Figures Circumscribing Circles

Tom M. Apostol and Mamikon A. Mnatsakanian

1. INTRODUCTION. The centroid of the boundary of an arbitrary triangle need not be at the same point as the centroid of its interior. But we have discovered that the two centroids are always collinear with the center of the inscribed circle, at distances in the ratio 3 : 2 from the center. We thought this charming fact must surely be known, but could find no mention of it in the literature. This paper generalizes this elegant and surprising result to any polygon that circumscribes a circle (Theorem 6). A key ingredient of the proof is a link to Archimedes' striking discovery concerning the area of a circular disk [**4**, p. 91], which for our purposes we prefer to state as follows:

Theorem 1 (Archimedes). *The area of a circular disk is equal to the product of its semiperimeter and its radius.*

Expressed as a formula, this becomes

$$A = \tfrac{1}{2}Pr, \tag{1}$$

where A is the area, P is the perimeter, and r is the radius of the disk.

First we extend (1) to a large class of plane figures circumscribing a circle that we call *circumgons*, defined in section 2. They include arbitrary triangles, all regular polygons, some irregular polygons, and other figures composed of line segments and circular arcs. Examples are shown in Figures 1 through 4. Section 3 treats *circumgonal rings*, plane regions lying between two similar circumgons. These rings have a constant width that replaces the radius in the corresponding extension of (1). We also show that all rings of constant width are necessarily circumgonal rings. Section 4 generalizes the relation of the two centroids of a triangle mentioned above to arbitrary circumgons, and section 5 explores several relations for centroids associated with circumgonal rings. Finally, section 6 mentions applications of circumgons to a class of isoperimetric problems that will be considered in detail elsewhere.

2. CIRCUMGONS. To pave the way for the general definition of a circumgon, we begin with some examples. The prototype is a triangle. Every triangle circumscribes a circle whose center is the point of intersection of the three angle bisectors. By dividing a triangle into three smaller triangles with a common vertex at the center of the inscribed circle, we easily see that (1) holds for any triangle of area A and perimeter P, where r is the radius of the inscribed circle.

A polygon with more than three edges may or may not circumscribe a circle. We are interested in those that do, because they provide examples of circumgons. Every regular polygon is a circumgon, but there are also *nonregular* circumgons, as illustrated in Figure 1b. Like a triangle, any polygon circumscribing a circle is a circumgon. The inscribed circle is called the *incircle*, its radius is called the *inradius*, and its center is called the *incenter*. All bisectors of the interior angles of a circumgon intersect at the incenter. By dividing the polygon into triangles with one common vertex at the incenter, it is easily seen that (1) holds for every circumgon whose boundary is a convex polygon.

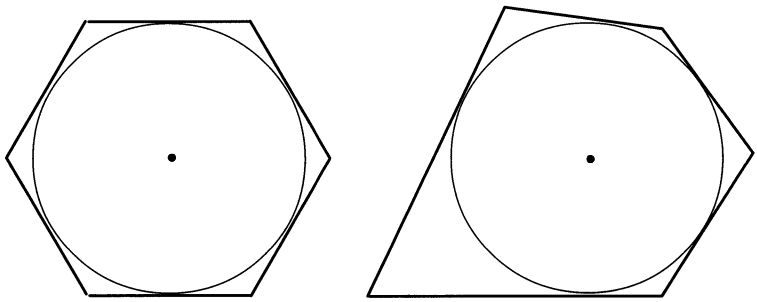
(a) (b)

Figure 1. Example of circumgons: (a) a regular hexagon and (b) a nonregular pentagon.

We will also extend (1) to more general circumgons, not necessarily convex, such as the polygon in Figure 2a, or the star-shaped polygon in Figure 2b, and to more general polygonal shapes, not necessarily closed, such as the example in Figure 4a. It may seem surprising that nonconvex polygons can circumscribe a circle. It's true that our examples are not ordinary garden variety circumscribing polygons, but when viewed appropriately, they do circumscribe a circle. For example, in Figure 2a only two edges of the polygon are tangent to the incircle. The other four edges do not even touch the incircle, but their extensions, shown by dotted lines, are tangent to the incircle. In Figure 2b, none of the edges of the pentagram touches the incircle, but each extended edge, shown by dotted lines, is tangent to the incircle.

