulas that we gave above for expressing $\varphi(n\beta)$ in $\varphi(\beta)$, it is clear that one can express $\varphi^2\left(m'.\frac{\omega'}{2n+1}\right)$ as a rational function of $\varphi^2\left(\frac{\omega'}{2n+1}\right)$. Therefore one can make

$$94. \quad p = \psi.\left[\varphi^2\left(\frac{\omega'}{2n+1}\right)\right] = \theta\left[\varphi^2\left(\frac{\omega'}{2n+1}\right), \varphi^2\left(\frac{2\omega'}{2n+1}\right), \ldots, \varphi^2\left(n\frac{\omega'}{2n+1}\right)\right],$$

$\theta$ designating a symmetric rational function. Putting $\nu\omega'$ in place of $\omega'$ it becomes

$$95. \quad \psi\left[\varphi^2\left(\frac{\nu\omega'}{2n+1}\right)\right] = \theta\left[\varphi^2\left(\frac{\nu\omega'}{2n+1}\right), \varphi^2\left(\frac{2\nu\omega'}{2n+1}\right), \ldots, \varphi^2\left(\frac{n\nu\omega'}{2n+1}\right)\right].$$

Setting

$$\alpha.\nu = (2n+1).k'_\alpha + k_\alpha,$$

where $k_\alpha$ is an integer between $-n$ and $n$, the series

$$k_1, k_2, \ldots, k_n$$

has the same terms (up to sign) as the following:

$$1, 2, 3, \ldots, n;$$

So it is clear that the right side of equation (95.) has the same value as $p$. Therefore:

$$96. \quad \psi\left[\varphi^2\left(\frac{\nu\omega'}{2n+1}\right)\right] = \psi\left[\varphi^2\left(\frac{\omega'}{2n+1}\right)\right].$$

Setting $\omega' = \omega$ and $\omega' = m\omega + \varpi i$ in the equation gives these two:

$$97. \quad \begin{cases} \psi\left[\varphi^2\left(\dfrac{\nu\omega'}{2n+1}\right)\right] = \psi\left[\varphi^2\left(\dfrac{\omega'}{2n+1}\right)\right], \\ \psi\left[\varphi^2\left(\nu.\dfrac{m\omega+\varpi i}{2n+1}\right)\right] = \psi\left[\varphi^2\left(\dfrac{m\omega+\varpi i}{2n+1}\right)\right], \end{cases}$$

for brevity set

$$98. \quad \varphi^2\left(\frac{\nu\omega}{2n+1}\right) = r_\nu, \ \varphi^2\left(\nu.\frac{m\omega+\varpi i}{2n+1}\right) = r_{\nu,m},$$

it becomes

$$99. \quad \psi.r_\nu = \psi r_1; \ \psi r_{\nu,m} = \psi r_{1,m}.$$

That stated, let

$$100. \quad \begin{cases} (p-\psi r_1)(p-\psi r_{1,0})(p-\psi r_{1,1})(p-\psi r_{1,2})\cdots(p-\psi r_{1,2n}) \\ = q_0 + q_1.p + q_2.p^2 + \cdots + q_{2n+1}.p^{2n+1} + p^{2n+2}. \end{cases}$$

I claim that one can express the coefficients $q_0, q_1$ etc. rationally in $e$ and $c$.

First, in virtue of the known formulas we can express these coefficients rationally in $t_1$, $t_2$, …, $t_{2n}$, if for brevity we set

$$101. \quad t_k = (\psi r_1)^k + (\psi r_{1,0})^k + (\psi r_{1,1})^k + \cdots + (\psi r_{1,2n})^k.$$

Now it remains, therefore, to find the quantities $t_1, t_2, \ldots$, which can easily be found by means of the relations in (99.). Indeed, successively setting $\nu = 1, 2, \ldots, n$, before having evaluated the two sides the $k$th power, we conclude at once that:

$$102. \quad \begin{cases} (\psi r_1)^k = \dfrac{1}{n}[(\psi r_1)^k + (\psi r_2)^k + \cdots + (\psi r_n)^k], \\ (\psi r_{1,m})^k = \dfrac{1}{n}[(\psi r_{1,m})^k + (\psi r_{2,m})^k + \cdots + (\psi r_{n,m})^k]. \end{cases}$$

So now substituting for $m$ all the numbers $0, 1, \ldots, 2n$, and then substituting in the expression for $t_k$, we get:

$$103. \quad \left\{ \begin{array}{ll} n.t_k & = (\psi r_1)^k + (\psi r_2)^k + \cdots + (\psi r_n)^k \\ & +(\psi r_{1,0})^k + (\psi r_{2,0})^k + \cdots + (\psi r_{n,0})^k \\ & +(\psi r_{1,1})^k + (\psi r_{2,1})^k + \cdots + (\psi r_{n,1})^k \\ & + \cdots \\ & +(\psi r_{1,2n})^k + (\psi r_{2,2n})^k + \cdots + (\psi r_{n,2n})^k. \end{array} \right.$$

As we will see, this value of $t_k$ is a symmetric rational function of $n(2n+2)$ quantities $r_1, r_2, \ldots,\ r_n, \ldots, r_{1,0}, r_{2,0}, \ldots,\ r_{n,0}, \ldots,\ r_{1,2n}, r_{2,2n}, \ldots, r_{n,2n}$, that are the $n(2n+2)$ roots of the equation $R = 0$. Therefore, as one knows, $t_k$ can be expressed rationally by the coefficients of this equation, and consequently by a rational function of $e$ and $c$. Having thus found the quantities $t_k$, one may deduce the values of $q_0, q_1, \ldots, q_{2n+1}$; which will also be rational functions of $e$ and $c$.

## 20.

That stated, supposing that

$$104. \quad 0 = q_0 + q_1.p + q_2.p^2 + \cdots + q_{2n+1}.p^{2n+1} + p^{2n+2},$$

one will have an equation of degree $2n+2$, the roots of which are

$$\psi r_1, \psi r_{1,0}, \psi r_{1,1}, \psi r_{1,2}, \cdots, \psi r_{1,2n}.$$

We will be able to find the function $\psi r_1$, that is, an arbitrary symmetric rational function of the roots $r_1, r_2, r_3, \ldots, r_n$, by means of an equation of degree $2n+2$.

Hence one will have in this manner the coefficients $p_0, p_1, \ldots, p_{n-1}$, by solving $n$ equations, each of degree $2n+2$.

Having determined $p_0, p_1, \ldots$, we will have by solving the equation

$$105. \quad 0 = p_0 + p_1 r + \cdots + p_{n-1}.r^{n-1} + r^n,$$

the value of the quantities

$$r_1, r_2, \ldots, r_n;\ r_{1,0}, r_{2,0}, \ldots, r_{n,0};\ r_{1,1}, r_{2,1}, \ldots, r_{n,1} \text{ etc. etc.},$$

the first of which is equal to $\varphi^2\left(\frac{\omega}{2n+1}\right)$. So determining this quantity, or solving the equation $R = 0$ that is of degree $(2n+2).n$, is reduced to that of equation of degree $(2n+2)$ and $n$.

But we can again simplify the previous procedure. Indeed, having the quantities $p_0, p_1, \ldots$, we see that it suffices to know an arbitrary one of them, and also that we can express the others rationally in that one.

Generally $p, q$ are two symmetric rational functions of the quantities $r_1, r_2, \ldots, r_n$, we may let, as we saw:

$$p = \psi r_1;\ q = \theta r_1,$$

$\psi r_1$ and $\theta r_1$ designating two rational functions in $r_1$ that have the property: if one changes $r_1$ arbitrarily into one of the other the quantities $r_1, r_2, \ldots, r_n$, it remains the same.

Now suppose:

$$s_k = (\psi r_1)^k.\theta r_1 + (\psi r_{1,0})^k.\theta r_{1,0} + (\psi r_{1,1})^k.\theta r_{1,1} + \cdots + (\psi r_{1,2n})^k.\theta r_{1,2n},$$

I claim that $s_k$ will be able to be expressed rationally in $e$ and $c$.

Indeed,

$$(\psi r_1)^k.\theta r_1 = (\psi r_\nu)^k.\theta r_\nu = \frac{1}{n}\left\{(\psi r_1)^k.\theta r_1 + (\psi r_2)^k.\theta r_2 + \cdots + (\psi r_n)^k.\theta r_n\right\},$$
$$(\psi r_{1,m})^k.\theta r_{1,m} = (\psi r_{\nu,m})^k.\theta r_{\nu,m} = \frac{1}{n}\left\{(\psi r_{1,m})^k.\theta r_{1,m} + (\psi r_{2,m})^k.\theta r_{2,m} + \cdots + (\psi r_{n,m})^k.\theta r_{n,m}\right\}.$$

Setting $m = 0, 1, 2, \ldots, 2n$, and substituting in the expression of $s_k$, we see that $s_k$ will be a symmetric rational function in the roots $r_0, r_1, \ldots, r_{1,0}$ etc., etc. of the equation $R = 0$; therefore $s_k$ can be expressed rationally in $e$ and $c$.

Knowing $s_k$, we obtain by setting $k = 0, 1, 2, \ldots, 2n$, $2n+1$ equations from which one easily deduces the value of $\theta r_1$ as a rational function of $\psi_1 r$. So given a function of the form $p$, we can express another arbitrary function of the same form as a rational function $p$. Thus as we claimed, we can express the coefficients $p_0, p_1, \ldots, p_{n-1}$ rationally using one of them arbitrarily. Therefore in conclusion, to find the values, it suffices to solve a single equation of degree $2n+2$, and as a consequence, to find the roots of the equation $R = 0$ it suffices to solve an equation of degree $2n+2$, and $2n+2$ equations of degree $n$.

## 21.

Now among the equations of which depend on the quantities $\varphi\left(\frac{\omega}{2m+1}\right); \varphi\left(\frac{\varpi i}{2n+1}\right)$ those of degree $n$ can be solved algebraically. The method by which we carried out the solution is entirely similar to that due to M. Gauss for the solution of the equation

$$\theta^{2n+1} - 1 = 0.$$

Let the equation

$$106. \qquad 0 = p_1 + p_1.r + p_2.r^2 + \cdots + p_{n-1}.r^{n-1} + r^n,$$

be given, of which the roots are:

$$\varphi^2\left(\frac{\omega'}{2n+1}\right), \varphi^2\left(\frac{2\omega'}{2n+1}\right), \ldots, \varphi^2\left(\frac{n\omega'}{2n+1}\right),$$

where $\omega'$ is one of the values $\omega$, $m\omega + \varpi i$. Designate by $\alpha$ one of the primitive roots of $2n+1$, that is to say, a integer such that $\mu = 2n+1$ is the smaller number that makes $\alpha^\mu - 1$ divisible by $2n+1$. I claim that the roots of equation (106.) can also be represented by

$$107. \quad \varphi^2(\epsilon), \varphi^2(\alpha\epsilon), \varphi^2(\alpha^2\epsilon), \varphi^2(\alpha^3\epsilon), \ldots, \varphi^2(\alpha^{n-1}.\epsilon),$$

where $\epsilon = \frac{\omega'}{2n+1}$.

Let

$$\alpha^m = (2n+1)k_m \pm a_m,$$

where $k$ is an integer and $a_m$ a positive integer less than $n+1$. I claim that the terms of the series

$$1, a_1, a_2, \ldots, a_{n-1}$$

will be all different.

Indeed, if one has

$$a_m = a_\mu,$$

this results in

$$\text{either } \alpha^m - a^\mu = (2n+1)(k_m - k_\mu),$$
$$\text{or } \alpha^m + \alpha^\mu = (2n+1)(k_m + k_\mu).$$

It is thus necessary that one of the quantities $\alpha^m - \alpha^\mu$, $\alpha^m + \alpha^\mu$ be divisible by $2n+1$; or supposing $m > \mu$, which is possible, it is necessary that $\alpha^{m-\alpha} - 1$ or $a^{m-\mu} + 1$ be divisible by $n$; however that is impossible, as $m - \mu$ is less than $n$.

Therefore the quantities $1, a_1, a_2, \ldots, a_{n-1}$ are distinct and as a consequence coincide with the numbers $1, 2, 3, 4, \ldots, n$, but in a different order, .

Thus noticing that

$$\varphi^2\{[(2n+1)k_m \pm a_m]\epsilon\} = \varphi^2(a_m\epsilon),$$

we see that the quantities (107.) are the same as these:

$$\varphi^2(\epsilon), \varphi^2(2\epsilon), \ldots, \varphi^2(n\epsilon),$$

that is to say, the roots of the equation (106.) QED.

What's more, since

$$\alpha^n = (2n+1)k_n - 1,$$

we will have

$$\alpha^{n+m} = (2n+1)k_n\alpha^m - \alpha^m,$$

so

$$a_{n+m} = +a_m$$

and

$$\varphi^2(\alpha^{n+m}\epsilon) = \varphi^2(\alpha^m\epsilon).$$

That stated, let an arbitrary imaginary root $\theta$ of the equation

$$\theta^n - 1 = 0$$

and

$$108. \quad \psi(\epsilon) = \varphi^2(\epsilon) + \varphi^2(\alpha\epsilon)\theta + \varphi^2(\alpha^2\epsilon)\theta^2 + \cdots + \varphi^2(\alpha^{n-1}\epsilon)\theta^{n-1}.$$

In virtue of what we saw previously, the right hand side of this equation can be transformed into a *rational* function of $\varphi^2(\epsilon)$.

Let us set

$$109. \quad \psi(\epsilon) = \chi\left[\varphi^2(\epsilon)\right].$$

Putting $\alpha^m\epsilon$ in place of $\epsilon$ in the first expression for $\psi(\epsilon)$, we get:

$$\begin{aligned}&\psi(\alpha^m\epsilon) = \varphi^2(\alpha^m\epsilon) + \varphi^2(\alpha^{m+1}\epsilon).\theta + \varphi^2(\alpha^{m+2}\epsilon).\theta^2 + \cdots \\ &+\varphi^2(\alpha^{n-1}\epsilon).\theta^{n-m-1} + \varphi^2(\alpha^n\epsilon).\theta^{n-m} + \cdots + \varphi^2(\alpha^{n+m-1}\epsilon).\theta^{n-1},\end{aligned}$$

but we saw that $\varphi^2(\alpha^{n+m}\epsilon) = \varphi^2(\alpha^m\epsilon)$; therefore:

$$\begin{aligned}&\psi(\alpha^m\epsilon) = \theta^{n-m}.\varphi^2(\epsilon) + \theta^{n-m+1}.\varphi^2(\alpha\epsilon) + \theta^{n-m+2}.\varphi^2(\alpha^2\epsilon) + \cdots \\ &+\theta^{n-1}.\varphi^2(\alpha^{m-1}\epsilon) + \varphi^2(\alpha^m\epsilon) + \theta.\varphi^2(\alpha^{m+1}\epsilon) + \cdots + \theta^{n-m-1}.\varphi^2(\alpha^{n-1}\epsilon).\end{aligned}$$

Multiplying by $\theta^m$, the right hand side becomes equal to $\psi(\epsilon)$, thus:

$$110. \quad \psi(\alpha^m\epsilon) = \theta^{-m}.\psi(\epsilon),$$

or:

$$\psi(\epsilon) = \theta^m.\chi\left[\varphi^2(\alpha^m\epsilon)\right],$$

from which raising both to the $n$th power, one concludes by virtue of the relation $\theta^{mn} = 1$:

$$111. \quad [\psi(\epsilon)]^n = \left\{\chi\left[\varphi^2(\alpha^m\epsilon)\right]\right\}^n.$$

By successively setting $m = 0, 1, 2, 3, \ldots, n-1, n$, the formula gives $n$ equations, which, adding them term by term, give the following:

$$112. \quad n\,[\psi(\epsilon)]^n = \left[\chi\left[\varphi^2(\epsilon)\right]\right]^n + \left[\chi\left[\varphi^2(\alpha\epsilon)\right]\right]^n + \left[\chi\left[\varphi^2(\alpha^2\epsilon)\right]\right]^n + \cdots + \left[\chi\left[\varphi^2(\alpha^{n-1}\epsilon)\right]\right]^n;$$

Now the right hand side of this equation is a symmetric rational function of $\varphi^2(\epsilon), \varphi^2(\alpha\epsilon), \ldots, \varphi^2(\alpha^{n-1}\epsilon)$, that is, the roots of equation (106.); therefore $(\psi\epsilon)^n$ can be expressed by rational functions of $p_0, p_1, \ldots, p_{n-1}$, and consequently by a rational function in any one of these quantities. Let $\nu$ be the value $\nu$ of $\psi(\epsilon)$, then we get

$$113. \quad \sqrt[n]{\nu} = \varphi^2(\epsilon) + \theta.\varphi^2(\alpha\epsilon) + \theta^2.\varphi^2(\alpha^2\epsilon) + \cdots + \theta^{n-1}.\varphi^2(\alpha^{n-1}\epsilon).$$

That established, let $\theta = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. The imaginary roots of the equation $\theta^n - 1$ can be represented by

$$\theta^1, \theta^2, \ldots, \theta^{n-1}.$$

Thus by successively setting $\theta$ equal to each of the roots and designating the corresponding values of $\nu$ by $\nu_1, \nu_2, \ldots, \nu_{n-1}$, we get

$$\begin{aligned}
&\sqrt[n]{\nu_1} = \varphi^2(\epsilon) + \theta.\varphi^2(\alpha\epsilon) + \cdots + \theta^{n-1}.\varphi^2(\alpha^{n-1}\epsilon)\\
&\sqrt[n]{\nu_2} = \varphi^2(\epsilon) + \theta^2.\varphi^2(\alpha\epsilon) + \cdots + \theta^{2n-2}.\varphi^2(\alpha^{n-1}\epsilon)\\
&\cdots\\
&\sqrt[n]{\nu_{n-1}} = \varphi^2(\epsilon) + \theta^{n-1}.\varphi^2(\alpha\epsilon) + \cdots +
\end{aligned}$$

By combining these eqautions with the following:

$$-p_{n-1} = \varphi^2(\epsilon) + \varphi^2(\alpha\epsilon) + \cdots + \varphi^2(\alpha^{n-1}\epsilon),$$

we easily conclude:

$$114. \quad \phi^2(\alpha^m\epsilon) = +\frac{1}{n}\left\{-p_{n-1} + \theta^{-m}.\sqrt[n]{\nu_1} + \theta^{-2m}.\sqrt[n]{\nu_2} + \theta^{-5m}.\sqrt[n]{\nu_3} + \cdots + \theta^{-(n-1)m}.\sqrt[n]{\nu_{n-1}}\right\},$$

and for $m = 0$:

$$115. \quad \varphi^2(\epsilon) = \frac{1}{n}\left\{-p_{n-1} + \sqrt[n]{\nu_1} + \sqrt[n]{\nu_2} + \cdots + \sqrt[n]{\nu_{n-1}}\right\}.$$

22.

