A $1 Problem

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1. INTRODUCTION: MAKING MONEY. With a title like this, one might expect to find described here some unsolved problem assigned a decidedly paltry bounty by a strapped mathematical institute, or perhaps a pet question from an eccentric, but thrifty, individual. Indeed, this is an article about how to make money, but it is not at all about some scheme to earn income. Rather, it concerns how actually to *make money*, specifically, how to design a one-dollar coin.

The United States has had little success in the last quarter century with its dollar coins. The Susan B. Anthony dollar was introduced in 1979, but with its round shape, silvery appearance, and milled edge, it was too similar to the quarter to gain much acceptance in everyday commerce. It was discontinued after only three years (though it appeared again briefly in 1999). In 2000, the Sacagawea dollar was put into circulation, and this coin is certainly more distinctive, with a golden color and a smooth edge. However, it has never gained wide acceptance in the U.S., no doubt in large part because one-dollar bills continue to be produced by the Bureau of Engraving and Printing.

In contrast, Canada replaced its dollar bills with a very successful coin in 1989, the "loonie." It is bronze in appearance, has no milled edge, and sports a polygonal shape. Its eleven sides give it a distinctive feel when fumbling for change at a bus stop or a newspaper stand. So perhaps redesigning the U.S. coin to have a unique *shape* would aid its acceptance in everyday circulation.

Suppose then that you have been appointed by the Department of the Treasury to oversee the creation of a new one-dollar coin for circulation in the U.S. You quickly realize that many parties have great interest in the design and shape of the new coin.

- The Secretary of the Treasury, mindful of the lukewarm reception for the two recent round coins, and aware of the success of the hendecagonal Canadian dollar, directs you to design a coin with a polygonal shape.
- The vending industry is very concerned with the diameter of the coin, so representatives lobby your office intensely to adopt a value that won't force them to retool their machines.
- Sculptors planning to submit designs for the faces would like to have the largest possible area for their work.
- Federal law requires that certain phrases appear on the coin, like E PLURIBUS UNUM and IN GOD WE TRUST, as well as the year of issuance. But to give the sculptors more room, you plan to inscribe as much of this information as possible on the *edge* of the coin, not on its faces. Thus, you need a coin with large perimeter.

To satisfy all these interests, you therefore need to design a coin with a polygonal shape, fixed diameter, maximal area, and large perimeter. Is it possible to satisfy all these demands? Are regular polygons optimal? Does the answer depend on the number of sides you choose for the coin?

Your worries would certainly be over if you could ignore both the restriction on the shape and the requirement on the diameter. In that case, you could first determine the perimeter from the length of the required inscriptions, then determine a shape with the largest possible area given this perimeter. This is the *isoperimetric problem* in the plane, and it is well known that the circle is the optimal configuration. This problem was first studied extensively by the great Swiss geometer Jakob Steiner (in [**25**], for instance), though it was Karl Weierstrass who supplied the first complete proof of the optimality of the circle (see also Viktor Blåsjö's piece "The Isoperimetric Problem" in the June–July 2005 issue of this MONTHLY [**5**]).

The requirement for a polygonal shape comes directly from the Secretary, however, so this demand is unlikely to be negotiable. Suppose then that you remove only the diameter restriction, fix the number of sides at n, and again determine the required perimeter first. Then you need to determine an n-gon with fixed perimeter and maximal area. This is the *polygonal isoperimetric problem*, and here again the solution is simple. It is well known that for any fixed n the regular polygon alone has maximal area among all n-gons with fixed perimeter. We supply a simple proof of this in the next section.

You realize, however, that shaking even a *single* vending machine is a risky proposition, so rattling the entire industry may not exactly be conducive to your health. You wisely decide therefore to attack the problem with the full set of constraints. You find then that you need to solve two *isodiametric problems* for polygons, one for the area and one for the perimeter. More precisely, you need to answer the following questions for a fixed integer n (≥ 3).

- What is the maximum area of a polygon with n sides and fixed diameter?
- What is the maximum perimeter of a convex polygon with n sides and fixed diameter?

Both of these questions are discussed in [**8**, Problem B6]. Solutions to the corresponding problems for planar curves of fixed diameter have long been known, and the circle is again the optimal configuration in both cases. The area problem was resolved by Ludwig Bieberbach in 1915 [**3**], the perimeter question by Artur Rosenthal and Otto Szász shortly thereafter [**23**].