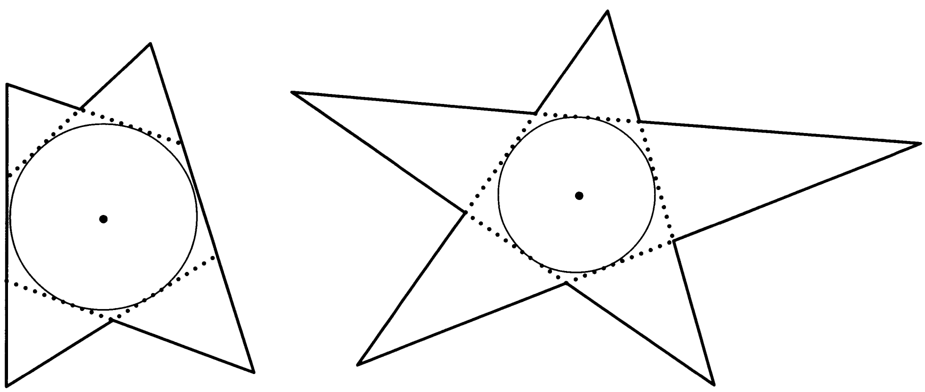
(a) (b)

Figure 2. Some circumgonal regions: the area of each is its semiperimeter times its inradius.

Building blocks of a circumgonal region. The definition of a general circumgonal region will be formulated in terms of simpler elements called *building blocks*, defined as follows. Start with a given circle, and consider a triangular wedge with one vertex at the center and with side opposite this vertex lying on a line tangent to the circle. We call this wedge a *building block* of the circumgonal region; the side opposite the center on the tangent line is called the *outer edge* of the block. The given circle is the *incircle*, its radius is the *inradius*, and its center is the *incenter*. An example is shown in Figure 3a. Because a circular arc can be regarded as a limiting case of circumscribing polygons, we also allow any sector of the incircle to be a building block of a circumgonal region, with its outer edge being the circular arc, as shown in Figure 3b. Thus,

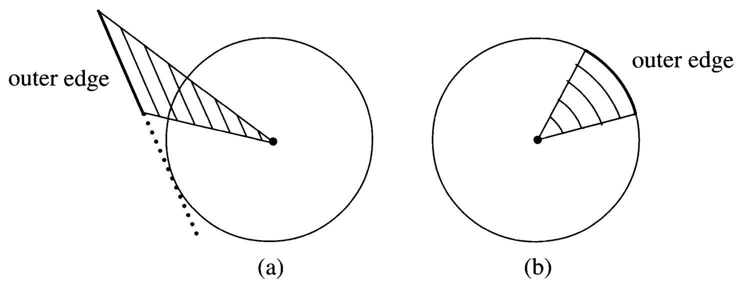

Figure 3. A building block of a circumgonal region is either (a) a triangular wedge or (b) a circular sector. Its perimeter is the length of the outer edge.

the area of each building block, whether it is a *triangular wedge* or a *circular sector*, is equal to half the length of its outer edge times the inradius. To extend Theorem 1, we simply define the *perimeter* of the block to be the *length of its outer edge*. This gives us:

Theorem 2. *The area of a circumgonal building block is equal to the product of its semiperimeter and its inradius.*

Definitions of circumgonal region and circumgon. A *circumgonal region* is the union of a finite set of nonoverlapping building blocks having the same incircle. The union of the corresponding outer edges is called a *circumgon*; the sum of the lengths of the outer edges is called the *perimeter* of the circumgon.

Note. The perimeter of a circumgon, as just defined, is not its perimeter in the usual Euclidean sense unless the circumgon is closed.