All of the roots of equation (106.) are contained in formula (115.), but since there are only $n$ of them, it still remains to give $\varphi^2(\epsilon)$ a form that does not contain roots to the question. Now this is easily done as follows:

Let

$$s_k = \frac{\sqrt[n]{\nu_k}}{(\sqrt[n]{\nu_1})^k}.$$

Using here $\epsilon$ instead of $\alpha^m\epsilon$, $\sqrt[n]{\nu_k}$ will become $\theta^{-km}.\sqrt[n]{\nu_k}$, and $\sqrt[n]{\nu_1}$ to $\theta^{-m}.\sqrt[n]{\nu_1}$, so $s_k$ becomes:

$$\frac{\theta^{-km}.\sqrt[n]{\nu_k}}{\left(\theta^{-m}\sqrt[n]{\nu_1}\right)^k} = \frac{\sqrt[n]{\nu_k}}{\left(\sqrt[n]{\nu_1}\right)^k}.$$

The function $s_k$, as can be seen, does not change value when one uses $\alpha^m\epsilon$ instead of $\epsilon$. Now $p_k$ is a rational function of $\varphi^2(\epsilon)$. Thus, by designating $p_k$ by $\lambda\left[\varphi^2(\epsilon)\right]$, we have

$$s_k = \lambda\left[\varphi^2(\alpha^m\epsilon)\right],$$

whatever the integer $m$. One will conclude in the same way as we have found for $(\psi\epsilon)^n$, that the value of $s_k$ is a rational function of one of the quantities $p_0, p_1, \ldots, p_{n-1}$. Knowing $s_k$, we get:

$$\sqrt[n]{\nu_k} = s_k\left(\sqrt[n]{\nu_1}\right)^k.$$

So putting $\nu$ in place of $\nu_1$, the expression for $\varphi^2(\alpha^m\epsilon)$ becomes:

116. $$\varphi^2(\alpha^m\epsilon) = \frac{1}{n}\left\{-p_{n-1} + \theta^{-m}.\nu^{\frac{1}{n}} + s_2\theta^{-2m}.\nu^{\frac{2}{n}} + \cdots + s_{n-1}\theta^{-(n-1)m}.\nu^{\frac{n-1}{n}}\right\},$$

for $m = 0$:

117. $$\varphi^2(\epsilon) = \frac{1}{n}\left\{-p_{n-1} + \nu^{\frac{1}{n}} + s_2.\nu^{\frac{2}{n}} + s_3.\nu^{\frac{3}{n}} + \cdots + s_{n-1}.\nu^{\frac{n-1}{n}}\right\}.$$

This value has only $n$ distinct values that correspond to the $n$ values of $\nu^{\frac{1}{n}}$. So ultimately the solution of the equation $P_{2n+1} = 0$ is reduced to one equation of degree $2n+2$; but this equation does not appear in general to be solvable algebraically. Nevertheless one can solve it completely in particular cases, e.g., when $e = c$, $e = c\sqrt{3}$, $e = c(2 \pm \sqrt{3})$, etc. In the course of this memoir I will deal with these cases, of which the first is especially remarkable, as much for the simplicity of the solution as for its beautiful applications in geometry.

Indeed, among other things I arrived at this theorem:

> One can divide the whole circumference of the lemniscate into $m$ equal parts using ruler and compass only if $m$ is of the form $2^n$ or $2^n + 1$, the latter number at the same time being prime; or if $m$ is a product of several numbers of the two forms.

This theorem is, one sees, is precisely the same as that of M. Guass relative to the circle.

## §. VI.
## Various expressions for the functions $\phi(n\beta)$, $f(n\beta)$, $F(n\beta)$.

### 23.

By using the known formulas that give the values of the coefficients of an algebraic equation as a function of the roots, one can derive various expressions for the functions $\phi(n\beta)$, $f(n\beta)$, $F(n\beta)$ from the formulas in the preceding paragraphs.

I will consider the most remarkable.

To shorten the formulas, the following notations will be useful: I will indicate:

1) By $\sum\limits_{m=k}^{k'} \psi(m)$ the sum, and by $\prod\limits_{m=k}^{k'} \psi(m)$ the product of all the quantities of the form $\psi(m)$ that one will obtain by giving $m$ all integer values between $k$ and $k'$, the limits $k$ and $k'$ included.

2) By $\sum\limits_{m=k}^{k'} \sum\limits_{\mu=\nu}^{\nu'} \psi(m,\mu)$ the sum, and by $\prod\limits_{m=k}^{k'} \prod\limits_{\mu=\nu}^{\nu'} \psi(m,\mu)$ the product of all the quantities of the form $\psi(m,\mu)$ that one obtains by giving $m$ all the integer values from $k$ to $k'$, and to $\mu$ the values from $\nu$ to $\nu'$, always including the limits.

According to that, it is clear that one will have:

$$119. \quad \sum_{m=k}^{k'} \psi(m) = \psi(k) + \psi(k+1) + \cdots + \psi(k'),$$

$$120. \quad \prod_{m=k}^{k'} \psi(m) = \psi(k) \cdot \psi(k+1) \cdots \psi(k'),$$

$$121. \quad \sum_{m=k}^{k'} \sum_{\mu=\nu}^{\nu'} \psi(m,\mu) = \sum_{\mu=\nu}^{\nu'} \psi(k,\mu) + \sum_{\mu=\nu}^{\nu'} \psi(k+1,\mu) + \cdots + \sum_{\mu=\nu}^{\nu'} \psi(k',\mu),$$

$$122. \quad \prod_{m=k}^{k'} \prod_{\mu=\nu}^{\nu'} \psi(m,\mu) = \prod_{\mu=\nu}^{\nu'} \psi(k,\mu). \prod_{\mu=\nu}^{\nu'} \psi(k+1,\mu) \cdots \prod_{\mu=\nu}^{\nu'} \psi(k',\mu).$$

That said, consider the equations

$$
123. \quad \begin{cases} \varphi(2n+1)\beta = \dfrac{P_{2n+1}}{Q_{2n+1}}, \\ f(2n+1)\beta = \dfrac{P'_{2n+1}}{Q_{2n+1}}, \\ F(2n+1)\beta = \dfrac{P''_{2n+1}}{Q_{2n+1}}. \end{cases}
$$

You will see, that $P_{2n+1}$ is a rational function of $x$ of degree $(2n+1)^2$ and of the form $x.\psi(x^2)$. Similarly, $P'_{2n+1}$ and $P''_{2n+1}$ are functions of the same form, the first in $y$, the second in $z$. Finally $Q_{2n+1}$ is a function that, expressed indistinctly in $x$, $y$, or $z$, will be of degree $(2n+1)^2-1$, and only contains even powers. So we have

$$
\begin{array}{ll}
P_{2n+1} = A.x^{(2n+1)^2} + \cdots + B.x; & P'_{2n+1} = A'.y^{(2n+1)^2} + \cdots + B'.y; \\
P''_{2n+1} = A''.z^{(2n+1)^2} + \cdots + B''.z; & Q_{2n+1} = C.x^{(2n+1)^2-1} + \cdots + D; \\
Q_{2n+1} = C'.y^{(2n+1)^2-1} + \cdots + D'; & Q_{2n+1} = C''.z^{(2n+1)^2-1} + \cdots + D''.
\end{array}
$$

Substituting these values into (123.), we get

$$
\begin{array}{l}
\left\{A.x^{(2n+1)^2} + \cdots + B.x\right\} - \varphi(2n+1)\beta.\left\{C.x^{(2n+1)^2-1} + \cdots + D\right\}, \\
\left\{A'.y^{(2n+1)^2} + \cdots + B'.y\right\} - f(2n+1)\beta.\left\{C'.y^{(2n+1)^2-1} + \cdots + D'\right\}, \\
\left\{A''.z^{(2n+1)^2} + \cdots + B''.z\right\} - F(2n+1)\beta.\left\{C''.z^{(2n+1)^2-1} + \cdots + D''\right\}.
\end{array}
$$

In the first of these equations $A$ is the coefficient of the first term, $-\varphi(2n+1)\beta.C$ that of the second, and $-\varphi(2n+1)\beta.D$ the last term. So $\frac{C}{A}\varphi(2n+1)\beta$ is equal to the sum and $\pm\frac{D}{A}\varphi(2n+1)\beta$ is equal to the product of the roots of the equations under discussion, an equation that is the same as this one:

$$
124. \quad \varphi(2n+1)\beta = \frac{P_{2n+1}}{Q_{2n+1}}.
$$

Thus while noticing that $A$, $C$ and $D$ (and in general all the coefficients) are independent of $\beta$, one sees that $\varphi(2n+1)\beta$ is (up to a constant coefficient) equal to the sum and product of all the roots of equation (124.).

In the same manner, one sees that $f(2n+1)\beta$ and $F(2n+1)\beta$ are respectively equal to the product or to the sum of the roots of the equations

$$
f(2n+1)\beta = \frac{P'_{2n+1}}{Q_{2n+1}}, F(2n+1)\beta = \frac{P''_{2n+1}}{Q_{2n+1}},
$$

by taking care to multiply the result by a suitably chosen constant coefficient.

Now the roots of equation (123.) according to No. 11 are respectively:

$$
\begin{aligned}
x &= (-1)^{m+\mu}.\varphi\left(\beta + \frac{m}{2n+1}\omega + \frac{\mu}{2n+1}\varpi i\right),\\
y &= (-1)^{m}.f\left(\beta + \frac{m}{2n+1}\omega + \frac{\mu}{2n+1}\varpi i\right),\\
z &= (-1)^{\mu}.F\left(\beta + \frac{m}{2n+1}\omega + \frac{\mu}{2n+1}\varpi i\right),
\end{aligned}
$$

where the limits of $m$ and $\mu$ are $-n$ and $n$.

Therefore in virtue of what we have just seen, and by using the notations adopted we will have the following formulas:

$$
125. \quad \left\{
\begin{aligned}
\varphi(2n+1)\beta &= A.\sum_{m=-n}^{+n}\sum_{\mu=-n}^{+n}(-1)^{m+\mu}.\varphi\left(\beta + \frac{m.\omega + \mu\varpi i}{2n+1}\right),\\
f(2n+1)\beta &= A'\sum_{m=-n}^{+n}\sum_{\mu=-n}^{+n}(-1)^{m}.f\left(\beta + \frac{m\omega + \mu\varpi i}{2n+1}\right),\\
F(2n+1)\beta &= A''\sum_{m=-n}^{+n}\sum_{\mu=-n}^{+n}(-1)^{\mu}.F\left(\beta + \frac{m\omega + \mu\varpi i}{2n+1}\right);\\
\varphi(2n+1)\beta &= B\prod_{m=-n}^{+n}\prod_{\mu=-n}^{+n}.\varphi\left(\beta + \frac{m\omega + \mu\varpi i}{2n+1}\right),\\
f(2n+1)\beta &= B'.\prod_{m=-n}^{+n}\prod_{\mu=-n}^{+n}.f\left(\beta + \frac{m\omega + \mu\varpi i}{2n+1}\right),\\
F(2n+1)\beta &= B''.\prod_{m=-n}^{+n}\prod_{\mu=-n}^{+n}.F\left(\beta + \frac{m\omega + \mu\varpi i}{2n+1}\right).
\end{aligned}
\right.
$$

To determine the constant quantities $A$, $A'$, $A''$, $B$, $B'$, $B''$, it will be necessary to give $\beta$ a particular value. Thus taking $\beta = \frac{\omega}{2} + \frac{\varpi}{2}i$ in the the first three formulas, after having divided the two sides by $\varphi\beta$, it will become, noticing that $\varphi\left(\frac{\omega}{2} + \frac{\varpi}{2}i\right) = \frac{1}{0}$:

$$
\left.
\begin{aligned}
A &= \frac{\varphi(2n+1)\beta}{\varphi\beta}\\
A' &= \frac{f(2n+1)\beta}{f\beta}\\
A'' &= \tfrac{F(2n+1)\beta}{f\beta}
\end{aligned}
\right\}\text{for } \beta = \frac{\omega}{2} + \frac{\varpi}{2}i.
$$

Given $\beta = \frac{\omega}{2} + \frac{\varpi}{2}i + \alpha$, one has:

$$\left.\begin{aligned}
A &= \frac{\varphi\left((2n+1)\alpha + n\omega + n\varpi i + \frac{\omega}{2} + \frac{\varpi}{2}i\right)}{\varphi\left(\alpha + \frac{\omega}{2} + \frac{\varpi}{2}i\right)}\\
&= \frac{\varphi\left((2n+1)\alpha + \frac{\omega}{2} + \frac{\varpi}{2}i\right)}{\varphi\left(\alpha + \frac{\omega}{2} + \frac{\varpi}{2}i\right)} = \frac{\varphi\alpha}{\varphi(2n+1)\alpha},\\
A' &= \frac{f\left((2n+1)\alpha + n\omega + n\varpi i + \frac{\omega}{2} + \frac{\varpi}{2}i\right)}{f\left(\alpha + \frac{\omega}{2} + \frac{\varpi}{2}i\right)}\\
&= (-1)^n.\frac{f\left((2n+1)\alpha + \frac{\omega}{2} + \frac{\varpi}{2}i\right)}{f\left(\alpha + \frac{\omega}{2} + \frac{\varpi}{2}i\right)} = (-1)^n.\frac{f\left(\alpha + \frac{\omega}{2}\right)}{f\left((2n+1)\alpha + \frac{\omega}{2}\right)},\\
A'' &= \frac{F\left((2n+1)\alpha + \frac{\omega}{2} + \frac{\varpi}{2}i\right)}{F\left(\alpha + \frac{\omega}{2} + \frac{\varpi}{2}i\right)}\\
&= (-1)^n.\frac{F\left((2n+1)\alpha + n\omega + n\varpi i + \frac{\omega}{2} + \frac{\varpi}{2}i\right)}{F\left(\alpha + \frac{\omega}{2} + \frac{\varpi}{2}i\right)} = (-1)^n.\frac{F\left(\alpha + \frac{\varpi}{2}i\right)}{F\left((2n+1)\alpha + \frac{\varpi}{2}i\right)}.
\end{aligned}\right\}\text{for } \alpha = 0.$$

These expressions for $A$, $A'$, $A''$ will be of the from $\frac{0}{0}$ by setting $\alpha = 0$. Thus we will find according to the known rules:

$$A = \frac{1}{2n+1},\ A' = A'' = \frac{(-1)^n}{2n+1}.$$

According to that the three first formulas will become:

$$126.\quad \left\{\begin{aligned}
\varphi(2n+1)\beta &= \frac{1}{2n+1}\sum_{m=-n}^{+n}\sum_{\mu=-n}^{+n}(-1)^{m+\mu}.\varphi\left(\beta + \frac{m\omega + \mu\varpi i}{2n+1}\right),\\
f(2n+1)\beta &= \frac{(-1)^n}{2n+1}.\sum_{m=-n}^{+n}\sum_{\mu=-n}^{+n}(-1)^{m}.f\left(\beta + \frac{m\omega + \mu\varpi i}{2n+1}\right),\\
F(2n+1)\beta &= \frac{(-1)^n}{2n+1}.\sum_{m=-n}^{+n}\sum_{\mu=-n}^{+n}(-1)^{\mu}.F\left(\beta + \frac{m\omega + \mu\varpi i}{2n+1}\right).
\end{aligned}\right.$$

To find the values of the constants $B$, $B'$, $B''$, I notice that we will have:

$$\begin{aligned}
127.\quad &\prod_{m=-n}^{+n}\prod_{\mu=-n}^{+n}\psi(m,\mu) = \psi(0,0).\prod_{m=1}^{n}\psi(m,0).\prod_{\mu=1}^{n}\psi(0,\mu)\\
&\times\prod_{m=1}^{n}\prod_{\mu=1}^{n}\psi(m,\mu).\psi(-m,-\mu).\prod_{m=1}^{n}\prod_{\mu=1}^{n}\psi(m,-\mu).\psi(-m,\mu).
\end{aligned}$$

Applying this transformation to formulas (126.), dividing the first by $\varphi\beta$, the second by $f\beta$ and the third by $F\beta$, at the same time making $\beta = 0$ in the first, $\beta = \frac{\omega}{2}$ in the second and $\beta = \frac{\varpi}{2}i$ in the third, and noticing that $\frac{\varphi(2n+1)\beta}{\varphi\beta} = 2n+1$, for $\beta = 0$, that $\frac{f(2n+1)\beta}{f\beta} = 2n+1$, for $\beta = \frac{\omega}{2}$, and that $\frac{F(2n+1)\beta}{F\beta} = 2n+1$, for $\beta = \frac{\varpi}{2}i$: we find:

$$
128. \quad \left\{
\begin{aligned}
(2n+1) \quad &= B. \prod_{m=1}^{n} \varphi^2\left(\frac{m\omega}{2n+1}\right) . \prod_{\mu=1}^{n} \varphi^2\left(\frac{\mu\varpi i}{2n+1}\right) \\
&\times \prod_{m=1}^{n}\prod_{\mu=1}^{n} \varphi^2\left(\frac{m\omega+\mu\varpi i}{2n+1}\right) .\varphi^2\left(\frac{m\omega-\mu\varpi i}{2n+1}\right), \\
(2n+1) \quad &= B'. \prod_{m=1}^{n} f^2\left(\frac{\omega}{2}+\frac{m\omega}{2n+1}\right) . \prod_{\mu=1}^{n} f^2\left(\frac{\omega}{2}+\frac{\mu\varpi i}{2n+1}\right) \\
&\times \prod_{m=1}^{n}\prod_{\mu=1}^{n} f^2\left(\frac{\omega}{2}+\frac{m\omega+\mu\varpi i}{2n+1}\right) .f^2\left(\frac{\omega}{2}+\frac{m\omega-\mu\varpi i}{2n+1}\right). \\
(2n+1) \quad &= B''. \prod_{m=1}^{n} F^2\left(\frac{\varpi}{2}i+\frac{m\omega}{2n+1}\right) . \prod_{\mu=1}^{n} F^2\left(\frac{\varpi}{2}i+\frac{\mu\varpi i}{2n+1}\right) \\
&\times \prod_{m=1}^{n}\prod_{\mu=1}^{n} F^2\left(\frac{\varpi}{2}i+\frac{m\omega+\mu\varpi i}{2n+1}\right) .F^2\left(\frac{\varpi}{2}i+\frac{m\omega-\mu\varpi i}{2n+1}\right).
\end{aligned}
\right.
$$