The isodiametric problems for polygons were first studied by Karl Reinhardt, Bieberbach's first student, in 1922 [**22**]. He solved the area problem for odd values of n, showing that the regular n-gon is best possible. Then, in an appendix that seems to have been missed in some of the later literature, he proved that the regular n-gon is never optimal when n is an even number and $n \geq 6$. He also solved the perimeter problem for all but a very thin set of positive integers. Some more details on the history of these problems appear later in this article, along with Reinhardt's proofs, including the sometimes overlooked result on the area for even n.

Later, we look more closely at the area problem and study how we might construct some better polygons when the number of sides is even. We investigate the perimeter problem, too, and examine the question of whether a single polygon can have both maximal area and optimal perimeter for its diameter. Then, armed with more knowledge about these isodiametric problems for polygons, maybe we can make a good recommendation to the Secretary on the shape of a new \$1 coin.

2. THE ISOPERIMETRIC PROBLEM FOR POLYGONS. We begin with a simple proof that the regular polygon is optimal in the isoperimetric problem. Many proofs of this are known (see, for instance, [**2**], [**6**], [**9**], or [**11**]), and several of these arguments are described very nicely by Blåsjö [**5**]. The proof here relies on two well-known formulas in classical geometry linking the perimeter and area of a polygon: Heron's formula for triangles and a generalization for quadrilaterals.

Heron's Formula. The area A of a triangle having side-lengths a, b, and c satisfies

$$A^2 = s(s-a)(s-b)(s-c),$$

where s denotes half the perimeter of the triangle, $s = (a+b+c)/2$.

Generalized Brahmagupta's Formula. The area A of a quadrilateral having side-lengths a, b, c, and d satisfies

$$A^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\theta,$$

where s denotes half the perimeter, $s = (a+b+c+d)/2$, and θ denotes the average of a pair of opposite interior angles of the quadrilateral.

The seventh-century Indian mathematician Brahmagupta obtained this formula in the special case when the last term vanishes. It is well known that this occurs precisely when the quadrilateral is cyclic, that is, when it can be inscribed in a circle. The general formula was apparently first noted in 1842 by Bretschneider [**7**] and Strehlke [**26**]; a succinct proof can be found in [**14**, p. 250].

Theorem. *Among all convex polygons with n sides and fixed perimeter, the regular polygon alone has the largest area.*

Proof. Suppose that P is a convex polygon with n sides and perimeter L. If P is not equilateral, then find two adjacent edges of different lengths, and adjust the middle vertex of $T = \triangle uvw$ as in Figure 1a to create a new triangle T' that corrects this imbalance. Using Heron's formula, we see that

$$\begin{aligned} 4A(T)^2 &= s(-a+b+c)(a-b+c)(s-c) \\ &= s(c^2-(a-b)^2)(s-c) \\ &< sc^2(s-c) = 4A(T')^2, \end{aligned}$$

so P does not have maximal area.

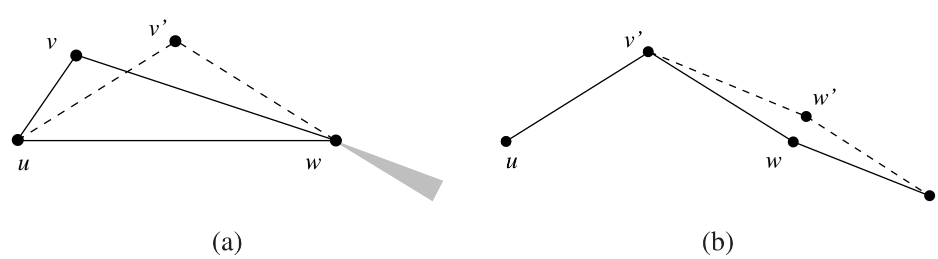


Figure 1.

Note that the polygon obtained here is not convex if the vertex that follows w in Figure 1a lies in the shaded zone shown, but if this is the case, the reflection illustrated in Figure 1b transforms it into a convex polygon with the same perimeter and larger area.

Next, suppose that P is equilateral but not equiangular, and find two adjacent vertices for which the interior angles in P are not equal. Adjust these vertices as in

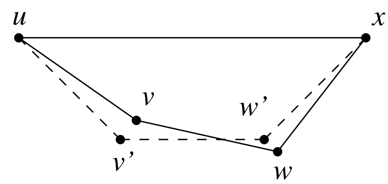


Figure 2.

Figure 2 to balance the angles. Then the parallelogram obtained has the property that two opposite angles sum to π, so the last term in the generalized Brahmagupta's formula now vanishes, while the rest of the formula is undisturbed. Once again P is not optimal.