This definition immediately gives the following extension of Theorem 2:

Theorem 3. *The area of any circumgonal region is equal to the product of its semiperimeter and its inradius.*

Both Theorems 2 and 3 can be expressed by the same formula used for Theorem 1:

$$A = \tfrac{1}{2}Pr, \tag{2}$$

where A is the area, P is the perimeter, and r is the inradius of the circumgon. Two examples satisfying (2) are shown in Figure 4.

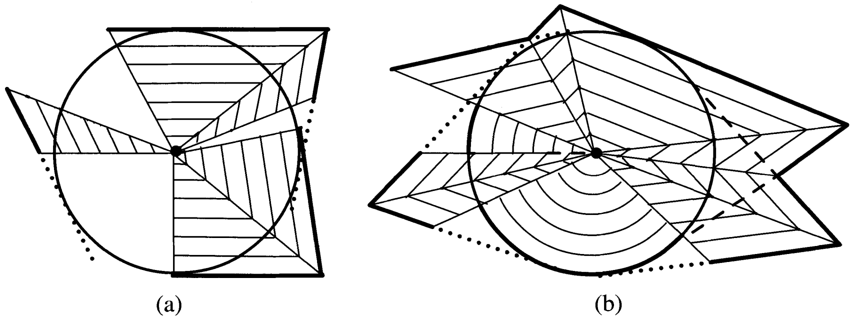

Figure 4. More examples of circumgonal regions: the area of each is its semiperimeter times its inradius.

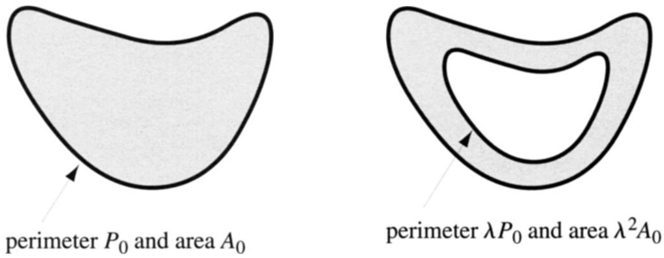

Figure 5. A simple closed curve with perimeter P_0 and area A_0 used to form a ring with size factor λ.

3. CIRCUMGONAL RINGS. Figure 5 shows a simple type of ring, the region between two similar nonoverlapping simple closed curves with similarity ratio λ, where $0 < \lambda < 1$. We call λ the *size factor* because it determines the size of the inner curve relative to the outer one. If the outer region has perimeter P_0 and area A_0, the inner region has perimeter λP_0 and area $\lambda^2 A_0$, regardless of the choice of center of scaling.

We are interested in rings formed by scaling a circumgonal region from its incenter. The inner and outer boundaries need not be closed curves because the circumgons need not be closed. For a general ring the perpendicular distance between the boundary curves need not be constant, even if portions of the boundaries are parallel, as in the case of two similar rectangles.

If the ring is formed by scaling a circumgonal region from its incenter, it is easy to show that the perpendicular distance between corresponding parallel segments (or circular arcs) is a constant, which we call the *width* of the ring.

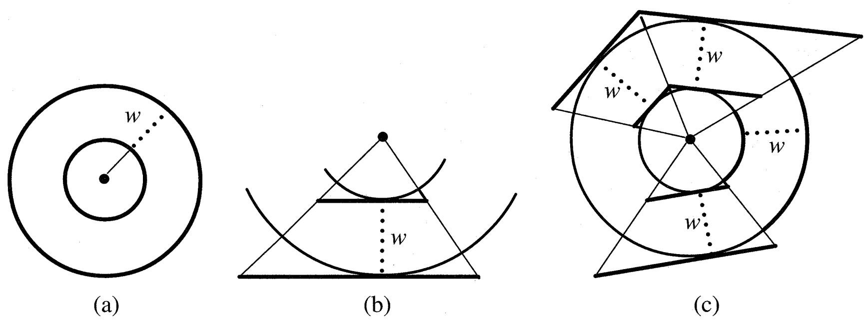

Figure 6. Examples of circumgonal rings. The annulus in (a) and the trapezoid in (b) are extreme cases.