From these equations one find the values of $B$, $B'$, $B''$, and then substituting these into the transformed formulas, we get:

$$
129.\quad \begin{cases}
\varphi(2n+1)\beta = \\
(2n+1)\varphi\beta.\prod\limits_{m=1}^{n}\dfrac{\varphi\left(\beta+\frac{m\omega}{2n+1}\right).\varphi\left(\beta-\frac{m\omega}{2n+1}\right)}{\varphi^2\left(\frac{m\omega}{2n+1}\right)}.\prod\limits_{\mu=1}^{n}\dfrac{\varphi\left(\beta+\frac{\mu\varpi i}{2n+1}\right).\varphi\left(\beta-\frac{\mu\varpi i}{2n+1}\right)}{\varphi^2\left(\frac{\mu\varpi i}{2n+1}\right)} \\
\times\prod\limits_{m=1}^{n}\prod\limits_{\mu=1}^{n}\dfrac{\varphi\left(\beta+\frac{m\omega+\mu\varpi i}{2n+1}\right).\varphi\left(\beta-\frac{m\omega+\mu\varpi i}{2n+1}\right)}{\varphi^2\left(\frac{m\omega+\mu\varpi i}{2n+1}\right)}.\dfrac{\varphi\left(\beta+\frac{m\omega-\mu\varpi i}{2n+1}\right).\varphi\left(\beta-\frac{m\omega-\mu\varpi i}{2n+1}\right)}{\varphi^2\left(\frac{m\omega-\mu\varpi i}{2n+1}\right)}, \\
f(2n+1)\beta = \\
(2n+1)f\beta.\prod\limits_{m=1}^{n}\dfrac{f\left(\beta+\frac{m\omega}{2n+1}\right).f\left(\beta-\frac{m\omega}{2n+1}\right)}{f^2\left(\frac{\omega}{2}+\frac{m\omega}{2n+1}\right)}.\prod\limits_{\mu=1}^{n}\dfrac{f\left(\beta+\frac{\mu\varpi i}{2n+1}\right).f\left(\beta-\frac{\mu\varpi i}{2n+1}\right)}{f^2\left(\frac{\omega}{2}+\frac{\mu\varpi i}{2n+1}\right)} \\
\times\prod\limits_{m=1}^{n}\prod\limits_{\mu=1}^{n}\dfrac{f\left(\beta+\frac{m\omega+\mu\varpi i}{2n+1}\right).f\left(\beta-\frac{m\omega+\mu\varpi i}{2n+1}\right)}{f^2\left(\frac{\omega}{2}+\frac{m\omega+\mu\varpi i}{2n+1}\right)}.\dfrac{f\left(\beta+\frac{m\omega-\mu\varpi i}{2n+1}\right).f\left(\beta-\frac{m\omega-\mu\varpi i}{2n+1}\right)}{f^2\left(\frac{\omega}{2}+\frac{m\omega-\mu\varpi i}{2n+1}\right)}, \\
F(2n+1)\beta = \\
(2n+1)F\beta.\prod\limits_{m=1}^{n}\dfrac{F\left(\beta+\frac{m\omega}{2n+1}\right).F\left(\beta-\frac{m\omega}{2n+1}\right)}{F^2\left(\frac{\varpi}{2}i+\frac{m\omega}{2n+1}\right)}.\prod\limits_{\mu=1}^{n}\dfrac{F\left(\beta+\frac{\mu\varpi i}{2n+1}\right).F\left(\beta-\frac{\mu\varpi i}{2n+1}\right)}{F^2\left(\frac{\varpi}{2}i+\frac{\mu\varpi i}{2n+1}\right)} \\
\times\prod\limits_{m=1}^{n}\prod\limits_{\mu=1}^{n}\dfrac{F\left(\beta+\frac{m\omega+\mu\varpi i}{2n+1}\right).F\left(\beta-\frac{m\omega+\mu\varpi i}{2n+1}\right)}{F^2\left(\frac{\varpi}{2}i+\frac{m\omega+\mu\varpi i}{2n+1}\right)}.\dfrac{F\left(\beta-\frac{m\omega-\mu\varpi i}{2n+1}\right).F\left(\beta-\frac{m\omega-\mu\varpi i}{2n+1}\right)}{F^2\left(\frac{\varpi}{2}i+\frac{m\omega-\mu\varpi i}{2n+1}\right)}.
\end{cases}
$$

One can give these formulas a very simple form by making use of the following formulas:

$$
\begin{aligned}
\frac{\varphi(\beta+\alpha).\varphi(\beta-\alpha)}{\varphi^2\alpha} &= -\frac{1-\frac{\varphi^2\beta}{\varphi^2\alpha}}{1-\frac{\varphi^2\beta}{\varphi^2\left(\alpha+\frac{\omega}{2}+\frac{\varpi}{2}i\right)}}, \\
\frac{f(\beta+\alpha).f(\beta-\alpha)}{f^2\left(\frac{\omega}{2}+\alpha\right)} &= -\frac{1-\frac{f^2\beta}{f^2\left(\frac{\omega}{2}+\alpha\right)}}{1-\frac{f^2\beta}{f^2\left(\alpha+\frac{\omega}{2}+\frac{\varpi}{2}i\right)}}, \\
\frac{F^2(\beta+\alpha).F(\beta-\alpha)}{F^2\left(\frac{\varpi}{2}i+\alpha\right)} &= -\frac{1-\frac{F^2\beta}{F^2\left(\frac{\varpi}{2}i+\alpha\right)}}{1-\frac{F^2\beta}{F^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\alpha\right)}},
\end{aligned}
$$

that one can easily verify using formulas (13.), (16.), (18.).
From these formulas, it is clear that one can put equations (129.) into the form:

$$
130.\quad \begin{cases}
\varphi(2n+1)\beta = \\
(2n+1)\varphi\beta.\prod\limits_{m=1}^{n}.\dfrac{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{m\omega}{2n+1}\right)}}{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega}{2n+1}\right)}}.\prod\limits_{\mu=1}^{n}\dfrac{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\mu\varpi i}{2n+1}\right)}}{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{\mu\varpi i}{2n+1}\right)}} \\
\times\prod\limits_{m=1}^{n}\prod\limits_{\mu=1}^{n}\dfrac{1-\frac{\varphi^\beta}{\varphi^2\left(\frac{m\omega+\mu\varpi i}{2n+1}\right)}}{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega+\mu\varpi i}{2n+1}\right)}}.\dfrac{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{m\omega-\mu\varpi i}{2n+1}\right)}}{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega-\mu\varpi i}{2n+1}\right)}}, \\
f(2n+1)\beta = \\
(2n+1)f\beta.\prod\limits_{m=1}^{n}\dfrac{1-\frac{f^2\beta}{f^2\left(\frac{m\omega}{2n+1}+\frac{\omega}{2}\right)}}{1-\frac{f^2\beta}{f^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega}{2n+1}\right)}}.\prod\limits_{\mu=1}^{n}\dfrac{1-\frac{f^2\beta}{f^2\left(\frac{\omega}{2}+\frac{\mu\varpi i}{2n+1}\right)}}{1-\frac{f^2\beta}{f^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{\mu\varpi i}{2n+1}\right)}} \\
\times\prod\limits_{m=1}^{n}\prod\limits_{\mu=1}^{n}\dfrac{1-\frac{f^2\beta}{f^2\left(\frac{\omega}{2}+\frac{m\omega+\mu\varpi i}{2n+1}\right)}}{1-\frac{f^2\beta}{f^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega+\mu\varpi i}{2n+1}\right)}}.\dfrac{1-\frac{f^2\beta}{f^2\left(\frac{\omega}{2}+\frac{m\omega-\mu\varpi i}{2n+1}\right)}}{1-\frac{f^2\beta}{f^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega-\mu\varpi i}{2n+1}\right)}}, \\
F(2n+1)\beta = \\
(2n+1)F\beta.\prod\limits_{m=1}^{n}\dfrac{1-\frac{F^2\beta}{F^2\left(\frac{\varpi}{2}i+\frac{m\omega}{2n+1}\right)}}{1-\frac{F^2\beta}{F^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega}{2n+1}\right)}}.\prod\limits_{\mu=1}^{n}\dfrac{1-\frac{F^2\beta}{F^2\left(\frac{\varpi}{2}i+\frac{\mu\varpi i}{2n+1}\right)}}{1-\frac{F^2\beta}{F^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{\mu\varpi i}{2n+1}\right)}} \\
\times\prod\limits_{m=1}^{n}\prod\limits_{\mu=1}^{n}\dfrac{1-\frac{F^2\beta}{F^2\left(\frac{\varpi}{2}i+\frac{m\omega+\mu\varpi i}{2n+1}\right)}}{1-\frac{F^2\beta}{F^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega+\mu\varpi i}{2n+1}\right)}}.\dfrac{1-\frac{F^2\beta}{F^2\left(\frac{\varpi}{2}i+\frac{m\omega-\mu\varpi i}{2n+1}\right)}}{1-\frac{F^2\beta}{F^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega-\mu\varpi i}{2n+1}\right)}}.
\end{cases}
$$

These formulas give, as one sees, the values of $\varphi(2n+1)\beta$, $f(2n+1)\beta$ and $F(2n+1)\beta$, expressed respectively as rational functions of $\varphi\beta$, $f\beta$ and $F\beta$, in the form of products.

We will also give the values of $f(2n+1)\beta$, $F(2n+1)\beta$ in another form that will be useful in what follows.

We have $f^2\beta = 1 - c^2\varphi^2\beta$. Hence

$$1-\frac{f^2\beta}{f^2\alpha} = \frac{c^2\left(\varphi^2\beta-\varphi^2\alpha\right)}{f^2\alpha} = \frac{c^2}{f^2\alpha}\varphi^2\beta - \frac{c^2\varphi^2\alpha}{f^2\alpha}$$

and

$$1-\frac{f^2\beta}{f^2\left(\frac{\varpi}{2}i+\alpha\right)} = \frac{c^2\left[\varphi^2\beta-\varphi^2\left(\frac{\varpi}{2}i+\alpha\right)\right]}{f^2\left(\frac{\varpi}{2}i+\alpha\right)},$$

now in virtue of (18.) one has:

$$f^2\left(\frac{\varpi}{2}i+\alpha\right) = \frac{e^2+c^2}{e^2}.\frac{1}{f^2\alpha};$$

therefore:

$$\frac{1-\frac{f^2\beta}{f^2\alpha}}{1-\frac{f^2\beta}{f^2\left(\frac{\varpi}{2}i+\alpha\right)}}=\frac{1}{f^4\alpha}.\frac{e^2+c^2}{e^2}.\frac{1-\frac{\varphi^2\beta}{\varphi^2\alpha}}{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\varpi}{2}i+\alpha\right)}}.$$

One similarly finds:

$$\frac{1-\frac{F^2\beta}{F^2\alpha}}{1-\frac{F^2\beta}{F^2\left(\frac{\omega}{2}+\alpha\right)}}=\frac{1}{F^4\alpha}.\frac{e^2+c^2}{e^2}.\frac{1-\frac{\varphi^2\beta}{\varphi^2\alpha}}{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\omega}{2}+\alpha\right)}}.$$

In virtue of these formulas, and setting $\beta=0$ to determine the constant factor, it is clear that one can write the expressions of $f(2n+1)\beta$, $F(2n+1)\beta$, as follows:

$$130'.\quad\left\{\begin{aligned}
&f(2n+1)\beta=f\beta.\prod_{m=1}^{n}\frac{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\omega}{2}+\frac{m\omega}{2n+1}\right)}}{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega}{2n+1}\right)}}.\prod_{\mu=1}^{n}\frac{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\omega}{2}i+\frac{\mu\varpi i}{2n+1}\right)}}{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{\mu\varpi i}{2n+1}\right)}}\\
&\times\prod_{m=1}^{n}\prod_{\mu=1}^{n}\frac{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\omega}{2}+\frac{m\omega+\mu\varpi i}{2n+1}\right)}}{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega+\mu\varpi i}{2n+1}\right)}}.\frac{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\omega}{2}+\frac{m\omega-\mu\varpi i}{2n+1}\right)}}{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega-\mu\varpi i}{2n+1}\right)}}\\
&F(2n+1)\beta=F\beta.\prod_{m=1}^{n}\frac{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\varpi}{2}i+\frac{m\omega}{2n+1}\right)}}{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega}{2n+1}\right)}}.\prod_{\mu=1}^{n}\frac{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\varpi}{2}i+\frac{\mu\varpi i}{2n+1}\right)}}{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{\mu\varpi i}{2n+1}\right)}}\\
&\times\prod_{m=1}^{n}\prod_{\mu=1}^{n}\frac{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\varpi}{2}i+\frac{m\omega+\mu\varpi i}{2n+1}\right)}}{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega+\mu\varpi i}{2n+1}\right)}}.\frac{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\varpi}{2}i+\frac{m\omega-\mu\varpi i}{2n+1}\right)}}{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega-\mu\varpi i}{2n+1}\right)}}.
\end{aligned}\right.$$

In this section we have considered the functions $\varphi(n\beta)$, $f(n\beta)$, $F(n\beta)$ in the case of odd values of $n$. One could find analogous expressions for the functions for even values of $n$; but since there is no difficulty in that and since the formulas which we arrived at are those that will be most useful to us later. I will not occupy myself with this [i.e., the even case].

## §. VII.
## Development of the functions $\varphi\alpha$, $f\alpha$, $F\alpha$ in series and in infinite products.

### 24.

In the formulas in the preceding sections, by setting $\beta=\frac{\alpha}{2n+1}$ one obtains expressions of the functions $\varphi\alpha$, $f\alpha$, $F\alpha$, that because of the indeterminate number $n$, can be varied in infinitely many ways. Among all the formulas that one thus obtains that result from in the assumption that $n$ is infinite are the most remarkable. In that case the functions $\varphi$, $f$, $F$ disappear from the values of $\varphi\alpha$, $f\alpha$, $F\alpha$, and one obtains for these functions algebraic expressions, but composed of infinitely many terms. To get these expressions, put $\beta = \frac{\alpha}{2n+1}$ into formulas (126.), (130.), and then find the limit of the right hand side of these equations for ever-increasing values of $n$. For brevity, let $\nu$ be a quantity for which the limit is zero for ever-increasing values of $n$. That said, consider successively the three formulas in (126.).

Making $\beta = \frac{\alpha}{2n+1}$ in the first of the formulas (126.), and noticing that

$$131. \quad \sum_{m=-n}^{+n} \sum_{\mu=-n}^{+n} \theta(m,\mu) = \theta(0,0) + \sum_{m=1}^{n} [\theta(m,0) + \theta(-m,0)] + \sum_{\mu=1}^{n} [\theta(0,\mu) + \theta(0,-\mu)] + \sum_{m=1}^{n} \sum_{\mu=1}^{n} [\theta(m,\mu) + \theta(-m,-\mu) + \theta(m,-\mu) + \theta(-m,\mu)],$$

it is clear that one can put the formula under consideration into the form:

$$132. \quad \varphi\alpha = \frac{1}{2n+1}.\varphi\left(\frac{\alpha}{2n+1}\right) + \frac{1}{2n+1} \sum_{m=1}^{n} (-1)^m \left\{\varphi\left(\frac{\alpha + m\omega}{2n+1}\right) + \varphi\left(\frac{\alpha - m\omega}{2n+1}\right)\right\} + \frac{1}{2n+1} \sum_{\mu=1}^{n} (-1)^\mu \left\{\varphi\left(\frac{\alpha + \mu\varpi i}{2n+1}\right) + \varphi\left(\frac{\alpha - \mu\varpi i}{2n+1}\right)\right\} - \frac{i}{ec}. \sum_{m=1}^{n} \sum_{\mu=1}^{n} (-1)^{m+\mu}.\psi(n-m, n-\mu) + \frac{i}{ec} \sum_{m=1}^{n} \sum_{\mu=1}^{n} (-1)^{m+\mu}.\psi_1(n-m, n-\mu),$$

where we have for brevity,

$$133. \quad \begin{cases} \psi(m,\mu) = \dfrac{1}{2n+1}.\left\{\dfrac{1}{\varphi\left(\frac{\alpha+(m+\frac{1}{2})\omega+(\mu+\frac{1}{2})\varpi i}{2n+1}\right)} + \dfrac{1}{\varphi\left(\frac{\alpha-(m+\frac{1}{2})\omega-(\mu+\frac{1}{2})\varpi i}{2n+1}\right)}\right\}, \\ \psi_1(m,\mu) = \dfrac{1}{2n+1}.\left\{\dfrac{1}{\varphi\left(\frac{\alpha+(m+\frac{1}{2})\omega-(\mu+\frac{1}{2})\varpi i}{2n+1}\right)} + \dfrac{1}{\varphi\left(\frac{\alpha-(m+\frac{1}{2})\omega+(\mu+\frac{1}{2})\varpi i}{2n+1}\right)}\right\}. \end{cases}$$

Now notice that

$$\varphi\left(\frac{\alpha+m\omega}{2n+1}\right)+\varphi\left(\frac{\alpha-m\omega}{2n+1}\right)=\frac{2\varphi\left(\frac{\alpha}{2n+1}\right).f\left(\frac{m\omega}{2n+1}\right).F\left(\frac{m\omega}{2n+1}\right)}{1+e^2c^2.\varphi^2\left(\frac{m\omega}{2n+1}\right).\varphi^2\left(\frac{\alpha}{2n+1}\right)}=\frac{A_m}{2n+1},$$

$$\varphi\left(\frac{\alpha+\mu\varpi i}{2n+1}\right)+\varphi\left(\frac{\alpha-\mu\varpi i}{2n+1}\right)=\frac{2\varphi\left(\frac{\alpha}{2n+1}\right).f\left(\frac{\mu\varpi i}{2n+1}\right).F\left(\frac{\mu\varpi i}{2n+1}\right)}{1+e^2c^2.\varphi^2\left(\frac{\mu\varpi i}{2n+1}\right).\varphi^2\left(\frac{\alpha}{2n+1}\right)}=\frac{B_\mu}{2n+1}$$

where $A_m$ and $B_\mu$ are finite quantities. The part of equation (132.) up to the $-$ sign, will take the form:

$$\frac{1}{2n+1}.\varphi\left(\frac{\alpha}{2n+1}\right)+\frac{1}{(2n+1)^2}.\sum_{m=1}^{n}(-1)^m(A_m+B_\mu);$$

now the limit of this quantity is evidently zero; therefore, on taking the limit of formula (132.), we have:

$$\begin{aligned}\varphi\alpha=&-\frac{i}{ec}\lim.\sum_{m=1}^{n}\sum_{\mu=1}^{n}(-1)^{m+\mu}.\psi(n-m,n-\mu)\\&+\frac{i}{ec}\lim.\sum_{m=1}^{n}\sum_{\mu=1}^{n}(-1)^{m+\mu}\psi_1(n-m,n-\mu),\end{aligned}$$

or:

$$\begin{aligned}134.\qquad \varphi\alpha=&-\frac{i}{ec}\lim.\sum_{m=0}^{n-1}\sum_{\mu=0}^{n-1}(-1)^{m+\mu}.\psi(m,\mu)\\&+\frac{i}{ec}\lim.\sum_{m=0}^{n-1}\sum_{\mu=0}^{n-1}(-1)^{m+\mu}.\psi_1(m,\mu).\end{aligned}$$

It suffices to consider one of these limits, because we will find the other by changing the sign of $i$.

Let us find the limit of

$$\sum_{m=0}^{n-1}\sum_{\mu=0}^{n-1}(-1)^{m+\mu}.\psi(m,\mu).$$

For that, it is necessary to try to put the preceding quantity into the form

$$P + \nu,$$

where $P$ is independent of $n$, and $\nu$ is a quantity that has zero as its limit; because then the quantity $P$ is precisely the limit under consideration.

25.