One still needs to establish that a maximum exists: this was the element missing from Steiner's proofs and was the piece first supplied by Weierstrass. A standard compactness argument takes care of this (Weierstrass's proof can be found in [**5**]). ■

After a quick computation of the area of a regular n-gon having perimeter L, we can rephrase this result as an inequality.

Isoperimetric Inequality for Polygons. The area A of a convex polygon with n sides having perimeter L satisfies

$$A \leq \frac{L^2}{4n} \cot(\pi/n),$$

with equality only for the regular n-gon.

In particular, then, the area satisfies $A < L^2/4\pi$. But the small gap between these two bounds proves useful later in our discussion of the isodiametric problems for polygons. Also, it's worth noting that some proofs of the full isoperimetric inequality for planar curves ($4\pi A \leq L^2$) use this result for polygons, employing a polygonal approximation at some step. Edler's proof [**12**] in fact uses such a strategy; a nice explanation can be found again in [**5**].

3. REULEAUX POLYGONS. Before turning to our isodiametric problems, we need to introduce some interesting geometric shapes that have very useful properties. A planar figure has *constant width* if its height never varies as it rolls across the floor. More formally, a convex, closed set C in the plane has constant width d if every pair of parallel lines supporting C are distance d apart. The circle certainly has this property, but there are many other shapes that enjoy it as well, including the *Reuleaux polygons*, named for the nineteenth-century German mechanical engineer Franz Reuleaux. A Reuleaux polygon is defined as a set of constant width whose boundary consists of a finite number of circular arcs of the same radius. Note that a Reuleaux polygon is not a polygon in the traditional sense, since its edges are not line segments, and that the circle is in fact a special case of a Reuleaux polygon. Reuleaux polygons are employed in coinage, too: the British twenty- and fifty-pence coins are regular Reuleaux heptagons.

We need three important facts about Reuleaux polygons. Additional properties of these shapes, and other sets of constant width, can be found in [**13**, chap. 7]. Note that since the diameter of a (closed) set is the largest possible distance between two points selected from the set, the diameter of a Reuleaux polygon is the same as its width.

(1) *The boundary of a Reuleaux polygon consists of an odd number of circular arcs.*

Proof. Suppose that R is a Reuleaux polygon with diameter d, and that v and w are adjacent vertices of R having distance d from another vertex u. Draw a circular arc of radius d centered at each of u, v, and w to obtain the dashed shape shown in Figure 3, which encompasses R. Draw the vertical dashed line from u to the middle of the arc vw. Then every vertex of R on one side of this line must have distance d from exactly two vertices on the other side of the line, except for v and w, which each have one such vertex on the line. It follows that R has the same number of vertices on each side of the line, and adding u makes an odd number. ■

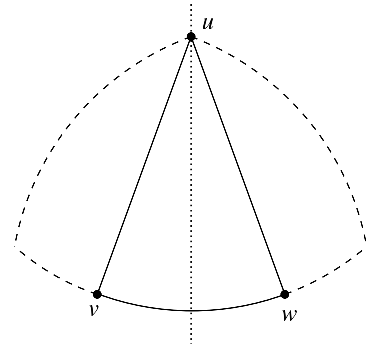


Figure 3.

(2) *A Reuleaux polygon with diameter d has perimeter πd.*

Proof. The set of line segments of length d that connect two vertices of such a Reuleaux polygon is a circuit forming a star with n points, and the sum of the angles at the points of such a star is π. ■

(3) *If P is a polygon with diameter d, then there exists a Reuleaux polygon with diameter d containing P.*

Proof. Let x and y be vertices of P that are distance d apart, and draw the line L that passes through x and y. Choose a side of L (henceforth called the "left" side), and draw circular arcs to that side of L with centers x, y, and those vertices of P to the right of L, as in Figure 4a. Mark the points of intersection. Then draw arcs of radius d on the right side of L centered at the intersection points of the *arcs* to the left side of L, as in Figure 4b. The figure composed of all the circular arcs is a Reuleaux polygon of diameter d that contains P. ■

Figure 5 shows the Reuleaux polygons constructed by this method using some regular polygons. It's evident that the figures obtained when the number of sides is even are quite different from those made when it is odd, and this discrepancy is ultimately the source of the difference in the even and odd cases of the isodiametric problem for the area. We next look at Reinhardt's results for these problems.

4. ISODIAMETRIC PROBLEMS AND KARL REINHARDT. Reinhardt presented a nice geometric solution to the perimeter problem when the number of sides is odd.