Figure 6 shows examples of circumgonal rings. The circular annulus in (a) and the trapezoid in (b) are extreme cases. A more general example is shown in (c). In each case, the constant width w is the perpendicular distance between its parallel edges. It is also true that circumgonal rings are the only rings having constant width. In fact, we have:

Theorem 4.

(a) *A circumgonal ring formed by scaling a circumgonal region from its incenter has constant width.*

(b) *Conversely, consider any ring formed by two similar contours, where the outer contour consists of a finite set of line segments and circular arcs. If the ring has constant width, then it is necessarily a circumgonal ring.*

Proof. The proof of (a) is an easy exercise, which shows that the constant width w is given by

$$w = (1 - \lambda)r,$$

where r is inradius of the larger circumgon and $\lambda < 1$ is the scaling factor.

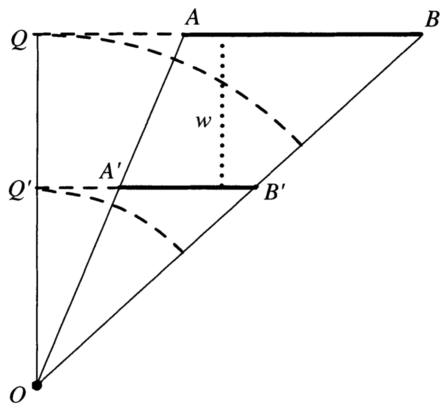

Figure 7. Illustrating the proof that every ring of constant width is a circumgonal ring.

To prove (b), refer to Figure 7, which shows a trapezoidal portion of the ring formed by parallel line segments AB and $A'B'$ having w as the perpendicular distance between them. The intersection O of the lines through AA' and BB' is the center of similarity, and $OA' = \lambda OA$, where λ (< 1) is the scaling factor. Let Q be the foot of a perpendicular from O to the line through AB. The circle with center O and radius OQ is tangent to the line through AB, hence AB is an outer edge of a circumgon with incenter O and inradius OQ. By similarity, point Q' on OQ satisfies $OQ' = \lambda OQ$, and the circle with center O and radius OQ' is tangent to the line through $A'B'$, so segment $A'B'$ is an outer edge of a circumgon with incenter O and inradius OQ'. But $w = OQ - OQ' = (1 - \lambda)OQ$, hence $OQ = w/(1 - \lambda)$ and $OQ' = \lambda w/(1 - \lambda)$. Thus the inradii and incenter O are completely determined by the width w and the scaling factor λ, as given above. This means that *every* trapezoidal portion of the ring circumscribes the same pair of circles. Consequently, the entire polygonal part of the ring is circumgonal with incenter O. The proof is even simpler for each portion of the ring that is a circular sector (of width w). ∎

The next result extends Theorem 3 to circumgonal rings.

Theorem 5. *The area of any circumgonal ring is equal to the product of its semiperimeter and its (constant) width.*

This can also be expressed as a formula resembling that in (2):

$$A = \tfrac{1}{2} P w, \tag{3}$$

where A is the area of the ring, P is its total perimeter, and w is its constant width.

Proof. If the outer boundary has perimeter P_0 and encloses a region of area A_0, then the ring has area $A = (1 - \lambda^2)A_0$ and total perimeter $P = (1 + \lambda)P_0$. For a circumgonal ring with inradius r of the larger circumgon we have $A_0 = P_0 r/2$, from which we find that

$$A = (1 - \lambda)(1 + \lambda)P_0 r/2 = (1 - \lambda)Pr/2 = Pw/2,$$

as asserted. ■

It is not surprising that (3) gives the formula for the area of a circular ring (Figure 6a). But it is reassuring to learn that (3) also becomes the well-known formula for the area of a trapezoid, average base times altitude. In fact, in Figure 6b the semiperimeter of the ring is the average length of the two parallel edges, and the width of the ring is the altitude of the trapezoid.