Let us initially consider the expression:

$$\sum_{\mu=0}^{n-1} (-1)^\mu . \psi(m, \mu).$$

Given

$$135. \quad \theta(m, \mu) = \frac{2\alpha}{\alpha^2 - (m\omega + \mu\varpi i)^2},$$

and setting

$$136. \quad \psi(m, \mu) - \theta(m, \mu) = \frac{2\alpha}{(2n+1)^2} . R_\mu,$$

one will have:

$$137. \quad \sum_{\mu=0}^{n-1} (-1)^\mu . \psi(m, \mu) - \sum_{\mu=0}^{n-1} (-1)^\mu . \theta(m, \mu) = 2\alpha . \sum_{\mu=0}^{n-1} (-1)^n . \frac{R_\mu}{(2n+1)^2}.$$

Then I say that the right hand side of this equation is a quantity of the form $\frac{\nu}{2n+1}$. According to (12.), (13.), we have

$$\frac{1}{\varphi(\beta+\epsilon)} + \frac{1}{\varphi(\beta-\epsilon)} = \frac{\varphi(\beta+\epsilon) + \varphi(\beta-\epsilon)}{\varphi(\beta+\epsilon).\varphi(\beta-\epsilon)} = \frac{2\varphi\beta.f\epsilon.F\epsilon}{\varphi^2\beta - \varphi^2\epsilon},$$

therefore setting $\beta = \frac{\alpha}{2n+1}$ and $\epsilon = \frac{m\omega + \mu\varpi i}{2n+1} = \frac{\epsilon_\mu}{2n+1}$ and $f\epsilon.F\epsilon = \theta\epsilon$, one has:

$$\psi(m, \mu) = \frac{1}{2n+1} . \frac{2\varphi\left(\frac{\alpha}{2n+1}\right).\theta\left(\frac{\epsilon_\mu}{2n+1}\right)}{\varphi^2\left(\frac{\alpha}{2n+1}\right) - \varphi^2\left(\frac{\epsilon_\mu}{2n+1}\right)}.$$

Now we have:

$$\varphi\left(\frac{\alpha}{2n+1}\right) = \frac{\alpha}{2n+1} + \frac{A\alpha^3}{(2n+1)^3},$$

so:

$$\psi(m,\mu) = \frac{\theta\left(\frac{\epsilon_\mu}{2n+1}\right)}{\varphi^2\left(\frac{\alpha}{2n+1}\right) - \varphi^2\left(\frac{\epsilon_\mu}{2n+1}\right)}.\left\{\frac{2A\alpha^3}{(2n+1)^4} + \frac{2\alpha}{(2n+1)^2}\right\},$$

and consequently:

$$\begin{aligned}\psi(m,\mu) - \theta(m,\mu) &= \frac{2\alpha}{(2n+1)^2}\left\{\frac{\theta\left(\frac{\epsilon_\mu}{2n+1}\right)}{\varphi^2\left(\frac{\alpha}{2n+1}\right) - \varphi^2\left(\frac{\epsilon_\mu}{2n+1}\right)} - \frac{1}{\left(\frac{\alpha}{2n+1}\right)^2 - \left(\frac{\epsilon_\mu}{2n+1}\right)^2}\right\}\\ &+ \frac{2A\alpha^3}{(2n+1)^4}.\frac{\theta\left(\frac{\epsilon_\mu}{2n+1}\right)}{\varphi^2\left(\frac{\alpha}{2n+1}\right) - \varphi^2\left(\frac{\epsilon_\mu}{2n+1}\right)}.\end{aligned}$$

Thus the value of $R_\mu$ becomes:

138. $$R_\mu = \frac{\theta\left(\frac{\epsilon_\mu}{2n+1}\right)}{\varphi^2\left(\frac{\alpha}{2n+1}\right) - \varphi^2\left(\frac{\epsilon_\mu}{2n+1}\right)}.\left\{1 + \frac{A\alpha^2}{(2n+1)^2}\right\} - \frac{1}{\left(\frac{\alpha}{2n+1}\right)^2 - \left(\frac{\epsilon_\mu}{2n+1}\right)^2}.$$

That posed, there are two cases to consider, to know whether $\frac{\epsilon_\mu}{2n+1}$ has limit zero or not.

a) If $\frac{\epsilon_\mu}{2n+1}$ has limit zero, we have:

$$\begin{gathered}\varphi^2\left(\frac{\epsilon_\mu}{2n+1}\right) = \frac{\epsilon_\mu^2}{(2n+1)^2} + \frac{B_\mu.\epsilon_\mu^4}{(2n+1)^4},\\ \theta\left(\frac{\epsilon_\mu}{2n+1}\right) = \sqrt{\left[1 - c^2\varphi^2\left(\frac{\epsilon_\mu}{2n+1}\right)\right]}.\sqrt{\left[1 + e^2\varphi^2\left(\frac{\epsilon_\mu}{2n+1}\right)\right]} = 1 + \frac{C_\mu\epsilon_\mu^2}{(2n+1)^2},\\ \varphi^2\left(\frac{\alpha}{2n+1}\right) = \frac{\alpha^2}{(2n+1)^2} + \frac{D.\alpha^4}{(2n+1)^4},\end{gathered}$$

where $B_\mu$, $C_\mu$, and $D$ have finite limits; thus in substituting:

139. $$\begin{aligned}R_\mu &= A\alpha^2.\frac{\frac{1}{\epsilon_\mu^2} + \frac{C_\mu}{(2n+1)^2}}{\frac{\alpha^2}{\epsilon_\mu^2} - 1 + \frac{D\alpha^4}{(2n+1)^2.\epsilon_\mu^2} - B_\mu\frac{\epsilon_\mu^2}{(2n+1)^2}}\\ &+ \frac{C_\mu.\frac{\alpha^2}{\epsilon_\mu^2} - \frac{D\alpha^4}{\epsilon_\mu^4} - C_\mu + B_\mu}{\left(1 - \frac{\alpha^2}{\epsilon_\mu^2}\right)^2 - \left(1 - \frac{\alpha^2}{\epsilon_\mu^2}\right)\left\{\frac{D\alpha^4}{(2n+1)^2.\epsilon_\mu^2} - B_\mu.\frac{\epsilon_\mu^2}{(2n+1)^2}\right\}},\end{aligned}$$

now whether $\epsilon_\mu$ is finite or infinite, it is clear that the quantity always converges towards a finite quantity for indefinitely increasing values of $n$. Therefore we have

$$140. \quad R_\mu = r_\mu + \nu_\mu,$$

where $r_\mu$ is a finite quantity independent of $n$.

b) If $\frac{\epsilon_\mu}{2n+1}$ has finite quantity for a limit, it is clear that in naming this limit $\delta_\mu$, one will have:

$$141. \quad R_\mu = -\frac{\theta(\delta_\mu)}{\varphi^2(\delta_\mu)} + \frac{1}{\delta_\mu^2} + \nu_\mu^1.$$

That posed, consider the expression $\sum_{\mu=1}^{n}(-1)^\mu . \frac{R_\mu}{(2n+1)^2}$. We have

$$142. \quad \sum_{\mu=1}^{n}(-1)^\mu . \frac{R_\mu}{(2n+1)^2} = \frac{1}{(2n+1)^2} . \{R_0 - R_1 + R_2 - R_3 + \cdots$$
$$\cdots + (-1)^{\nu-1} . R_{\nu-1} + (-1)^{\nu+1}[R_{\nu+1} - R_{\nu+1} + R_{\nu+2} - R_{\nu+3} + \cdots + (-1)^{n-\nu-1} . R_{n-1}]\}.$$

First suppose that $\frac{\epsilon_\mu}{2n+1}$ has a finite quantity for a limit, whatever the value of $\mu$. Then notice that

$$\delta_{\mu+1} = \delta_\mu,$$

we have

$$R_\mu - R_{\mu+1} = \nu_\mu^1 - \nu_{\mu+1}^1,$$

so:

$$\sum_{\mu=0}^{n-1}(-1)^\mu . \frac{R_\mu}{(2n+1)^2} = \frac{1}{(2n+1)^2}\left(\nu_0^1 - \nu_1^1 + \nu_2^1 - \nu_3^1 + \cdots + \nu_{k-2}^1 - \nu_{k-1}^1\right) + \frac{B}{(2n+1)^2},$$

where $k = n$ or $n-1$ according to whether $n$ is even or odd.

The quantity $B$ always has a finite quantity for a limit, namely $B = 0$ if $n$ is even and $B = R_{n-1}$ if $n$ is odd.

Now we know that a sum such as

$$\nu_0^1 - \nu_1^1 + \nu_2^1 - \cdots + \nu_{k-2}^1 - \nu k - 1^1,$$

can be put into form $k.\nu$, $\nu$ having limit zero. So substituting:

$$\sum_{\mu=0}^{n-1}(-1)^{\mu}.\frac{R_\mu}{(2n+1)^2} = \frac{k.\nu+B}{(2n+1)^2};$$

now, $k$ is equal to $n$ or to $n-1$, and $B$ is finite, the limit of $\frac{k.\nu+B}{2n+1}$ will be zero. Thus:

$$143. \qquad \sum_{\mu=0}^{n-1}(-1)^{\mu}.\frac{R_\mu}{(2n+1)^2} = \frac{\nu}{2n+1}.$$

Now suppose that $\frac{m}{2n+1}$ has limit zero. The $\frac{\epsilon_\mu}{2n+1}$ is also zero at the limit, unless at the same time $\frac{\mu}{2n+1}$ does not have a finite quantity for a limit. Given in this case $\nu$, the integer immediately smaller than $\sqrt{n}$, consider the sum

$$R_0 - R_1 + R_2 - R_3 + \cdots + (-1)^{\nu-1}R_{\nu-1}.$$

Supposing that $\mu$ is one of the numbers $0, 1, \ldots, \nu$, it is clear that $\frac{\epsilon_\mu}{(2n+1)} = \frac{m\omega+\mu\varpi i}{2n+1}$ has zero for a limit; hence, according to what we have seen, $R_\mu$ will be a finite quantity and as a consequence

$$R_0 - R_1 + R_2 - \cdots + (-1)^{\nu-1}R_{\nu-1} = \nu.R,$$

where $R$ is also a finite quantity.

Now consider the sum

$$(-1)^{\nu}\{R_\nu - R_{\nu+1} + R_{\nu+2} - \cdots + (-1)^{n-\nu-1}.R_{n-1}\}.$$

If $\frac{\epsilon_\mu}{2n+1}$ has for itslimit a quantity different than zero, we have as we will see:

$$R_\mu - R_{\mu+1} = \nu^1_\mu - \nu^1_{\mu+1};$$

if on the contrary $\frac{\epsilon_\mu}{2n+1}$ has limit zero, we have:

$$R_\mu = \nu_\mu + \nu^1_\mu;$$

now, if at the same time $\mu > \sqrt{n}$, it is clear, that in virtue of the value of $R_\mu$

$$\nu_\mu = B_\mu - C_\mu;$$

now it is clear that $B_\mu$ and $C_\mu$ both have for limits quantities independent of $\mu$. So naming the limits $B$ and $C$, we have:

$$R_\mu = B - C + \nu_\mu,$$

and consequently, also in this case,

$$R_\mu - R_{\mu+1} = \nu_\mu - \nu_{\mu+1}.$$

Therefore, as in the case where $\frac{\epsilon_\mu}{2n+1}$ has a limit different that zero for all the values of $\mu$, we show that

$$\frac{(-1)^\nu}{(2n+1)^2}\{R_\nu - R_{\nu-1} + \cdots + (-1)^{n-\nu-1}R_{n-1}\} = \frac{\nu}{(2n+1)}.$$

Now by combining the equations above, one will conclude

$$\sum_{\mu=0}^{n-1}(-1)^\mu.\frac{R_\mu}{(2n+1)^2} = \frac{1}{(2n+1)^2}.(\nu R) + \frac{\nu}{2n+1};$$

now $\frac{\nu}{2n+1}$ has zero for a limit, so

$$\sum_{\mu=0}^{n-1}(-1)^\mu.\frac{R_\mu}{(2n+1)^2} = \frac{\nu}{2n+1}.$$

Thus this formula always holds, and as a consequence of formula (137.) one deduces:

$$144. \quad \sum_{\mu=0}^{n-1}(-1)^\mu\psi(m,\mu) - \sum_{\mu=0}^{n-1}(-1)^\mu.\theta(m,\mu) = \frac{\nu}{2n+1}.$$

That posed, it remains to put $\sum_{\mu=0}^{n-1}(-1)^\mu.\theta(m,\mu)$ into the form $P + \frac{\nu}{2n+1}$.
Now this can be done as follows. We have:

$$145. \quad \begin{cases} \displaystyle\sum_{\mu=0}^{n-1}(-1)^\mu\theta(m,\mu) = \sum_{\mu=0}^{\frac{1}{0}}\theta(m,\mu) - \sum_{\mu=n}^{\frac{1}{0}}(-1)^\mu\theta(m,\mu) \text{ and} \\ \displaystyle\sum_{\mu=n}^{\frac{1}{0}}(-1)^\mu.\theta(m,\mu) = (-1)^n\{\theta(m,n) - \theta(m,n+1) + \theta(m,n+2) - \cdots \text{etc.}\}. \end{cases}$$

Now according to a known formula we have:

$$\begin{aligned} & \theta(m,n) - \theta(m,n+1) + \theta(m,n+2) - \cdots \\ = \; & \frac{1}{2}\theta(m,n) + A\frac{\partial\theta(m,n)}{\partial n} + B.\frac{\partial^3\theta(m,n)}{\partial^3 n} + \cdots, \end{aligned}$$

where $A, B, \ldots$ are numbers; now

$$\theta(m,n) = \frac{2\alpha}{\alpha^2 - \left[\left(m+\frac{1}{2}\right)\omega + \left(n+\frac{1}{2}\right)\varpi i\right]^2},$$

so substituting:

$$\theta(m,n) - \theta(m,n+1) + \cdots = \frac{\alpha}{\alpha^2 - \left[\left(m+\frac{1}{2}\right)\omega + \left(n+\frac{1}{2}\right)\varpi i\right]^2} + \frac{4A\alpha\varpi i.n}{\left[\alpha^2 - \left(\left(m+\frac{1}{2}\right)\omega + \left(n+\frac{1}{2}\right)\varpi i\right)^2\right]^2} + \cdots$$

From there it follows that

$$\theta(m,n) - \theta(m,n+1) + \cdots = \frac{\alpha}{\varpi^2.n^2} + \frac{\nu}{n^2} = \frac{\nu}{2n+1}.$$

Therefore by virtue of equations (145.)

$$\sum_{\mu=0}^{n-1}(-1)^\mu\theta(m,\mu) = \sum_{\mu=0}^{\frac{1}{0}}(-1)^\mu \cdot \theta(m,\mu) + \frac{\nu}{2n+1}.$$

and as a consequence

$$146. \quad \sum_{\mu=0}^{n-1}(-1)^\mu\psi(m,\mu) = \sum_{\mu=0}^{\infty}(-1)^\mu \cdot \theta(m,\mu) + \frac{\nu}{2n+1}.$$

26.

Having transformed of this way the quantity $\sum_{\mu=0}^{n-1}(-1)^\mu, \psi(m,\mu)$, one deduces equation (146.):

$$147. \quad \sum_{m=0}^{n-1}\sum_{\mu=0}^{n-1}(-1)^{m+\mu}\psi(m,\mu) = \sum_{m=0}^{n-1}(-1)^m.\varrho_m + \sum_{m=0}^{n-1}\frac{\nu_m}{2n+1},$$

setting

$$148. \quad \varrho_m = \sum_{\mu=0}^{\infty}(-1)^\mu.\theta(m,\mu);$$

now

$$\sum_{m=0}^{n-1}.\frac{\nu_m}{2n+1} = \frac{\nu_0+\nu_1+\nu_2+\cdots+\nu_{n-1}}{2n+1} = \frac{n\nu}{2n+1} = \frac{\nu}{2},$$

$\nu$ having limit zero. Hence equation (147.) gives, by making $n$ infinite:

$$149. \quad \lim . \sum_{m=0}^{n-1} \sum_{\mu=0}^{n-1} (-1)^{m+\mu}.\psi(m,\mu) = \sum_{m=0}^{\infty} (-1)^m \varrho_m.$$

Similarly, if for brevity we make:

$$150. \quad \begin{cases} \theta_1(m,\mu) = \dfrac{2\alpha}{\alpha^2 - (m\omega - \mu\varpi i)^2}, \\ \varrho'_m = \displaystyle\sum_{\mu}^{\infty} (-1)^\mu.\theta_1(m,\mu), \end{cases}$$

one has:

$$151. \quad \lim . \sum_{m=0}^{n-1} \sum_{\mu=0}^{n-1} (-1)^{m+\mu}.\psi_1(m,\mu) = \sum_{m=0}^{\infty} (-1)^m \varrho'_m.$$

Having found these two quantities that make up the expression of $\varphi\alpha$, we will have upon substituting:

$$\varphi\alpha = -\frac{i}{ec}.\sum_{m=0}^{\infty} (-1)^m.\varrho_m + \frac{i}{ec}.\sum_{m=0}^{\infty} (-1)^m.\varrho'_m = \frac{i}{ec}.\sum_{m=0}^{\infty} (-1)^m \{\varrho'_m - \varrho_m\},$$

or, by giving the values of $\varrho'_m$ and $\varrho_m$,

$$152. \quad \varphi\alpha = \frac{i}{ec} \sum_{m=0}^{\infty} (-1)^m \left\{ \sum_{\mu=0}^{\infty} \left\{ \frac{2\alpha}{\alpha^2 - \left[\left(m+\frac{1}{2}\right)\omega - \left(\mu+\frac{1}{2}\right)\varpi i\right]^2} - \frac{2\alpha}{\alpha^2 - \left[\left(m+\frac{1}{2}\right)\omega + \left(\mu+\frac{1}{2}\right)\varpi i\right]^2} \right\} \right\}.$$