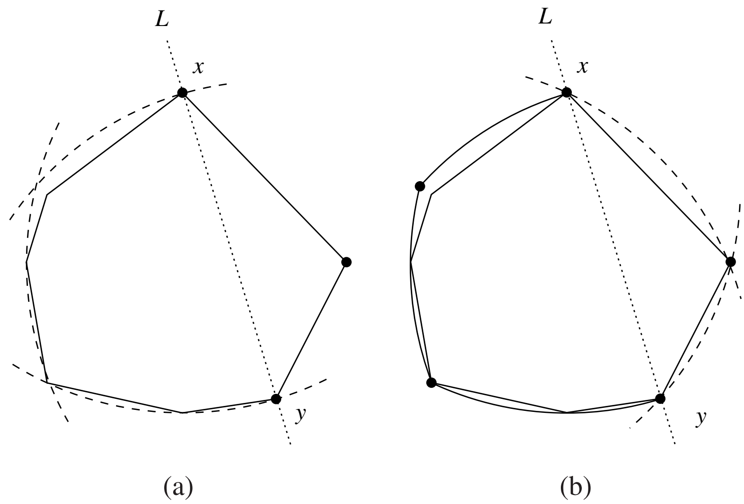


Figure 4. Constructing a Reuleaux polygon.

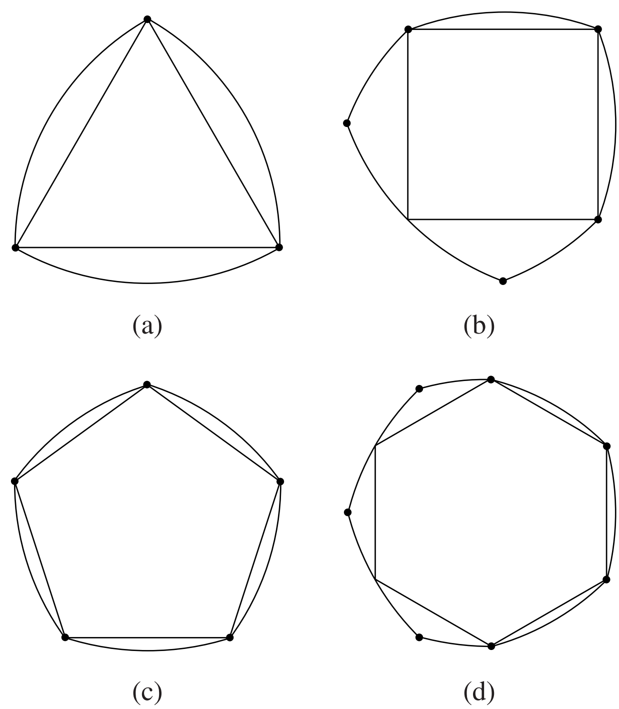


Figure 5. Reuleaux polygons constructed from some regular polygons.

Theorem (Reinhardt). *Suppose that n is an odd integer. Among all convex polygons with n sides and fixed diameter, the regular polygon has the largest perimeter.*

Proof. Assume that P is a convex polygon with diameter d and an odd number n of sides. Construct a Reuleaux polygon R of diameter d containing P using the method we described in section 3. We can assume that P is inscribed in R, since otherwise we could create a polygon with the same diameter and larger perimeter by moving a vertex to the boundary of R. Since R has perimeter πd, we can form a semicircle of radius d by realigning all the arcs on the boundary of R in succession into one large arc. (Start with a vertex of R that is also a vertex of P.) Mark the vertices of P where they occurred on the component arcs of the semicircle, and connect these points with line segments to form a polygonal path ℓ consisting of n line segments. Figure 6 shows the semicircle and path obtained when P is the regular hexagon of Figure 5d. Here, we've started at the top vertex of the hexagon and proceeded around the boundary clockwise to form the semicircle.

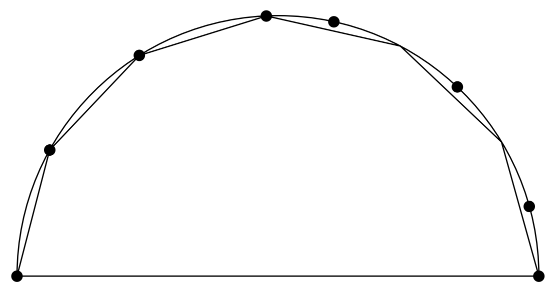


Figure 6.