4. CENTROIDS OF CIRCUMGONAL REGIONS. This section uncovers a simple but surprising relation that always holds between the area centroid of a circumgonal region and the centroid of its boundary. Specifically, denote by $\boldsymbol{C}(A)$ the vector from the incenter O to the area centroid, and by $\boldsymbol{C}(B)$ the vector from O to the centroid of the boundary curve (with respect to arclength). Figure 8b illustrates these for a triangle. We will prove that, for a given circumgon, the location of one of the centroids determines the location of the other. In fact, we have:

Theorem 6. *The area centroid $\boldsymbol{C}(A)$ of any circumgonal region and the centroid $\boldsymbol{C}(B)$ of its boundary are collinear with the incenter, and are related by the equation*

$$\boldsymbol{C}(B) = \tfrac{3}{2}\boldsymbol{C}(A). \tag{4}$$

Proof. A classical result of Archimedes [**4**, p. 201] states that the area centroid of a triangle is at the intersection of its medians. It is also known [**2**, p. 11] that the distance from each vertex to the centroid is two-thirds the length of the median from that vertex. Apply this to the triangular block with incenter O and outer edge of length a shown in Figure 8a. In vector notation, $\boldsymbol{C}(A) = (2/3)\boldsymbol{C}(B)$, where $\boldsymbol{C}(B)$ is the midpoint of the outer edge. Hence

$$\boldsymbol{C}(B) = \tfrac{3}{2}\boldsymbol{C}(A), \tag{5}$$

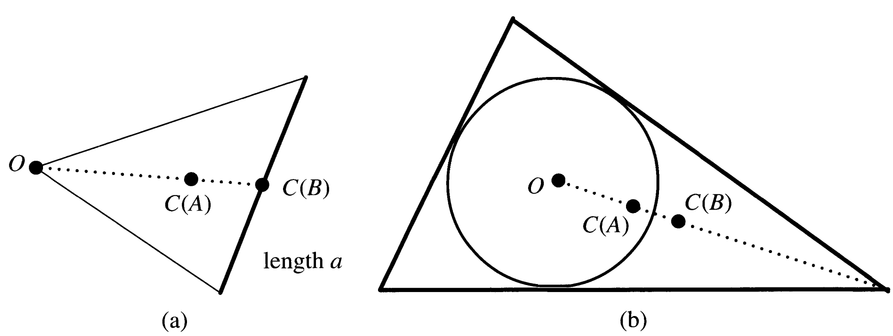

Figure 8. Centroid $\boldsymbol{C}(A) = \frac{2}{3}\boldsymbol{C}(B)$ for (a) a triangular block, and (b) for any triangle with incenter O.

which proves (4) for a triangular block. Now take any polygonal circumgon with triangular building blocks having outer edges of lengths $a_1, \ldots, a_n$, a common vertex at the incenter O, and respective areas $A_1, \ldots, A_n$. Figure 8b shows the case of a triangle. Denote by $\boldsymbol{C}(A_1), \ldots, \boldsymbol{C}(A_n)$ the corresponding vectors from the incenter O to the area centroid of each triangular block. The area centroid of their union is at the point described by the vector

$$\boldsymbol{C}(A) = \frac{\sum_{k=1}^{n} A_k \boldsymbol{C}(A_k)}{\sum_{k=1}^{n} A_k}. \tag{6}$$

In (6), write $A_k = a_k r/2$, where r is the inradius. The common factor $r/2$ cancels, and (6) becomes

$$\boldsymbol{C}(A) = \frac{\sum_{k=1}^{n} a_k \boldsymbol{C}(A_k)}{\sum_{k=1}^{n} a_k}. \tag{7}$$

On the other hand, the vector $\boldsymbol{C}(B)$ from O to the centroid of the boundary is

$$\boldsymbol{C}(B) = \frac{\sum_{k=1}^{n} a_k \boldsymbol{C}(B_k)}{\sum_{k=1}^{n} a_k},$$

where $\boldsymbol{C}(B_k)$ denotes the vector from O to the midpoint of the kth outer edge. Apply (5) to each triangular block to find that $\boldsymbol{C}(B_k) = (3/2)\boldsymbol{C}(A_k)$. Use this in the last equation and compare with (7) to obtain (4) for a polygonal circumgon. Because a circular arc can be regarded as a limiting case of a circumscribing polygon, formula (4) also holds for circumgons that include circular arcs as part of their boundaries. ■

We can also deduce (4) for a circular sector in a different manner. It is known [**2**, p. 12] that the area centroid of a circular sector of radius r subtending a central angle 2α lies on the radial line that bisects the sector at a distance $(2/3)r(\sin\alpha)/\alpha$ from the center, and that the centroid of the outer arc is at a distance $r(\sin\alpha)/\alpha$ from the center. Consequently, (4) holds for every circular sectorial building block of a circumgon.