Now

$$\frac{2\alpha}{\alpha^2 - \left[\left(m+\frac{1}{2}\right)\omega \pm \left(\mu+\frac{1}{2}\right)\varpi i\right]^2} = \frac{1}{\alpha - \left(m+\frac{1}{2}\right)\omega \mp \left(\mu+\frac{1}{2}\right)\varpi i} + \frac{1}{\alpha + \left(m+\frac{1}{2}\right)\omega \pm \left(\mu+\frac{1}{2}\right)\varpi i},$$

so:

$$\begin{aligned} &\frac{2\alpha}{\alpha^2 - \left[\left(m+\frac{1}{2}\right)\omega - \left(\mu+\frac{1}{2}\right)\varpi i\right]^2} - \frac{2\alpha}{\alpha^2 - \left[\left(m+\frac{1}{2}\right)\omega + \left(\mu+\frac{1}{2}\right)\varpi i\right]^2} \\ = \quad &+\frac{(2\mu+1)\varpi i}{\left[\alpha + \left(m+\frac{1}{2}\right)\omega\right]^2 + \left(\mu+\frac{1}{2}\right)^2\varpi^2} - \frac{(2\mu+1)\varpi i}{\left[\alpha - \left(m+\frac{1}{2}\right)\omega\right]^2 + \left(\mu+\frac{1}{2}\right)^2\varpi^2}, \end{aligned}$$

thus the expression of $\varphi\alpha$ becomes a real form:

$$153. \quad \varphi\alpha = \frac{1}{ec}.\sum_{m=0}^{\infty}(-1)^m.\sum_{\mu=0}^{\infty}(-1)^\mu.\left\{\frac{(2\mu+1)\varpi}{\left[\alpha-\left(m+\frac{1}{2}\right)\omega\right]^2+\left(\mu+\frac{1}{2}\right)^2\varpi^2}-\frac{(2\mu+1)\varpi}{\left[\alpha+\left(m+\frac{1}{2}\right)\omega\right]^2+\left(\mu+\frac{1}{2}\right)^2\varpi^2}\right\}$$

that is, we have:

$$\begin{aligned} 154. \quad \varphi\alpha &= \frac{\varpi}{ec}\left(\delta_0-\delta_1+\delta_2-\delta_3+\cdots+(-1)^m\delta_m-\cdots\right) \\ &- \frac{\varpi}{ec}\left(\delta'_0-\delta'_1+\delta'_2-\delta'_3+\cdots+(-1)^m\delta'_m-\cdots\right), \end{aligned}$$

or:

$$155. \quad \begin{cases} \delta_m = \frac{1}{\left[\alpha-\left(m+\frac{1}{2}\right)\omega\right]^2+\frac{\varpi^2}{4}} - \frac{3}{\left[\alpha-\left(m+\frac{1}{2}\right)\right]\omega\right]^2+\frac{9\varpi^2}{4}} + \frac{5}{\left[\alpha-\left(m+\frac{1}{2}\right)\omega\right]^2+\frac{25\varpi^2}{4}} - \cdots \\ \delta'_\mu = \frac{1}{\left[\alpha+\left(m+\frac{1}{2}\right)\omega\right]^2+\frac{\varpi^2}{4}} - \frac{3}{\left[\alpha+\left(m+\frac{1}{2}\right)\omega\right]^2+\frac{9\varpi^2}{4}} + \frac{5}{\left[\alpha+\left(m+\frac{1}{2}\right)\omega\right]^2+\frac{25\varpi^2}{4}} - \cdots \end{cases}$$

If we start the study of the limit of the function $\sum_{m=0}^{n-1}\sum_{\mu=0}^{n-1}(-1)^{m+\mu}.\psi(m,\mu)$ by that of $\sum_{m=0}^{n-1}(-1)^m\psi(m,\mu)$ instead of that of $\sum_{\mu=0}^{n-1}(-1)^\mu\psi(m,\mu)$, as we have done, one finds instead of the formula (153.) the following:

$$156. \quad \varphi\alpha = \frac{1}{ec}\sum_{\mu=0}^{\infty}.(-1)^\mu\sum_{m=0}^{\infty}(-1)^m.\left\{\frac{(2\mu+1)\varpi}{\left[\alpha-\left(m+\frac{1}{2}\right)\omega\right]^2+\left(\mu+\frac{1}{2}\right)^2\varpi^2}-\frac{(2\mu+1)\varpi}{\left[\alpha+\left(m+\frac{1}{2}\right)\omega\right]^2+\left(\mu+\frac{1}{2}\right)^2\varpi^2}\right\},$$

that is:

$$\begin{aligned} 157. \quad \varphi\alpha &= \frac{\varpi}{ec}\left(\epsilon_0-3\epsilon_1+5\epsilon_2-7\epsilon_3+\cdots+(-1)^\mu.(2\mu+1)\epsilon_\mu-\cdots\right) \\ &- \frac{\varpi}{ec}\left(\epsilon'_0-3\epsilon'_1+5\epsilon'_2-7\epsilon'_3+\cdots+(-1)^\mu.(2\mu+1)\epsilon'_\mu-\cdots\right), \end{aligned}$$

where

$$158. \quad \begin{cases} \epsilon_\mu = \frac{1}{\left(\alpha-\frac{\omega}{2}\right)^2+\left(\mu+\frac{1}{2}\right)^2\varpi^2} - \frac{1}{\left(\alpha-\frac{3\omega}{2}\right)^2+\left(\mu+\frac{1}{2}\right)^2\varpi^2} + \frac{1}{\left(\alpha-\frac{5\omega}{2}\right)^2+\left(\mu+\frac{1}{2}\right)^2\varpi^2} - \cdots \\ \epsilon'_\mu = \frac{1}{\left(\alpha+\frac{\omega}{2}\right)^2+\left(\mu+\frac{1}{2}\right)^2\varpi^2} - \frac{1}{\left(\alpha+\frac{3\omega}{2}\right)^2+\left(\mu+\frac{1}{2}\right)^2\varpi^2} + \frac{1}{\left(\alpha+\frac{5\omega}{2}\right)^2+\left(\mu+\frac{1}{2}\right)^2\varpi^2} - \cdots \end{cases}$$

Now to find the expression of $f\alpha$ with means of the second formula in (126.). Setting $\beta = \frac{\alpha}{2n+1}$ there and noticing that then the limit of the quantities contained in the first two lines becomes equal to zero, we have:

$$\begin{aligned} 159. \quad f\alpha &= \lim.(-1)^n \sum_{m=1}^{n}\sum_{\mu=1}^{n}(-1)^m.\psi(n-m, n-\mu) \\ &+ \lim.(-1)^n \sum_{m=1}^{n}\sum_{\mu=1}^{n}(-1)^m.\psi_1(n-m, n-\mu), \end{aligned}$$

where for brevity we define:

$$\begin{aligned} \psi(n-m, n-\mu) &= \frac{1}{2n+1}\left\{f\left(\frac{\alpha+m\omega+\mu\varpi i}{2n+1}\right)+f\left(\frac{\alpha-m\omega-\mu\varpi i}{2n+1}\right)\right\}, \\ \psi_1(n-m, n-\mu) &= \frac{1}{2n+1}\left\{f\left(\frac{\alpha+m\omega-\mu\varpi i}{2n+1}\right)+f\left(\frac{\alpha-m\omega+\mu\varpi i}{2n+1}\right)\right\}. \end{aligned}$$

Now we have:

$$f(\beta-\epsilon)+f(\beta-\epsilon) = \frac{2f\beta.f\epsilon}{1+e^2c^2\varphi^2\epsilon.\varphi^2\beta} = \frac{f\epsilon}{e^2c^2\varphi^2\epsilon}.\frac{f\beta}{\varphi^2\beta+\frac{1}{e^2c^2\varphi^2\epsilon}}.$$

Given

$$\epsilon = \frac{m\omega+\mu\varpi i}{2n+1},$$

we will have:

$$\begin{aligned} \frac{1}{\varphi\epsilon} &= -iec.\varphi\left(\frac{\omega}{2}+\frac{\varpi}{2}-\epsilon\right) = -iec.\varphi\left(\frac{\left(n-m+\frac{1}{2}\right)\omega+\left(n-\mu+\frac{1}{2}\right)\varpi i}{2n+1}\right), \\ \frac{f\epsilon}{\varphi\epsilon} &= -\frac{i\sqrt{(e^2+c^2)}}{F\left(\epsilon-\frac{\varpi}{2}i\right)} = -i\sqrt{(e^2+c^2)}.\frac{c}{\sqrt{(e^2+c^2)}}.F\left(\epsilon-\frac{\omega}{2}-\frac{\varpi}{2}i\right) \\ &= -ci.F.\left(\frac{\left(n-m+\frac{1}{2}\right)\omega+\left(n-\mu+\frac{1}{2}\right)\varpi i}{2n+1}\right). \end{aligned}$$

Therefore we have, by substituting and putting $m$ and $\mu$ respectively instead of $n-m$ and $n-\mu$:

$$\psi(m,\mu) = \frac{1}{e}.\frac{2\varphi\left(\frac{\left(m+\frac{1}{2}\right)\omega+\left(\mu+\frac{1}{2}\right)\varpi i}{2n+1}\right).F\left(\frac{\left(m+\frac{1}{2}\right)\omega+\left(\mu+\frac{1}{2}\right)\varpi i}{2n+1}\right)}{(2n+1)\left[\varphi^2\left(\frac{\alpha}{2n+1}\right)-\varphi^2\left(\frac{\left(m+\frac{1}{2}\right)\omega+\left(\mu+\frac{1}{2}\right)\varpi i}{2n+1}\right)\right]}$$

We get the value of $\psi_1(m,\mu)$ just by changing the sign of $i$.
Now setting

$$\theta(m,\mu) = \frac{(2m+1)\omega+(2\mu+1)\varpi i}{\alpha^2-\left[\left(m+\frac{1}{2}\right)\omega+\left(\mu+\frac{1}{2}\right)\varpi i\right]^2}$$

and

$$\theta_1(m,\mu) = \frac{(2m+1)\omega-(2\mu+1)\varpi i}{\alpha^2-\left[\left(m+\frac{1}{2}\right)\omega-\left(\mu+\frac{1}{2}\right)\varpi i\right]^2},$$

and then finding the limit of the function

$$\sum_{m=0}^{n-1}\sum_{\mu=0}^{n-1}(-1)^m.\psi(m,\mu),$$

in the same manner as before, we find:

$$\lim.\sum_{m=0}^{n-1}\sum_{\mu=0}^{n-1}(-1)^m.\psi(m,\mu) = \frac{1}{e}.\sum_{\mu=0}^{\infty}\left\{\sum_{m=0}^{\infty}(-1)^m.\theta(m,\mu)\right\}$$

and

$$\lim.\sum_{m=0}^{n-1}\sum_{\mu=0}^{n-1}(-1)^m.\psi_1(m,\mu) = \frac{1}{e}.\sum_{\mu=0}^{\infty}\left\{\sum_{m=0}^{\infty}(-1)^m.\theta_1(m,\mu)\right\};$$

so by substituting in (159.), and giving the values of $\theta(m,\mu)$ and $\theta_1(m,\mu)$, we have:

160. $\quad f\alpha =$

$$\frac{1}{e}.\sum_{\mu=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m\left\{\frac{(2m+1)\omega+(2\mu+1)\varpi i}{\alpha^2-\left[\left(m+\frac{1}{2}\right)\omega+\left(\mu+\frac{1}{2}\right)\varpi i\right]^2}+\frac{(2m+1)\omega-(2\mu+1)\varpi i}{\alpha^2-\left[\left(m+\frac{1}{2}\right)\omega-\left(\mu+\frac{1}{2}\right)\varpi i\right]^2}\right\}.$$

The quantity contained by the brace-brackets can also be put into the form:

$$\frac{2\left[\alpha-\left(m+\frac{1}{2}\right)\omega\right]}{\left[\alpha-\left(m+\frac{1}{2}\right)\omega\right]^2+\left(\mu+\frac{1}{2}\right)^2\varpi^2}-\frac{2\left[\alpha+\left(m+\frac{1}{2}\right)\omega\right]}{\left[\alpha+\left(m+\frac{1}{2}\right)\omega\right]^2+\left(\mu+\frac{1}{2}\right)\varpi^2},$$

so also

$$161.\quad f\alpha =$$
$$\frac{1}{e}.\sum_{\mu=0}^{\infty}\left\{\sum_{m=0}^{\infty}(-1)^m.\frac{2\left[\alpha-\left(m+\frac{1}{2}\right)\omega\right]}{\left[\alpha-\left(m+\frac{1}{2}\right)\omega\right]^2+\left(\mu+\frac{1}{2}\right)\omega^2}-\sum_{m=0}^{\infty}(-1)^m\frac{2\left[\alpha+\left(m+\frac{1}{2}\right)\omega\right]}{\left[\alpha+\left(m+\frac{1}{2}\right)\omega\right]^2+\left(\mu+\frac{1}{2}\right)^2\varpi^2}\right\}.$$

In the same manner one will have:

$$162.\quad F(\alpha) =$$
$$\frac{1}{e}.\sum_{m=0}^{\infty}\left\{\sum_{\mu=0}^{\infty}(-1)^\mu\frac{(2\mu+1)\varpi}{\left[\alpha+\left(m+\frac{1}{2}\right)\omega\right]^2+\left(\mu+\frac{1}{2}\right)\varpi^2}+\sum_{\mu=0}^{\infty}.\frac{(2\mu+1)\varpi}{\left[\alpha+\left(m+\frac{1}{2}\right)\omega\right]^2+\left(\mu+\frac{1}{2}\right)\varpi^2}\right\}.$$

28.

Let us come now to formulas (130.). To find the value of the second, after having set $\beta = \frac{\alpha}{2n+1}$, and supposing $n$ is infinite, we go initially to find the limit of the following expression:

$$163.\quad t = \prod_{m=1}^{n}\prod_{\mu=1}^{n}\left\{\frac{1-\frac{\varphi^2\left(\frac{\alpha}{2n+1}\right)}{\varphi^2\left(\frac{m\omega+\mu\varpi i+k}{2n+1}\right)}}{1-\frac{\varphi^2\left(\frac{\alpha}{2n+1}\right)}{\varphi^2\left(\frac{m\omega+\mu\varpi i+l}{2n+1}\right)}}\right\},$$

where $k$ and $l$ are two quantities independent of $n, m, \mu$.
Taking the logarithm and putting for short:

$$164.\quad \psi(m,\mu) = \log\left\{\frac{1-\frac{\varphi^2\left(\frac{\alpha}{2n+1}\right)}{\varphi^2\left(\frac{m\omega+\mu\varpi i+k}{2n+1}\right)}}{1-\frac{\varphi^2\left(\frac{\alpha}{2n+1}\right)}{\varphi^2\left(\frac{m\omega+\mu\varpi i+l}{2n+1}\right)}}\right\},$$

we have:

$$165.\quad \log t = \sum_{m=1}^{n}\sum_{\mu=1}^{n}\psi(m,\mu).$$

Initially consider the expression: $\sum_{\mu=1}^{n}\psi(m,\mu)$. Given

$$166.\quad \theta(m,\mu) = \log\left\{\frac{1-\frac{\alpha^2}{(m\omega+\mu\varpi i+k)^2}}{1-\frac{\alpha^2}{(m\omega+\mu\varpi i+l)^2}}\right\},$$

we have:

$$\psi(m,\mu)-\theta(m,\mu)=\log\left\{\frac{1-\frac{\varphi^2\left(\frac{\alpha}{2n+1}\right)}{\varphi^2\left(\frac{m\omega+\mu\varpi i+k}{2n+1}\right)}}{1-\frac{\alpha^2}{(m\omega+\mu\varpi i+k)^2}}.\frac{1-\frac{\alpha^2}{(m\omega+\mu\varpi i+l)^2}}{1-\frac{\varphi^2\left(\frac{\alpha}{2n+1}\right)}{\varphi^2\left(\frac{m\omega+\mu\varpi i+l}{2n+1}\right)}}\right\}.$$

That posed, I say the right hand side of that equation is, for all values of $m$ and $\mu$, of the form:

$$\psi(m,\mu)-\theta(m,\mu)=\frac{\nu}{(2n+1)^2}.$$

To show that, it is necessary to distinguish two cases, if the limit of $\frac{m\omega+\mu\varpi i}{2n+1}$ is a quantity different from zero or if it is equal to zero.

a) In the first case under consideration, naming the limit $a$, one will have:

$$\varphi^2\left(\frac{m\omega+\mu\varpi i+k}{2n+1}\right)=\varphi^2 a+\nu,$$
$$\varphi^2\left(\frac{m\omega+\mu\varpi i+l}{2n+1}\right)=\varphi^2 a+\nu',$$
$$\varphi^2\left(\frac{\alpha}{2n+1}\right)=\frac{\alpha^2}{(2n+1)^2}+\frac{\nu'}{(2n+1)^2},$$

so:

$$1-\frac{\varphi^2\left(\frac{\alpha}{2n+1}\right)}{\varphi^2\left(\frac{m\omega+\mu\varpi i+k}{2n+1}\right)}=1-\frac{\alpha^2}{(2n+1)^2\varphi^2 a}+\frac{\nu}{(2n+1)^2},$$
$$1-\frac{\varphi^2\left(\frac{\alpha}{2n+1}\right)}{\varphi^2\left(\frac{m\omega+\mu\varpi i+l}{2n+1}\right)}=1-\frac{\alpha^2}{(2n+1)^2.\varphi^2 a}+\frac{\nu'}{(2n+1)^2}.$$

Similarly

$$1-\frac{\alpha^2}{(m\omega+\mu\varpi i+k)^2}=1-\frac{\alpha^2}{(2n+1)^2\left\{\frac{m\omega+\mu\varpi i+k}{(2n+1)}\right\}}=1-\frac{\alpha^2}{(2n+1)^2a^2}+\frac{\nu}{(2n+1)^2},$$
$$1-\frac{\alpha^2}{(m\omega+\mu\varpi i+l)^2}=1-\frac{\alpha^2}{(2n+1)^2a^2}+\frac{\nu'}{(2n+1)^2}.$$

By substituting these values, the expression of $\psi(m,\mu)-\theta(m,\mu)$ takes the form:

$$\psi(m,\mu)-\theta(m,\mu)=\log\left\{\frac{1-\frac{\nu}{(2n+1)^2}}{1-\frac{\nu'}{(2n+1)^2}}.\frac{1-\frac{\nu_1}{(2n+1)^2}}{1-\frac{\nu_1'}{(2n+1)^2}}\right\},$$

the quantities $\nu, \nu', \nu_1, \nu_1'$ all having limit zero.
We thus have:

$$\log\left(1-\frac{\nu}{(2n+1)^2}\right)=\frac{\nu}{(2n+1)^2} \text{ etc.},$$

and consequently:

$$\psi(m,\mu)-\theta(m,\mu)=\frac{\nu}{(2n+1)^2}.$$

b) If the limit of the quantity $\frac{m\omega+\mu\varpi i}{2n+1}$ is equal to zero, one will have:

$$\begin{aligned}\varphi^2\left(\frac{m\omega+\mu\varpi i+k}{2n+1}\right) &= \frac{(m\omega+\mu\varpi i+k)^2}{(2n+1)^2}+A.\frac{(m\omega+\mu\varpi i+k)^4}{(2n+1)^4}+\cdots,\\ \varphi^2\left(\frac{\alpha}{2n+1}\right) &= \frac{\alpha^2}{(2n+1)^2}+A'.\frac{\alpha^4}{(2n+1)^4}+\cdots,\end{aligned}$$

so:

$$1-\frac{\varphi^2\left(\frac{\alpha}{2n+1}\right)}{\varphi^2\left(\frac{m\omega+\mu\varpi i+k}{2n+1}\right)}=1-\frac{\alpha^2+A'.\frac{\alpha^4}{(2n+1)^4}+\cdots}{(m\omega+\mu\varpi i+k)^2+A.\frac{(m\omega+\mu\varpi i+k)^4}{(2n+1)^4}}.$$

Now if $m\omega+\mu\varpi i$ is not indefinitely increasing with $n$, we have:

$$1-\frac{\varphi^2\left(\frac{\alpha}{2n+1}\right)}{\varphi^2\left(\frac{m\omega+[\mu]\varpi i+k}{2n+1}\right)}=1-\frac{\alpha^2}{(m\omega+\mu\varpi i+k)^2}+\frac{B}{(2n+1)^2},$$

similarly:

$$1-\frac{\varphi^2\left(\frac{\alpha}{2n+1}\right)}{\varphi^2\left(\frac{m\omega+\mu\varpi i+l}{2n+1}\right)}=1-\frac{\alpha^2}{(m\omega+\mu\varpi i+l)^2}+\frac{C}{(2n+1)^2},$$

so in this case:

$$\psi(m,\mu)-\theta(m,\mu)=\log\left\{\frac{1-\frac{B'}{(2n+1)^2}}{1-\frac{C'}{(2n+1)^2}}\right\},$$

$B'$ and $C'$ having finite limits, or:

$$\psi(m,\mu) - \theta(m,\mu) = \frac{D}{(2n+1)^2},$$

the limit of $D$ also being a finite quantity.