Relation (4) holds in particular for any triangle, and also for *any polygon circumscribing a circle*. Because these two cases are so basic, we restate them here as corollaries:

Corollary 7.

(a) *The area centroid $\boldsymbol{C}(A)$ of any triangle and the centroid $\boldsymbol{C}(B)$ of its boundary are collinear with the incenter and are related by the equation*

$$\boldsymbol{C}(B) = \tfrac{3}{2}\boldsymbol{C}(A). \tag{8}$$

(b) *The same relation holds for any polygon circumscribing a circle.*

The results for these two classical cases are so simple that we thought they must surely be known, but we could find neither of them in the literature. In fact, the analysis in [**1**] suggests that even for a triangle the result was not previously recorded.

Another derivation of Corollary 7(a) can be given by referring to Figure 9. It is known that the centroid of the boundary of a triangle is the incenter O' of the medial triangle shown. Both triangles have common median lines, hence a common area centroid at point M. The incenter O of the larger triangle is collinear with O' and M, and its inradius is twice that of the smaller triangle, so $OM = 2MO'$. Consequently $OO' = OM + MO' = (3/2)OM$, in agreement with (8).

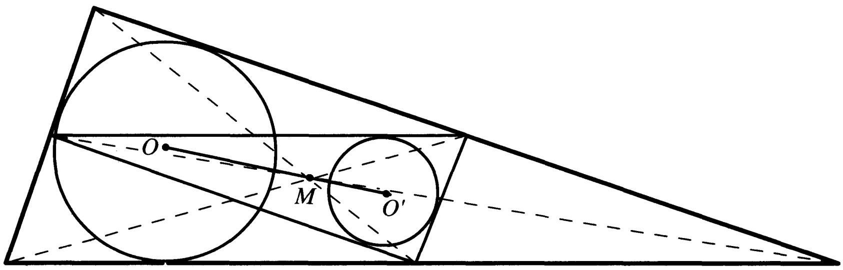

Figure 9. Another argument showing that $C(B) = (3/2)C(A)$ for a triangle.

5. CENTROIDS OF CIRCUMGONAL RINGS.

There are companion results for the centroid of a circumgonal ring. For simplicity, we refer to a ring with size factor λ simply as a λ-ring. First we have:

Theorem 8. *The area centroid $\boldsymbol{C}(A_{\text{ring}})$ of any circumgonal λ-ring is related to the area centroid $\boldsymbol{C}(A_{\text{outer}})$ of the outer circumgon by the equation*

$$\boldsymbol{C}(A_{\text{ring}}) = \frac{1-\lambda^3}{1-\lambda^2}\boldsymbol{C}(A_{\text{outer}}). \tag{9}$$

Proof. Let $\boldsymbol{C}(A_{\text{inner}})$ denote the area centroid of the inner circumgon, and let A_{outer} and A_{inner} denote the areas of the outer and inner circumgons, respectively. Equating moments we have

$$(A_{\text{outer}} - A_{\text{inner}})\boldsymbol{C}(A_{\text{ring}}) + A_{\text{inner}}\boldsymbol{C}(A_{\text{inner}}) = A_{\text{outer}}\boldsymbol{C}(A_{\text{outer}}).$$

Because of the relations

$$\boldsymbol{C}(A_{\text{inner}}) = \lambda\boldsymbol{C}(A_{\text{outer}}), \qquad A_{\text{inner}} = \lambda^2 A_{\text{outer}}$$

the foregoing equation reduces to (9). ■

Note that (4) is obtained as a limiting case of (9) as $\lambda \to 1$. In fact, our original discovery of (4) was obtained as this limiting case of (9).