If on the contrary the quantity $m\omega + \mu\varpi i$ increases indefinitely with $n$, one has:

$$\frac{1 - \frac{\varphi^2\left(\frac{\alpha}{2n+1}\right)}{\varphi^2\left(\frac{m\omega+\mu\varpi i+k}{2n+1}\right)}}{1 - \frac{\alpha^2}{(m\omega+\mu\varpi i+k)^2}} = 1 + \frac{\alpha^2}{(2n+1)^2}.\left\{\frac{A - A'\frac{\alpha^2}{(m\omega+\mu\varpi i+k)^2}}{1 + A\left(\frac{m\omega+\mu\varpi i+k}{2n+1}\right)}\right\} \times \frac{1}{1 - \frac{\alpha^2}{(m\omega+\mu\varpi i+k)^2}};$$

where the quantities $\frac{1}{m\omega+\mu\varpi i+k}$, $\frac{m\omega+\mu\varpi i+k}{2n+1}$ have limit zero; thus the preceding quantity is of the form:

$$1 + \frac{\alpha^2}{(2n+1)^2}.A'',$$

$A''$ having a finite quantity for a limit. By changing $k$ to $l$ and designating the corresponding value of $A''$ by $A''_1$, the value of $\psi(m,\mu) - \theta(m,\mu)$ becomes:

$$\psi(m,\mu) - \theta(m,\mu) = \log\left\{\frac{1 + \frac{\alpha^2}{(2n+1)^2}A''}{1 + \frac{\alpha^2}{(2n+1)^2}A''_1}\right\} = \frac{\alpha^2\left(A'' - A''_1\right)}{(2n+1)^2} + \frac{\nu}{(2n+1)^2}.$$

Now the limit of $A''$ is the same as that of $A$; now it is clear that this last limit is independent of $k, m, \mu$ (indeed, it is equal to the coefficient of $\alpha^4$ in the development of $\varphi^2\alpha$). Therefore we have:

$$A'' = M + \nu,$$

and by changing $k$ to $l$:

$$A''_1 = M + \nu',$$

from where $A'' - A''_1 = \nu - \nu' = \nu$. Hence $A'' - A''_1$ has limit zero, and as a consequence we have:

$$\psi(m,\mu) - \theta(m,\mu) = \frac{\nu}{(2n+1)^2}.$$

Hence we have demonstrated, that while taking

$$167. \quad \psi(m,\mu) - \theta(m,\mu) = \frac{A_{m,\mu}}{(2n+1)^2},$$

the limit of $A_{m,\mu}$ will be always be equal to zero as $m\omega + \mu\varpi i$ increases indefinitely with $n$, and that it will be equal to a finite quantity in the contrary case.

29.

That posed, consider the quantity

$$\sum_{\mu=1}^{n} \psi(m,\mu).$$

By substituting the value of $\psi(m,\mu)$, it becomes:

$$168. \quad \sum_{\mu=1}^{n} \psi(m,\mu) = \sum_{\mu=1}^{n} \theta(m,\mu) + \frac{1}{(2n+1)^2}.\sum_{\mu=1}^{n}(A_{m,\mu}).$$

Given the greatest integer $\nu$ less than $\sqrt{n}$, we can make:

$$\begin{aligned} \sum_{\mu=1}^{n} A_{m,\mu} &= A_{m,1} + A_{m,2} + \cdots + A_{m,\nu} \\ &+ A_{m,\nu+1} + A_{m,\nu+2} + \cdots + A_{m,n}. \end{aligned}$$

Now according to the nature of the quantities $A_{m,\mu}$, the sum contained in the first line will be equal to $\nu.A_m$, and the second equal to $A'_m(n-\nu-1)$, where $A_m$ is a finite quantity and $A'_m$ a quantity that has limit zero. Therefore:

$$\sum_{\mu=n}^{n} A_{m,\mu} = \nu.A_m + (n-\nu)A'_m = (2n+1).B_m,$$

where

$$B_m = \frac{\nu}{2n+1}A_m + \frac{n-\nu-1}{2n+1}A'_m.$$

Thus the quantity $B_m$ has limit zero, noticing that $\nu$ is not larger than $\sqrt{n}$.

From there, the expression of $\sum_{\mu=1}^{n} \psi(m,\mu)$ changes into:

$$169. \quad \sum_{\mu=1}^{n} \psi(m,\mu) = \sum_{\mu=1}^{n} \theta(m,\mu) + \frac{B_m}{2n+1}.$$

To get the limit of $\sum_{\mu=1}^{n} \theta(m,\mu)$, I write

$$\sum_{\mu=1}^{n} \theta(m,\mu) = \sum_{\mu=1}^{\infty} \theta(m,\mu) - \sum_{\mu=n+1}^{\infty} \theta(m,\mu) = \sum_{\mu=1}^{\infty} \theta(m,\mu) - \sum_{\mu=1}^{\infty} \theta(m,\mu+n).$$

Now one can find the value of $\sum_{\mu=1}^{\infty} \theta(m,\mu+n)$ as follows.
One has:

$$\begin{aligned}
\theta(m,\mu+n) &= \log\left\{\frac{1-\frac{\alpha^2}{[m\omega+(\mu+n)\varpi i+k]^2}}{1-\frac{\alpha^2}{[m\omega+(\mu+n)\varpi i+l]^2}}\right\} \\
&= \alpha^2\left\{\frac{1}{[m\omega+(\mu+n)\varpi i+l]^2}-\frac{1}{m\omega+(\mu+n)\varpi i+k]^2}\right\} \\
&- \frac{1}{2}\alpha^2\left\{\frac{1}{[m\omega+(\mu+n)\varpi i+l]^2}-\frac{1}{[m\omega+(\mu+n)\varpi i+k]^2}\right\} \\
&+ \text{ etc.}
\end{aligned}$$

From which one deduces:

$$\sum_{\mu=1}^{\mu} \theta(m,\mu+n) = \frac{\alpha^2}{n}.\sum \frac{1}{n}.\theta\left(\frac{\mu}{n}\right) - \frac{\alpha^4}{2n^3}.\sum \frac{1}{n}.\theta_1\left(\frac{\mu}{n}\right) + \cdots,$$

where

$$\begin{aligned}
\theta\left(\frac{\mu}{n}\right) &= \frac{1}{\left(\frac{m\omega+l}{n}+\varpi i+\frac{\mu}{n}\varpi i\right)^2} - \frac{1}{\left(\frac{m\omega+k}{n}+\varpi i+\frac{\mu}{n}\varpi i\right)^2}, \\
\theta_1\left(\frac{\mu}{n}\right) &= \frac{1}{\left(\frac{m\omega+l}{n}+\varpi i+\frac{\mu}{n}\varpi i\right)^4} - \frac{1}{\left(\frac{m\omega+k}{n}+\varpi i+\frac{\mu}{n}\varpi i\right)^4}, \\
&\text{etc.}
\end{aligned}$$

However it is known that the limit of $\sum_{\mu=1}^{\mu} \frac{1}{n}\theta\left(\frac{\mu}{n}\right)$is equal to $\int_0^x \theta(x)\partial x$, thus

$$\sum_{\mu=1}^{\mu} \frac{1}{n}\theta\left(\frac{\mu}{n}\right) = \int_0^x \theta(x)\partial x + \nu$$

$$\sum_{\mu=1}^{\mu} \frac{1}{n}\theta_1\left(\frac{\mu}{n}\right) = \int_0^x \theta_1(x)\partial x + \nu_1,$$

and as a consequence on substituting

$$\sum_{\mu=1}^{\mu}\theta(m,\mu+n)=\frac{\alpha^2}{n}\int_0^x\theta(x)\partial x+\frac{\alpha^4}{2n^3}\int_0^x\theta_1(x)\partial x+\cdots+\frac{\nu\alpha^2}{n}+\frac{\nu_1\alpha^4}{2n^3}+\cdots;$$

now

$$\theta(x) \;=\; \frac{1}{\left(\frac{m\omega+l}{n}+\varpi i+x\varpi i\right)^2}-\frac{1}{\left(\frac{m\omega+k}{n}+\varpi i+x\varpi i\right)^2}$$
$$\text{etc.}$$

thus we have

$$\begin{aligned}\int_0^x\theta(x)\partial x &= \frac{1}{\varpi i}.\left\{\frac{1}{\left(\frac{m\omega+k}{n}+\varpi i+x\varpi i\right)}-\frac{1}{\left(\frac{m\omega+l}{n}+\varpi i+x\varpi i\right)}\right\}\\ &- \frac{1}{\varpi i}.\left\{\frac{1}{\left(\frac{m\omega+k}{n}+\varpi i\right)}-\frac{1}{\left(\frac{m\omega+l}{n}+\varpi i\right)}\right\}\\ &= \frac{1}{\varpi i}.\frac{k-l}{n}.\frac{1}{\left(\frac{m\omega+l}{n}+\varpi i+x\varpi i\right)\left(\frac{m\omega+k}{n}+\varpi i+x\varpi i\right)}\\ &- \frac{1}{\varpi i}.\frac{k-l}{n}\frac{1}{\left(\frac{m\omega+l}{n}+\varpi i\right)\left(\frac{m\omega+k}{n}+\varpi i\right)}.\end{aligned}$$

The limit of this expression for $\int_0^x\theta(x)\partial x$ is zero for an arbitrary value of $x$. Similarly one finds that the limit of $\sum_0^x\theta_1\partial x$ is zero. Hence:

$$\begin{aligned}\sum_{\mu=1}^{\mu}\theta(m,\mu+n) &= \frac{\alpha^2}{n}\nu-\frac{\alpha^4}{2n^3}.\nu'+\frac{\alpha^6}{3n^5}.\nu''-\cdots\\ &= \frac{\alpha^2}{2n+1}.\left\{\nu-\frac{\alpha^2}{2n^2}\nu'+\frac{a^4}{3n^4}\nu''-\cdots\right\}\frac{2n+1}{n}=\frac{\nu}{2n+1},\end{aligned}$$

so by also making $\mu=\infty$:

$$\sum_{\mu=1}^{\infty}\theta(m,\mu+n)=\frac{\nu}{2n+1},$$

from which:

$$\sum_{\mu=1}^{n}\psi(m,\mu+n)=\sum_{\mu=1}^{\infty}\theta(m,\mu)-\frac{\nu}{2n+1},\text{ and}$$

$$170. \qquad \sum_{\mu=1}^{n}\psi(m,\mu)=\sum_{\mu=1}^{\infty}\theta(m,\mu)+\frac{\nu_m}{2n+1},$$

$\nu_m$ having limit zero. From there we deduce that

$$\sum_{m=1}^{n}\sum_{\mu=1}^{n}\psi(m,\mu)=\sum_{m=1}^{n}\left\{\sum_{\mu=1}^{\infty}\theta(m,\mu)\right\}+\sum_{m=1}^{n}\frac{\nu_m}{2n+1}.$$

By taking the limit of the both sides and noticing that

$$\sum_{m=1}^{n}\frac{\nu}{2n+1}=\frac{\nu_1+\nu_2+\cdots+\nu_m}{2n+1}=\nu,$$

we have:

$$171. \qquad \lim.\sum_{m=1}^{n}\sum_{\mu=1}^{n}\psi(m,\mu)=\sum_{m=1}^{\infty}\left\{\sum_{\mu=1}^{\infty}\theta(m,\mu)\right\}.$$

By giving the values of $\psi(m,\mu)$ and $\theta(m,\mu)$, and passing from the logarithms to the numbers, we conclude:

$$172. \qquad \lim.\prod_{m=1}^{n}\prod_{\mu=1}^{n}\left\{\frac{1-\frac{\varphi^2\left(\frac{\alpha}{2n+1}\right)}{\varphi^2\left(\frac{m\omega+\mu\varpi i+k}{2n+1}\right)}}{1-\frac{\varphi^2\left(\frac{\alpha}{2n+1}\right)}{\varphi^2\left(\frac{m\omega+\mu\varpi i+l}{2n+1}\right)}}\right\}=\prod_{m=1}^{\infty}\left\{\prod_{\mu=1}^{\infty}\left(\frac{1-\frac{\alpha^2}{(m\omega+\mu\varpi i+k)^2}}{1-\frac{\alpha^2}{(m\omega+\mu\varpi i+l)^2}}\right)\right\}.$$

By an analysis entirely similar to the preceding, but simpler, we likewise find:

$$173. \qquad \lim.\prod_{m=1}^{n}\left\{\frac{1-\frac{\varphi^2\left(\frac{\alpha}{2n+1}\right)}{\varphi^2\left(\frac{m\omega+k}{2n+1}\right)}}{1-\frac{\varphi^2\left(\frac{\alpha}{2n+1}\right)}{\varphi^2\left(\frac{m\omega+l}{2n+1}\right)}}\right\}=\prod_{m=1}^{\infty}\left\{\frac{1-\frac{\alpha^2}{(m\omega+k)^2}}{1-\frac{\alpha^2}{(m\omega+l)^2}}\right\},$$

$$174. \qquad \lim.\prod_{\mu=1}^{n}\left\{\frac{1-\frac{\varphi^2\left(\frac{\alpha}{2n+1}\right)}{\varphi^2\left(\frac{\mu\varpi i+k}{2n+1}\right)}}{1-\frac{\varphi^2\left(\frac{\alpha}{2n+1}\right)}{\varphi^2\left(\frac{\mu\varpi i+k}{2n+1}\right)}}\right\}=\prod_{\mu=1}^{\infty}\left\{\frac{1-\frac{\alpha^2}{(\mu\varpi i+k)^2}}{1-\frac{\alpha^2}{(\mu\varpi i+l)^2}}\right\}.$$

Now nothing is easier than to find the values of $\varphi\alpha$, $f\alpha$, $F\alpha$.
To begin, consider the first formula (130.). We have:

$$\begin{aligned} e^2c^2\varphi^2\left(\frac{m\omega+\mu\varpi i}{2n+1}\right) &= -\frac{1}{\varphi^2\left(\frac{m\omega+\mu\varpi i}{2n+1}-\frac{\omega}{2}-\frac{\varpi}{2}i\right)} \\ &= -\frac{1}{\varphi^2\left(\frac{\left(n-m+\frac{1}{2}\right)\omega+\left(n-\mu+\frac{1}{2}\right)\varpi i}{2n+1}\right)}, \end{aligned}$$

thus:

$$\begin{aligned} \prod_{m=1}^{n}\prod_{1}^{\mu}\left\{\frac{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{m\omega+\mu\varpi i}{2n+1}\right)}}{1+e^2c^2.\varphi^2\left(\frac{m\omega+\mu\varpi i}{2n+1}\right)\varphi^2\beta}\right\} &= \frac{\prod_{m=1}^{n}\prod_{1}^{\mu}\left\{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{m\omega+\mu\varpi i}{2n+1}\right)}\right\}}{\prod_{m=1}^{n}\prod_{1}^{\mu}.\left\{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\left(n-m+\frac{1}{2}\right)\omega+\left(n-\mu+\frac{1}{2}\right)\varpi i}{2n+1}\right)}\right\}} \\ &= \prod_{m=1}^{n}\prod_{\mu=1}^{\mu}\left\{\frac{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{m\omega+\mu\varpi i}{2n+1}\right)}}{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{m\omega+\mu\varpi i+\frac{\omega}{2}+\frac{\varpi}{2}i}{2n+1}\right)}}\right\} \end{aligned}$$

That established, if we set $\beta=\frac{\alpha}{2n+1}$ and suppose that $n$ is infinite, we get, using formulas (172.), (173.), (174.), and noting that $\alpha$ is the limit of $(2n+1)\varphi\left(\frac{\alpha}{2n+1}\right)$ and equal to $\alpha$[sic.]:

$$175. \quad \varphi\alpha=\alpha\prod_{m=1}^{\infty}\left\{1-\frac{\alpha^2}{(m\omega)^2}\right\}.\prod_{\mu=1}^{\infty}\left\{1+\frac{\alpha^2}{(\mu\varpi)^2}\right\}$$
$$\times\prod_{m=1}^{\infty}\left\{\prod_{\mu=1}^{\infty}\left(\frac{1-\frac{\alpha^2}{(m\omega+\mu\varpi i)^2}}{1-\frac{\alpha^2}{\left[\left(m-\frac{1}{2}\right)\omega+\left(\mu-\frac{1}{2}\right)\varpi i\right]^2}}\right).\prod_{\mu=1}^{\infty}\left(\frac{1-\frac{\alpha^2}{(m\omega-\mu\varpi i)^2}}{1-\frac{\alpha^2}{\left[\left(m-\frac{1}{2}\right)\omega-\left(\mu-\frac{1}{2}\right)\varpi i\right]^2}}\right)\right\}.$$

The two formulas (130) will give in the same manner, by setting $\beta=\frac{\alpha}{2n+1}$ and noticing that $f(0)=1$, $F(0)=1$:

$$176. \quad f\alpha = \prod_{m=1}^{\infty}\left\{\prod_{\mu=1}^{\infty}\left(\frac{1-\frac{\alpha^2}{\left[\left(m-\frac{1}{2}\right)\omega+\mu\varpi i\right]^2}}{1-\frac{\alpha^2}{\left[\left(m-\frac{1}{2}\right)\omega+\left(\mu-\frac{1}{2}\right)\varpi i\right]^2}}\right)\left(\frac{1-\frac{\alpha^2}{\left[\left(m-\frac{1}{2}\right)\omega-\mu\varpi i\right]^2}}{1-\frac{\alpha^2}{\left[\left(m-\frac{1}{2}\right)\omega-\left(\mu-\frac{1}{2}\right)\varpi i\right]^2}}\right)\right\}$$
$$\times\prod_{\mu=0}^{\infty}\left\{1-\frac{\alpha^2}{\left(m+\frac{1}{2}\right)\omega^2}\right\},$$

$$177. \quad F\alpha =$$
$$\prod_{\mu=0}^{\infty}\left\{1+\frac{\alpha^2}{\left(\mu+\frac{1}{2}\right)\varpi^2}\right\}.\prod_{m=1}^{\infty}\left\{\prod_{\mu=1}^{\infty}\left(\frac{1-\frac{\alpha^2}{\left[m\omega+\left(\mu-\frac{1}{2}\right)\varpi i\right]^2}}{1-\frac{\alpha^2}{\left[\left(m-\frac{1}{2}\right)\omega+\left(\mu-\frac{1}{2}\right)\varpi i\right]^2}}\right)\left(\frac{1-\frac{\alpha^2}{\left[m\omega+\left(\mu-\frac{1}{2}\right)\varpi i\right]^2}}{1-\frac{\alpha^2}{\left[\left(m-\frac{1}{2}\right)\omega+\left(\mu-\frac{1}{2}\right)\varpi i\right]^2}}\right)\right\}.$$

One can also give a real form to the preceding expressions as follows:

$$178. \quad \varphi\alpha = \alpha.\prod_{\mu=1}^{\infty}\left(1+\frac{\alpha^2}{\mu^2\varpi^2}\right).\prod_{m=1}^{\infty}\left(1-\frac{\alpha^2}{m^2\omega^2}\right)$$
$$\times\prod_{m=1}^{\infty}\prod_{\mu=1}^{\infty}\frac{1+\frac{(\alpha+m\omega)^2}{\mu^2\varpi^2}}{1+\frac{\left[\alpha+\left(m-\frac{1}{2}\right)\omega\right]^2}{\mu^2\varpi^2}}.\frac{1+\frac{(\alpha-m\omega)^2}{\mu^2\varpi^2}}{1+\frac{\left[\alpha-\left(m-\frac{1}{2}\right)\omega\right]^2}{\mu^2\varpi^2}}.\left(\frac{1+\frac{\left(m-\frac{1}{2}\right)^2\omega^2}{\mu^2\varpi^2}}{1+\frac{m^2\omega^2}{\mu^2\varpi^2}}\right)^2,$$

$$179. \quad f\alpha = \prod_{m=1}^{\infty}\left(1-\frac{\alpha^2}{\left(m-\frac{1}{2}\right)^2\omega^2}\right)$$
$$\times\prod_{m=1}^{\infty}\prod_{\mu=1}^{\infty}\frac{1+\frac{\left[\alpha+\left(m-\frac{1}{2}\right)\right]^2}{\mu^2\varpi^2}}{1+\frac{\left[\alpha+\left(m-\frac{1}{2}\right)\omega\right]^2}{\left(\mu-\frac{1}{2}\right)^2\varpi^2}}.\frac{1+\frac{\left[\alpha-\left(m-\frac{1}{2}\right)\omega\right]^2}{\mu^2\varpi^2}}{1+\frac{\left[\alpha-\left(m-\frac{1}{2}\right)\omega\right]^2}{\left(\mu-\frac{1}{2}\right)^2\varpi^2}}.\left(\frac{1+\frac{\left(m-\frac{1}{2}\right)^2\omega^2}{\left(\mu-\frac{1}{2}\right)^2\varpi^2}}{1+\frac{\left(m-\frac{1}{2}\right)^2\omega^2}{\mu^2\varpi^2}}\right)^2,$$

$$180. \quad F\alpha = \prod_{\mu=1}^{\infty}\left(1+\frac{\alpha^2}{\left(\mu+\frac{1}{2}\right)^2\varpi^2}\right)$$
$$\times\prod_{m=1}^{\infty}\prod_{\mu=1}^{\infty}\frac{1+\frac{(\alpha+m\omega)^2}{\left(\mu-\frac{1}{2}\right)^2\varpi^2}}{1+\frac{\left[\alpha+\left(m-\frac{1}{2}\right)\omega\right]^2}{\left(\mu-\frac{1}{2}\right)^2\varpi^2}}.\frac{1+\frac{(\alpha-m\omega)^2}{\left(\mu-\frac{1}{2}\right)^2\varpi^2}}{1+\frac{\left[\alpha-\left(m-\frac{1}{2}\right)\omega\right]^2}{\left(\mu-\frac{1}{2}\right)^2\varpi^2}}.\left(\frac{1+\frac{\left(m-\frac{1}{2}\right)^2\omega^2}{\left(\mu-\frac{1}{2}\right)^2\varpi^2}}{1+\frac{m^2\omega^2}{\left(\mu-\frac{1}{2}\right)^2\varpi^2}}\right)^2.$$

These transformations easily take place by means of the formula:

$$\left(1-\frac{\alpha^2}{(a+bi)^2}\right)\left(1-\frac{\alpha^2}{(a-bi)^2}\right)=\left(1+\frac{\alpha}{a+bi}\right)\left(1+\frac{\alpha}{a-bi}\right)\left(1-\frac{\alpha}{a+bi}\right)\left(1-\frac{\alpha}{a-bi}\right)$$
$$=\frac{(\alpha+a)^2+b^2}{a^2+b^2}.\frac{(\alpha-a)^2+b^2}{a^2+b^2}=\left(1+\frac{(\alpha+a)^2}{b^2}\right)\left(1+\frac{(\alpha-a)^2}{b^2}\right).\frac{1}{\left(1+\frac{a^2}{b^2}\right)^2}.$$

## 31.

In what preceded we arrived at two types of expressions for the functions $\varphi\alpha$, $f\alpha$, $F\alpha$: those that give the decomposition of the functions as partial fractions, all in the form of double infinite series, the others gave the same functions decomposed into a number of infinite factors, each one in its turn composed of infinitely many factors.

Now we can greatly simplify the proceeding formulas by means of exponential and circular functions. This is what we will see by what follows:

First consider equations (178.), (179.), (180.). In virtue of known formulas, we have:

$$\frac{\sin y}{y}=\prod_{\mu=1}^{\infty}\left(1-\frac{y^2}{\mu^2\pi^2}\right);\quad \cos y=\prod_{\mu=1}^{\infty}\left(1-\frac{y^2}{\left(\mu-\frac{1}{2}\right)^2\pi^2}\right);$$

so:

$$\prod_{\mu=1}^{\infty}\frac{\left(1-\frac{z^2}{\mu^2\pi^2}\right)}{\left(1-\frac{y^2}{\left(\mu-\frac{1}{2}\right)\pi^2}\right)}=\frac{\sin z}{z\cos y};\quad \prod_{\mu=1}^{\infty}\left(\frac{1-\frac{z^2}{\left(\mu-\frac{1}{2}\right)\pi^2}}{1-\frac{y^2}{\left(\mu-\frac{1}{2}\right)\pi^2}}\right)=\frac{\cos z}{\cos y}.$$

In virtue of these formulas it is clear that the expressions of $\varphi\alpha$, $f\alpha$, $F\alpha$ can be put into the form:

$$\varphi\alpha=\frac{\varpi}{\pi}.\frac{\sin\left(\alpha\frac{\pi}{\varpi}i\right)}{i}.\prod_{m=1}^{\infty}\left(1-\frac{\alpha^2}{m^2\omega^2}\right)$$
$$\times\prod_{m=1}^{\infty}\left(\frac{\sin(\alpha+m\omega)\frac{\pi}{\varpi}i.\sin(\alpha-m\omega)\frac{\pi}{\varpi}i.\cos^2\left(m-\frac{1}{2}\right)\omega\frac{\pi}{\varpi}i}{\cos\left[\alpha+\left(m-\frac{1}{2}\right)\omega\right]\frac{\pi}{\varpi}i.\cos\left[\alpha-\left(m-\frac{1}{2}\right)\omega\right]\frac{\pi}{\varpi}i.\sin^2 m\omega\frac{\pi}{\varpi}i}.\frac{\left(m\omega\frac{\pi}{\varpi}i\right)^2}{(\alpha+m\omega)(\alpha-m\omega).\frac{\pi^2}{\omega^2}i^2}\right),$$
$$f\alpha=\prod_{m=1}^{\infty}\left(1-\frac{\alpha^2}{\left(m-\frac{1}{2}\right)^2\omega^2}\right)$$
$$\times\prod_{m=1}^{\infty}\left(\tan\left[\alpha+\left(m-\frac{1}{2}\right)\omega\right]\frac{\pi}{\varpi}i.\tan\left[\alpha-\left(m-\frac{1}{2}\right)\omega\right]\frac{\pi}{\varpi}i.\cot^2\left(m-\frac{1}{2}\right)\omega\frac{\pi}{\varpi}i\frac{\left(m-\frac{1}{2}\right)^2\omega^2}{\alpha^2-\left(m-\frac{1}{2}\right)^2\omega^2}\right),$$

$$F\alpha = \cos\left(\alpha\frac{\pi}{\varpi}i\right).\prod_{m=1}^{\infty}\left(\frac{\cos(\alpha+m\omega)\frac{\pi}{\varpi}i.\cos(\alpha-m\omega)\frac{\pi}{\varpi}i.\cos^2\left(m-\frac{1}{2}\right)\omega\frac{\pi}{\varpi}i}{\cos\left[\alpha+\left(m-\frac{1}{2}\right)\omega\right]\frac{\pi}{\varpi}i.\cos\left[\alpha-\left(m-\frac{1}{2}\right)\omega\right]\frac{\pi}{\varpi}i.\cos^2 m\omega.\frac{\pi}{\varpi}i}\right).$$

One will find the real expressions by substituting exponential functions for the circular functions in the expressions.

We have:

$$\sin(a-b).\sin(a+b) = \sin^2 a - \sin^2 b,$$
$$\cos(a+b).\cos(a-b) = \cos^2 a - \sin^2 b,$$

so:

$$\frac{\sin(\alpha+m\omega)\frac{\pi}{\varpi}i.\sin(\alpha-m\omega)\frac{\pi}{\varpi}i}{\sin^2 m\omega\frac{\pi}{\varpi}i} = -\left\{1-\frac{\sin^2\alpha\frac{\pi}{\varpi}i}{\sin^2 m\omega\frac{\pi}{\varpi}i}\right\},$$
$$\frac{\cos\left[\alpha+\left(m-\frac{1}{2}\right)\omega\right]\frac{\pi}{\varpi}i.\cos\left[\alpha-\left(m-\frac{1}{2}\right)\omega\right]\frac{\pi}{\varpi}i}{\cos^2\left(m-\frac{1}{2}\right)\omega\frac{\pi}{\varpi}i} = 1-\frac{\sin^2\alpha\frac{\pi}{\varpi}i}{\cos^2\left(m-\frac{1}{2}\right)\omega\frac{\pi}{\varpi}i},$$
$$\tan\left[\alpha+\left(m-\frac{1}{2}\right)\omega\right]\frac{\pi}{\varpi}i.\tan\left[\alpha-\left(m-\frac{1}{2}\right)\omega\right]\frac{\pi}{\varpi}i.\cot^2\left(m-\frac{1}{2}\right)\omega\frac{\pi}{\varpi}i$$
$$= -\frac{1-\frac{\sin^2\alpha\frac{\pi}{\varpi}i}{\sin^2\left(m-\frac{1}{2}\right)\omega\frac{\pi}{\varpi}i}}{1-\frac{\sin^2\alpha\frac{\pi}{\varpi}i}{\cos^2\left(m-\frac{1}{2}\right)\omega\frac{\pi}{\varpi}i}}$$

From this and noticing that

$$\frac{m^2\omega^2}{\alpha^2-m^2\omega^2} = -\frac{1}{1-\frac{\alpha^2}{m^2\omega^2}} \text{ and}$$
$$\frac{\left(m-\frac{1}{2}\right)^2\omega^2}{\alpha^2-\left(m-\frac{1}{2}\right)^2\omega^2} = -\frac{1}{1-\frac{\alpha^2}{\left(m-\frac{1}{2}\right)\omega^2}},$$

it is clear that we have:

$$181. \quad \varphi\alpha = \frac{\pi}{\varpi}.\frac{\sin\left(\frac{\alpha}{\varpi}\pi i\right)}{i}.\prod_{m=1}^{\infty}.\frac{1-\frac{\sin^2\alpha\frac{\pi}{\varpi}i}{\sin^2 m\omega\frac{\pi}{\varpi}i}}{1-\frac{\sin^2\alpha\frac{\pi}{\varpi}i}{\cos^2\left(m-\frac{1}{2}\right)\omega\frac{\pi}{\varpi}i}},$$

$$182.\quad f\alpha = \prod_{m=0}^{\infty} \cdot \frac{1-\frac{\sin^2\alpha\frac{\pi}{\varpi}i}{\sin^2\left(m+\frac{1}{2}\right)\omega\frac{\pi}{\varpi}i}}{1-\frac{\sin^2\alpha\frac{\pi}{\varpi}i}{\cos^2\left(m+\frac{1}{2}\right)\omega\frac{\pi}{\varpi}i}},$$

$$183.\quad F\alpha = \cos\left(\frac{\alpha}{\varpi}\pi i\right)\cdot\prod_{m=1}^{\infty}\cdot\frac{1-\frac{\sin^2\frac{\alpha}{\varpi}\pi i}{\cos^2 m\frac{\omega}{\varpi}\pi i}}{1-\frac{\sin^2\frac{\alpha}{\varpi}\pi i}{\cos^2\left(m-\frac{1}{2}\right)\frac{\omega}{\varpi}\pi i}}.$$

Substituting for the cosines and sines of imaginary arcs their values in exponential quantitites, the formulas become:

$$184.\quad \varphi\alpha = \frac{1}{2}\cdot\frac{\varpi}{\pi}\left(h^{\frac{\alpha}{\varpi}\pi}-h^{-\frac{\alpha}{\varpi}\pi}\right)\cdot\prod_{m=1}^{\infty}\frac{1-\left(\frac{h^{\frac{\alpha}{\varpi}\pi}-h^{-\frac{\alpha}{\varpi}\pi}}{h^{m\frac{\omega}{\varpi}\pi}-h^{-m\frac{\omega}{\varpi}\pi}}\right)^2}{1+\left(\frac{h^{\frac{\alpha}{\varpi}\pi}-h^{-\frac{\alpha}{\varpi}\pi}}{h^{\left(m-\frac{1}{2}\right)\frac{\omega}{\varpi}\pi}+h^{-\left(m-\frac{1}{2}\right)\frac{\omega}{\varpi}\pi}}\right)^2},$$

$$185.\quad f\alpha = \prod_{m=0}^{\infty}\cdot\frac{1-\left(\frac{h^{\frac{\alpha}{\varpi}\pi}-h^{-\frac{\alpha}{\varpi}\pi}}{h^{\left(m+\frac{1}{2}\right)\frac{\omega}{\varpi}\pi}-h^{-\left(m+\frac{1}{2}\right)\frac{\omega}{\varpi}\pi}}\right)^2}{1+\left(\frac{h^{\frac{\alpha}{\varpi}\pi}-h^{-\frac{\alpha}{\varpi}\pi}}{h^{\left(m+\frac{1}{2}\right)\frac{\omega}{\varpi}\pi}+h^{-\left(m+\frac{1}{2}\right)\frac{\omega}{\varpi}\pi}}\right)^2},$$

$$186.\quad F\alpha = \frac{1}{2}\left(e^{\frac{\alpha}{\varpi}\pi}+e^{-\frac{\alpha}{\varpi}\pi}\right)\cdot\prod_{m=1}^{\infty}\frac{1+\left(\frac{h^{\frac{\alpha}{\varpi}\pi}-h^{-\frac{\alpha}{\varpi}\pi}}{h^{m\frac{\omega}{\varpi}\pi}+h^{-m\frac{\omega}{\varpi}\pi}}\right)^2}{1+\left(\frac{h^{\frac{\alpha}{\varpi}\pi}-h^{-\frac{\alpha}{\varpi}\pi}}{h^{\left(m-\frac{1}{2}\right)\frac{\omega}{\varpi}\pi}+h^{-\left(m-\frac{1}{2}\right)\frac{\omega}{\varpi}\pi}}\right)^2}$$

where $h$ is the number $2.1718218\ldots$.[7]

We can also transform this formula in the following manner.

If one replaces $\alpha$ by $\alpha i$, we will have the values of $\varphi(\alpha i)$, $f(\alpha i)$, $F(\alpha i)$. By now changing $c$ to $e$ and $e$ to $c$, the quantities:

$$\omega;\ \varpi;\ \varphi(\alpha i);\ f(\alpha i);\ F(\alpha i)$$

will resepectivley change to:

$$\varpi;\ \omega;\ i\varphi\alpha;\ F\alpha;\ f\alpha,$$

so the preceding formulas give:

[7]There is a typo here. The number we now call $e$ is equal to $2.71828128\ldots$.

$$187.\quad \varphi\alpha = \frac{\omega}{\pi}.\sin\frac{\alpha\pi}{\omega}.\prod_{m=1}^{\infty}.\frac{1+\frac{4\sin^2\left(\frac{\alpha\pi}{\omega}\right)}{\left(h^{\frac{m\varpi\pi}{\omega}}-h^{-\frac{m\varpi\pi}{\omega}}\right)^2}}{1-\frac{4\sin^2\left(\frac{\alpha\pi}{\omega}\right)}{\left(h^{\frac{(2m-1)\varpi\pi}{2\omega}}+h^{-\frac{(2m-1)\varpi\pi}{2\omega}}\right)^2}},$$

$$188.\quad F\alpha = \prod_{m=0}^{\infty}\frac{1+\frac{4\sin^2\left(\frac{\alpha\pi}{\omega}\right)}{\left(h^{\frac{(2m+1)\varpi\pi}{2\omega}}-h^{-\frac{(2m+1)\varpi\pi}{2\omega}}\right)^2}}{1-\frac{4\sin^2\left(\frac{\alpha\pi}{\omega}\right)}{\left(h^{\frac{(2m+1)\varpi\pi}{2\omega}}+h^{-\frac{(2m+1)\varpi\pi}{2\omega}}\right)^2}}$$

$$189.\quad f\alpha = \cos\left(\frac{\alpha\pi}{\omega}\right).\prod_{m=1}^{\infty}\frac{1-\frac{4\sin^2\left(\frac{\alpha\pi}{\omega}\right)}{\left(h^{\frac{m\varpi\pi}{\omega}}+h^{-\frac{m\varpi\pi}{\omega}}\right)^2}}{1-\frac{4\sin^2\left(\frac{\alpha\pi}{\omega}\right)}{\left(h^{\frac{(2m+1)\varpi\pi}{2\omega}}+h^{-\frac{(2m+1)\varpi\pi}{2\omega}}\right)^2}}.$$

## 32.

Now consider formulas (160.), (161.), (162.).
We have:

$$\sum_{\mu=0}^{\infty}(-1)^{\mu}.\frac{(2\mu+1)\pi}{y^2+\left(\mu+\frac{1}{2}\right)^2\pi^2}=\frac{2}{h^y+h^{-y}},$$

so setting:

$$y=\left[\alpha\pm\left(m+\frac{1}{2}\right)\omega\right]\frac{\pi}{\varpi}:$$

$$\sum_{\mu=0}^{\infty}(-1)^{\mu}.\frac{(2\mu+1)\varpi}{\left[\alpha\pm\left(m+\frac{1}{2}\right)\omega\right]^2+\left(\mu+\frac{1}{2}\right)^2\varpi^2}=\frac{2\pi}{\varpi}.\frac{1}{h^{\left[\alpha\pm\left(m+\frac{1}{2}\right)\omega\right]\frac{\pi}{\varpi}}+h^{-\left[\alpha\pm\left(m+\frac{1}{2}\right)\omega\right]\frac{\pi}{\varpi}}}.$$

In virtue of this formula it is easy to see that expressions (153.), (162.) of $\varphi\alpha$ and $F\alpha$ become:

$$190.\quad \varphi\alpha =$$
$$\frac{2}{ec}\frac{\pi}{\varpi}\sum_{m=0}^{\infty}(-1)^m.\left\{\frac{1}{h^{\left[\alpha-\left(m+\frac{1}{2}\right)\omega\right]\frac{\pi}{\varpi}}+h^{-\left[\alpha-\left(m-\frac{1}{2}\right)\omega\right]\frac{\pi}{\varpi}}}-\frac{1}{h^{\left[\alpha+\left(m+\frac{1}{2}\right)\omega\right]\frac{\pi}{\varpi}}+h^{-\left[\alpha+\left(m+\frac{1}{2}\right)\omega\right]\frac{\pi}{\varpi}}}\right\},$$