Next, we extend Theorem 6 by relating the area centroid $\boldsymbol{C}(A_{\text{ring}})$ of any circumgonal ring with the centroid $\boldsymbol{C}(B_{\text{total}})$ of its full boundary.

Theorem 9. *The area centroid $\boldsymbol{C}(A_{\text{ring}})$ of any circumgonal λ-ring is related to the centroid $\boldsymbol{C}(B_{\text{total}})$ of its boundary by the equation*

$$\boldsymbol{C}(A_{\text{ring}}) = \frac{2}{3}\,\frac{\lambda^2+\lambda+1}{\lambda^2+1}\boldsymbol{C}(B_{\text{total}}). \tag{10}$$

Proof. Denote the vectors from the incenter to the centroids of the inner and outer boundaries, respectively, by $\boldsymbol{C}(B_{\text{inner}})$ and $\boldsymbol{C}(B_{\text{outer}})$. Let P_{in} and P_{out} denote the corresponding inner and outer perimeters. The definition of centroid states that

$$C(B_{\text{total}}) = \frac{P_{\text{in}}C(B_{\text{inner}}) + P_{\text{out}}C(B_{\text{outer}})}{P_{\text{in}} + P_{\text{out}}}.$$

Because of the relation $P_{\text{in}} = \lambda P_{\text{out}}$ this becomes

$$C(B_{\text{total}}) = \frac{\lambda C(B_{\text{inner}}) + C(B_{\text{outer}})}{\lambda + 1}. \tag{11}$$

But $C(B_{\text{inner}}) = \lambda C(B_{\text{outer}})$, and $C(B_{\text{outer}}) = (3/2)C(A_{\text{outer}})$ by (4), so (11) can be written as

$$C(B_{\text{total}}) = \frac{3}{2}\,\frac{\lambda^2 + 1}{\lambda + 1}C(A_{\text{outer}}),$$

which, together with (9), gives

$$C(A_{\text{ring}}) = \frac{2}{3}\,\frac{\lambda + 1}{\lambda^2 + 1}\,\frac{1 - \lambda^3}{1 - \lambda^2}C(B_{\text{total}}) = \frac{2}{3}\,\frac{\lambda^2 + \lambda + 1}{\lambda^2 + 1}C(B_{\text{total}}),$$

as asserted. ■

The next theorem relates the area centroid $C(A_{\text{ring}})$ of a circumgonal ring to the centroids of its outer and inner boundary curves.

Theorem 10. *For any circumgonal λ-ring the following hold:*

$$C(A_{\text{ring}}) = \frac{2}{3}\,\frac{\lambda^2 + \lambda + 1}{\lambda + 1}C(B_{\text{outer}}), \tag{12}$$

$$C(A_{\text{ring}}) = \frac{2}{3}\,\frac{\lambda^2 + \lambda + 1}{\lambda(\lambda + 1)}C(B_{\text{inner}}), \tag{13}$$

$$C(A_{\text{ring}}) = \frac{2}{3}\,\frac{\lambda^2 + \lambda + 1}{1 - \lambda^2}\big(C(B_{\text{outer}}) - C(B_{\text{inner}})\big), \tag{14}$$

$$C(A_{\text{ring}}) - C(B_{\text{inner}}) = \frac{\lambda + 2}{1 + 2\lambda}\big(C(B_{\text{outer}}) - C(A_{\text{ring}})\big). \tag{15}$$

Proof. Theorem 9 and (11) yield (12), which implies (13). From (12) and (13) we infer (14). From (13) we obtain

$$C(A_{\text{ring}}) - C(B_{\text{inner}}) = \frac{(1 - \lambda)(\lambda + 2)}{3\lambda(\lambda + 1)}C(B_{\text{inner}}) = \frac{(1 - \lambda)(\lambda + 2)}{3(\lambda + 1)}C(B_{\text{outer}}),$$

whereas (12) gives us

$$C(B_{\text{outer}}) - C(A_{\text{ring}}) = \frac{(1 - \lambda)(1 + 2\lambda)}{3(\lambda + 1)}C(B_{\text{outer}}).$$

Comparing the last two equations we get (15). ■

For a trapezoid, the result in (15) was known to Archimedes [**4**, Proposition 15, p. 201].