191. $F\alpha =$
$$\frac{2}{c}.\frac{\pi}{\varpi}.\sum_{m=0}^{\infty}\left\{\frac{1}{h^{\left[\alpha-\left(m+\frac{1}{2}\right)\omega\right]\frac{\pi}{\varpi}}+h^{-\left[\alpha-\left(m+\frac{1}{2}\right)\omega\right]\frac{\pi}{\omega}}}+\frac{1}{h^{\left[\alpha+\left(m+\frac{1}{2}\right)\omega\right]\frac{\pi}{\varpi}}+h^{-\left[\alpha+\left(m+\frac{1}{2}\right)\omega\right]\frac{\pi}{\varpi}}}\right\}.$$

The preceding expression of $\varphi\alpha$, $F\alpha$ can also be put in many other forms: I will recall the most remarkable. Joining initially the terms of the right side, one finds:

$$192.\quad \varphi\alpha = \frac{2}{ec}.\frac{\pi}{\varpi}.\sum_{m=0}^{\infty}(-1)^m.\left\{\frac{\left(h^{\frac{\alpha\pi}{\varpi}}-h^{-\frac{\alpha\pi}{\varpi}}\right)\left(h^{\left(m+\frac{1}{2}\right)\frac{\omega\pi}{\varpi}}-h^{-\left(m+\frac{1}{2}\right)\frac{\omega\pi}{\varpi}}\right)}{h^{\frac{2\alpha\pi}{\varpi}}+h^{-\frac{2\alpha\pi}{\varpi}}+h^{(2m+1)\frac{\omega\pi}{\varpi}}+h^{-(2m+1)\frac{\omega\pi}{\varpi}}}\right\},$$
$$193.\quad F\alpha = \frac{2}{c}.\frac{\pi}{\varpi}.\sum_{m=0}^{\infty}.\left\{\frac{\left(h^{\frac{\alpha\pi}{\varpi}}+h^{-\frac{\alpha\pi}{\varpi}}\right)\left(h^{\left(m+\frac{1}{2}\right)\frac{\omega\pi}{\varpi}}+h^{-\left(m+\frac{1}{2}\right)\frac{\omega\pi}{\varpi}}\right)}{h^{\frac{2\alpha\pi}{\varpi}}+h^{-\frac{2\alpha\pi}{\varpi}}+h^{(2m+1)\frac{\omega\pi}{\varpi}}+h^{-(2m+1)\frac{\omega\pi}{\varpi}}}\right\}.$$

If for brevity one supposes:

$$194.\quad h^{\frac{\alpha\pi}{\varpi}} = \epsilon \text{ and } h^{\frac{\omega\pi}{\varpi}} = r^2,$$

these formulas, by developing the right hand side, become:

195. $\varphi\alpha =$
$$\frac{2}{ec}.\frac{\pi}{\varpi}\left(\epsilon-\frac{1}{\epsilon}\right).\left\{\frac{r-\frac{1}{r}}{r^2+\epsilon^2+\frac{1}{\epsilon^3}+\frac{1}{r^2}}-\frac{r^3-\frac{1}{r^2}}{r^6+\epsilon^2+\frac{1}{\epsilon^2}+\frac{1}{r^6}}+\frac{r^5-\frac{1}{r^5}}{r^{10}+\epsilon^2+\frac{1}{\epsilon^2}+\frac{1}{r^{10}}}-\cdots\right\}$$
196. $F\alpha =$
$$\frac{2}{c}.\frac{\pi}{\varpi}.\left(\epsilon+\frac{1}{\epsilon}\right).\left\{\frac{r+\frac{1}{r}}{r^2+\epsilon^2+\frac{1}{\epsilon^2}+\frac{1}{r^2}}+\frac{r^3+\frac{1}{r^3}}{r^6+\epsilon^2+\frac{1}{\epsilon^2}+\frac{1}{r^6}}+\frac{r^5+\frac{1}{r^5}}{r^{10}+\epsilon^2+\frac{1}{\epsilon^2}+\frac{1}{r^{10}}}-\cdots\right\}.$$

Putting $\alpha i$ instead of $\alpha$ in formulas (192.), (193.), changing also $c$ to $e$ and $e$ to $c$, and noticing that the quantities

$$\omega,\ \varpi,\ \varphi(\alpha i),\ F(\alpha i),\ h^{\alpha i\frac{\pi}{\varpi}}-h^{-\alpha i\frac{\pi}{\varpi}},\ h^{\alpha i\frac{\pi}{\varpi}}+h^{\alpha i\frac{\pi}{\varpi}},$$

respectively change to:

$$\varpi,\ \omega,\ i\varphi(\alpha),\ f\alpha,\ 2i.\sin\alpha\frac{\pi}{\omega},\ 2\cos\alpha\frac{\pi}{\omega},$$

they become:

$$197. \quad \varphi\alpha = \frac{4}{ec}.\frac{\pi}{\omega}.\sum_{m=0}^{\infty}(-1)^m.\left\{\frac{\sin\frac{\alpha\pi}{\omega}\left(h^{\left(m+\frac{1}{2}\right)\frac{\varpi\pi}{\omega}} - h^{-\left(m+\frac{1}{2}\right)\frac{\varpi\pi}{\omega}}\right)}{h^{(2m+1)\frac{\varpi\pi}{\omega}} + 2\cos 2\alpha\frac{\pi}{\omega} + h^{-(2m+1)\frac{\varpi\pi}{\omega}}}\right\},$$

$$198. \quad f\alpha = \frac{4}{e}.\frac{\pi}{\omega}.\sum_{m=0}^{\infty}.\left\{\frac{\cos\frac{\alpha\pi}{\omega}.\left(h^{\left(m+\frac{1}{2}\right)\frac{\varpi\pi}{\omega}} + h^{-\left(m+\frac{1}{2}\right)\frac{\varpi\pi}{\omega}}\right)}{h^{(2m+1)\frac{\varpi\pi}{\omega}} + 2\cos 2\alpha\frac{\pi}{\omega} + h^{-(2m+1)\frac{\varpi\pi}{\omega}}}\right\}.$$

Putting for short

$$199. \quad h^{\frac{\varpi\pi}{2\omega}} = \varrho,$$

and expanding, we obtain:

$$200. \quad \varphi\left(\alpha\frac{\omega}{2}\right) =$$
$$\frac{4}{ec}.\frac{\pi}{\omega}.\sin\left(\alpha\frac{\pi}{2}\right).\left\{\frac{\varrho-\frac{1}{\varrho}}{\varrho^2+2\cos(\alpha\pi)+\frac{1}{\varrho^2}} - \frac{\varrho^3-\frac{1}{\varrho^3}}{\varrho^6+2\cos(\alpha\pi)+\frac{1}{\varrho^6}} + \frac{\varrho^5-\frac{1}{\varrho^5}}{\varrho^{10}+2\cos(\alpha\pi)+\frac{1}{\varrho^{10}}} - \cdots\right\},$$

$$201. \quad f\left(\alpha\frac{\omega}{2}\right) =$$
$$\frac{4}{e}\frac{\pi}{\omega}.\cos\left(\alpha\frac{\pi}{2}\right).\left\{\frac{\varrho+\frac{1}{\varrho}}{\varrho^2+2\cos(\alpha\pi)+\frac{1}{\varrho^2}} + \frac{\varrho^3+\frac{1}{\varrho^3}}{\varrho^6+2\cos(\alpha\pi)+\frac{1}{\varrho^6}} + \frac{\varrho^5+\frac{1}{\varrho^5}}{\varrho^{10}+2\cos(\alpha\pi)+\frac{1}{\varrho^{10}}} + \cdots\right\}.$$

Substituting into formulas (190.), (191.) the values of $\epsilon$ and $r$ in place of $h^{\alpha\frac{\pi}{\varpi}}$ and $h^{\frac{\omega\pi}{2\varpi}}$, it becomes:

$$202. \quad \varphi\alpha = \frac{2}{ec}.\frac{\pi}{\varpi}.\sum_{m=0}^{\infty}(-m).\left\{\frac{1}{\epsilon.r^{-(2m+1)}+\epsilon^{-1}.r^{2m+1}} - \frac{1}{\epsilon.r^{2m+1}+\epsilon^{-1}.r^{-(2m+1)}}\right\},$$

$$203. \quad F\alpha = \frac{2}{c}.\frac{\pi}{\varpi}.\sum_{m=0}^{\infty}\left\{\frac{1}{\epsilon.r^{-2m-1}+\epsilon^{-1}.r^{2m+1}} + \frac{1}{\epsilon.r^{2m+1}+\epsilon^{-1}.r^{-2m-1}}\right\}.$$

Suppose now that $\alpha < \frac{\omega}{2}$, we have:

$$\frac{1}{\epsilon.r^{-2m-1}+\epsilon^{-1}r^{2m+1}} = \frac{\epsilon^{-1}.r^{-2m-1}}{1+\epsilon^2.r^{-4m-2}} = \epsilon.r^{-2m-1} - \epsilon^3.r^{-6m-3} + \epsilon^5.r^{-10m-5} - \cdots,$$
$$\frac{1}{\epsilon.r^{2m+1}+\epsilon^{-1}.r^{-2m-1}} = \frac{\epsilon^{-1}.r^{-2m-1}}{1+\epsilon^{-2}.r^{-4m-2}} = \epsilon^{-1}.r^{-2m-1} - \epsilon^{-3}.r^{-6m-3} + \epsilon^{-5}.r^{-10m-5} - \cdots,$$

thus:

$$\begin{aligned}
\varphi\alpha &= \frac{2}{ec}.\frac{\pi}{\omega}.\left\{\sum_{m=0}^{\infty}(-1)^m\left\{(\epsilon-\epsilon^{-1}).r^{-2m-1}-(\epsilon^3-\epsilon^{-3}).r^{-6m-3}+(\epsilon^5-\epsilon^{-5}).r^{-10m-5}-\cdots\right\}\right\}\\
&= \frac{2}{ec}.\frac{\pi}{\omega}.\left\{(\epsilon-\epsilon^{-1}).\sum_{m=0}^{\infty}(-1)^m r^{-2m-1}-(\epsilon^3-\epsilon^{-3}).\sum_{m=0}^{\infty}(-1)^m r^{-6m-3}+\cdots\right\}.
\end{aligned}$$

Now:

$$\begin{aligned}
\sum_{m=0}^{\infty}(-1)^m.r^{-2m-1} &= r^{-1}-r^{-3}+r^{-5}-\cdots=\frac{r^{-1}}{1+r^{-2}}=\frac{r}{r^2+1},\\
\sum_{m=0}^{\infty}(-1)^m r^{3m-3} &= r^{-3}-r^{-6}+r^{-9}-\cdots=\frac{r^{-3}}{1+r^{-6}}=\frac{r^3}{r^6+1}
\end{aligned}$$

etc., hence:

$$204.\quad \varphi\alpha=\frac{2}{ec}.\frac{\pi}{\varpi}.\left\{\frac{\epsilon-\epsilon^{-1}}{r+r^{-1}}-\frac{\epsilon^3-\epsilon^{-3}}{r^3+r^{-3}}+\frac{\epsilon^5-\epsilon^{-5}}{r^5+r^{-5}}-\cdots\right\}.$$

In the same manner we find:

$$205.\quad F\alpha=\frac{2}{c}.\frac{\pi}{\varpi}.\left\{\frac{\epsilon+\epsilon^{-1}}{r-r^{-1}}-\frac{\epsilon^3+\epsilon^{-3}}{r^3-r^{-3}}+\frac{\epsilon^5+\epsilon^{-5}}{r^5-r^{-5}}-\cdots\right\}.$$

By putting $\alpha\frac{\varpi}{2}i$ in place of $\alpha$, and also changing $e$ to $c$ and $c$ to $e$,

$$\epsilon;\ \varpi;\ \varphi\left(\alpha\frac{\varpi}{2}i\right);\ F\left(\alpha\frac{\varpi}{2}i\right);\ r;\ \epsilon^m+\epsilon^{-m};\ \epsilon^m-\epsilon^{-m}$$

become:

$$\varpi;\ \omega;\ i\varphi\left(\alpha\frac{\omega}{2}\right);\ f\left(\alpha\frac{\omega}{2}\right);\varrho;\ 2\cos\left(m\alpha\frac{\pi}{2}\right);\ 2i\sin\left(m\alpha\frac{\pi}{2}\right);$$

thus:

$$\begin{aligned}
206.\quad \varphi\left(\alpha\frac{\omega}{2}\right) &= \frac{4}{ec}.\frac{\pi}{\omega}.\left\{\frac{\sin\left(\alpha\frac{\pi}{2}\right)}{\varrho+\frac{1}{\varrho}}-\frac{\sin\left(3\alpha\frac{\pi}{2}\right)}{\varrho^3+\frac{1}{\varrho^3}}+\frac{\sin\left(5\alpha\frac{\pi}{2}\right)}{\varrho^5+\frac{1}{\varrho^5}}-\cdots\right\},\\
207.\quad f\left(\alpha\frac{\omega}{2}\right) &= \frac{4}{e}.\frac{\pi}{\omega}.\left\{\frac{\cos\left(\alpha\frac{\pi}{2}\right)}{\varrho-\frac{1}{\varrho}}-\frac{\cos\left(3\alpha\frac{\pi}{2}\right)}{\varrho^3-\frac{1}{\varrho^3}}+\frac{\cos\left(5\alpha\frac{\pi}{2}\right)}{\varrho^5-\frac{1}{\varrho^5}}-\cdots\right\}.
\end{aligned}$$

These last four formulas offer very simple expressions for $\varphi\alpha$, $f\alpha$, $F\alpha$. By differentiating and integrating one can deduce a host of others more or less remarkable.

In the case $e = c$, the preceding formulas take very simple forms because of the relation $\omega = \varpi$ that takes place in this case. For simplicity, set $e = c = 1$. We have:

$$r = h^{\frac{\omega\pi}{2\varpi}} = h^{\frac{\pi}{2}},\ \varrho = h^{\frac{\varpi\pi}{2\omega}} = h^{\frac{\pi}{2}},$$

so by substituting and putting $\alpha = \alpha\frac{\varpi}{2}$ into (204.), (205.), we get:

$$\varphi\left(\alpha\tfrac{\omega}{2}\right) = 2.\tfrac{\pi}{\omega}.\left\{\tfrac{h^{\frac{\alpha\pi}{2}} - h^{-\frac{\alpha\pi}{2}}}{h^{\frac{\pi}{2}} + h^{-\frac{\pi}{2}}} - \tfrac{h^{\frac{3\alpha\pi}{2}} - h^{-\frac{3\alpha\pi}{2}}}{h^{\frac{3\pi}{2}} + h^{-\frac{3\pi}{2}}} + \tfrac{h^{\frac{5\alpha\pi}{2}} - h^{-\frac{5\alpha\pi}{2}}}{h^{\frac{5\pi}{2}} + h^{-\frac{5\pi}{2}}} - \cdots\right\},$$
$$F\left(\alpha\tfrac{\omega}{2}\right) = 2\tfrac{\pi}{\omega}.\left\{\tfrac{h^{\frac{\alpha\pi}{2}} + h^{-\frac{\alpha\pi}{2}}}{h^{\frac{\pi}{2}} - h^{-\frac{\pi}{2}}} - \tfrac{h^{\frac{3\alpha\pi}{2}} + h^{-\frac{3\alpha\pi}{2}}}{h^{\frac{3\pi}{2}} - h^{-\frac{3\pi}{2}}} + \tfrac{h^{\frac{5\alpha\pi}{2}} + h^{-\frac{5\alpha\pi}{2}}}{h^{\frac{5\pi}{2}} - h^{-\frac{5\pi}{2}}} - \cdots\right\},$$
$$\varphi\left(\alpha\tfrac{\omega}{2}\right) = \tfrac{4\pi}{\omega}\left\{\sin\left(\alpha\tfrac{\pi}{2}\right).\tfrac{h^{\frac{\pi}{2}}}{1+h^{\pi}} - \sin\left(3\alpha\tfrac{\pi}{2}\right).\tfrac{h^{3\frac{\pi}{2}}}{1+h^{3\pi}} + \sin\left(5\alpha\tfrac{\pi}{2}\right).\tfrac{h^{5\frac{\pi}{2}}}{1+h^{5\pi}} - \cdots\right\},$$
$$f\left(\alpha\tfrac{\omega}{2}\right) = \tfrac{4\pi}{\omega}.\left\{\cos\left(\alpha\tfrac{\pi}{2}\right).\tfrac{h^{\frac{\pi}{2}}}{h^{\pi}-1} - \cos\left(3\alpha\tfrac{\pi}{2}\right).\tfrac{h^{3\frac{\pi}{2}}}{h^{3\pi}-1} + \sin\left(5\alpha\tfrac{\pi}{2}\right).\tfrac{h^{5\frac{\pi}{2}}}{h^{5\pi}-1} - \cdots\right\}.$$

The functions $\varphi$, $f$, $F$ are determined by the equations

$$\alpha.\tfrac{\omega}{2} = \int_0^x \tfrac{\partial x}{\sqrt{(1-x^4)}};\ \tfrac{\omega}{2} = \int_0^1 \tfrac{\partial x}{\sqrt{(1-x^4)}};$$
$$x = \varphi\left(\alpha\tfrac{\omega}{2}\right) = \sqrt{(1-x^2)} = f\left(\alpha\tfrac{\omega}{2}\right) = \sqrt{(1+x^2)} = F\left(\alpha\tfrac{\omega}{2}\right).$$

If in the two last equations one sets $\alpha = 0$, and if one notices that then the value of $\frac{\varphi\left(\alpha\frac{\omega}{2}\right)}{\sin\left(\alpha\frac{\pi}{2}\right)}$ is $= \frac{\omega}{\pi}$ and that of $\frac{\sin\left(m\alpha\frac{\pi}{2}\right)}{\sin\left(\alpha\frac{\pi}{2}\right)} = m$, one will find:

$$\frac{\omega}{2} = \frac{\pi}{2}.\left\{\frac{h^{\frac{\pi}{2}}}{h^{\pi}-1} - \frac{h^{\frac{3\pi}{2}}}{h^{3\pi}-1} + \frac{h^{\frac{5\pi}{2}}}{h^{5\pi}-1} - \cdots\right\} = \int_0^1 \frac{\partial x}{\sqrt{(1-x^4)}},$$
$$\frac{\omega^2}{2} = \pi^2.\left\{\frac{h^{\frac{\pi}{2}}}{h^{\pi}+1} - 3\frac{h^{\frac{3\pi}{2}}}{h^{3\pi}+1} + 5\frac{h^{\frac{5\pi}{2}}}{h^{5\pi}+1} - \cdots\right\} = \left(\int_0^1 \frac{\partial x}{\sqrt{(1-x^4)}}\right)^2.$$

(Continued in the next issue.)