Corollary 11 (Archimedes). *The area centroid of a trapezoid lies on the segment joining the midpoints of its parallel edges and divides this segment in the ratio* $(\lambda + 2)/(1 + 2\lambda)$ *when taken from the shorter parallel edge to the longer, the ratio of whose lengths is* λ.

Note. Archimedes does not state explicitly from where the division point is measured, but this is implicit in his accompanying diagram.

6. APPLICATIONS OF CIRCUMGONS TO ISOPERIMETRIC PROBLEMS.

Traditional isoperimetric problems compare different plane regions having equal perimeters and ask for the region of maximal area. It is known [**3**, p. 373], [**5**, p. 83] that among all plane regions with a given perimeter, the circle encloses the largest area. Equivalently, among all plane regions with a given area, the circle has the smallest perimeter.

Many isoperimetric problems deal with specific contours, such as polygons. For example, among all polygons with a given number of sides, the regular ones have maximal area for given perimeter, or minimal perimeter for given area.

A polygon is defined by its sides and its angles. We can imagine a flexible polygon with fixed sides hinged at the vertices. Its perimeter is fixed but its area can be varied by changing the angles. An elegant well-known theorem [**5**, Theorem 12.5a], states:

> *A polygon inscribed in a circle has larger area than any other polygon with sides of the same lengths (therefore of the same perimeter) in the same order.*

Using properties of circumgons obtained in this article, we can establish a dual to this theorem:

> *A polygon circumscribing a circle has a smaller area than any other polygon with the same perimeter and the same interior angles in a given order.*

A proof of this and more general results, including extensions to 3-space, will be given elsewhere.

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TOM M. APOSTOL joined the Caltech mathematics faculty in 1950 and became professor emeritus in 1992. He is director of *Project MATHEMATICS!* (http://www.projectmathematics.com) an award-winning series of videos he initiated in 1987. His long career in mathematics is described in the September 1997 issue of *The College Mathematics Journal.* He is currently working with colleague Mamikon Mnatsakanian to produce materials demonstrating Mamikon's innovative and exciting approach to mathematics.
California Institute of Technology, 253-37 Caltech, Pasadena, CA 91125
apostol@caltech.edu

MAMIKON A. MNATSAKANIAN received a Ph.D. in physics in 1969 from Yerevan University, where he became professor of astrophysics. As an undergraduate he began developing innovative geometric methods for solving many calculus problems by a dynamic and visual approach that makes no use of formulas. He is currently working with Tom Apostol under the auspices of *Project MATHEMATICS!* to present his methods in a multimedia format.
California Institute of Technology, 253-37 Caltech, Pasadena, CA 91125
mamikon@caltech.edu

The Infinitude of the Primes

We assume that there are only finitely many primes $p_1, \ldots, p_k$. If m is a non-negative integer, then each (ordered) partition of m into k parts ($m = \alpha_1 + \cdots + \alpha_k$, with $0 \le \alpha_i \le m$ for $i = 1, 2, \ldots, k$) gives rise to a natural number $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$. Every natural number n is of this form for some m. Therefore,

$$\sum_n \frac{1}{n} = \sum_m \sum \frac{1}{p_1^{\alpha_1} \cdots p_k^{\alpha_k}},$$

where the inner sum is extended over all k-tuples $(\alpha_1, \ldots, \alpha_k)$ with the property that $\alpha_1 + \cdots + \alpha_k = m$. There are at most m^k such k-tuples. Combined with the fact that $p_i \ge 2$, this gives

$$\sum_n \frac{1}{n} \le \sum_m \frac{m^k}{2^m} < \infty.$$

This contradiction shows the set of primes is not finite.

Remark. A more delicate argument would reveal that, in fact,

$$\sum_p \frac{1}{p} = \infty,$$

where the sum is taken over all primes p.

—Submitted by Dinesh Singh
University of Delhi, India