Notice that the first three line segments in ℓ here have exactly the same length as the corresponding edges of the original hexagon. The last three segments in ℓ, however, are slightly longer than their counterparts in the polygon, since the realignment of arcs in these cases stretches these segments slightly. Figure 7 illustrates this stretching phenomenon, which occurs whenever adjacent vertices of P lie on different arcs of R.

Next, it is straightforward to show that a path composed of a fixed number of line segments inscribed in a semicircle has maximal length precisely when the segments have equal length. It follows therefore that P has maximal perimeter when P is equilateral and each vertex of R is also a vertex of P. This is certainly the case for the regular n-gon. ■

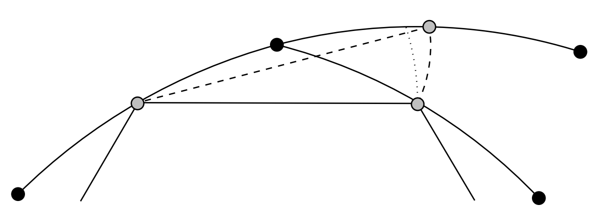


Figure 7.

Reinhardt's construction gives us quite a bit more than just the solution to the perimeter problem for odd n. More broadly, it resolves this problem whenever n has an odd factor. To see this, suppose that m is a nontrivial odd factor of n. Construct the regular Reuleaux polygon R with m sides, subdivide each of its bounding arcs into n/m subarcs of equal length, and let P be the convex hull of these points. Then P is equilateral, and R has no extra vertices lying outside P, so P has maximal perimeter. Figure 8a illustrates the case $n = 6$ and $m = 3$.

Also, we see that when n is odd the optimal configuration in this problem is unique only when n is prime. Figures 8b and 8c show the two enneagons (nine-sided polygons) with unit diameter and maximal perimeter, one inscribed in a regular Reuleaux

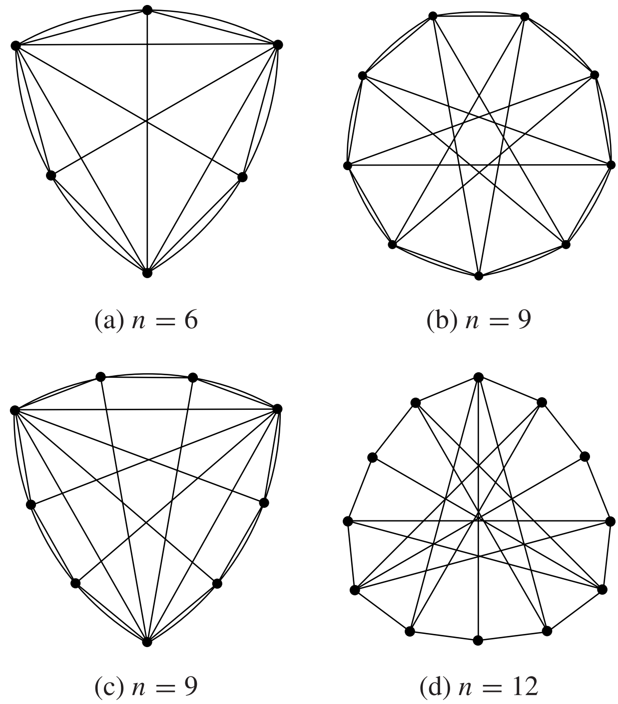


Figure 8. Some polygons with maximal perimeter.

enneagon and the other in a Reuleaux triangle, and Figure 8d exhibits one of the two dodecagons with this property (the other is obtained by subdividing the Reuleaux triangle). The Reuleaux polygon for this last figure is not even equilateral! But when n is even, the regular n-gon is never among the optimal configurations.

In addition, we can solve the isodiametric problem for the area when n is odd by combining the perimeter result with the isoperimetric theorem for polygons.

Theorem (Reinhardt). *Suppose that n is an odd integer. Among all convex polygons with n sides and fixed diameter, the regular polygon has the largest area.*

Proof. Assume that P has n sides and diameter d but that P is not regular. Then the regular n-gon with diameter d has equal or larger perimeter. If it is larger, then dilate P to create a polygon P' with the same shape as P and with perimeter equal to that of the regular n-gon with diameter d. Because the regular n-gon alone has maximal area among all polygons with the same perimeter, the area of P is strictly less than the area of the regular n-gon. ∎

Finally, we obtain two useful inequalities for polygons. The first bounds the perimeter of an n-gon in terms of its diameter and comes from computing the length of an equilateral path with n segments inscribed in a semicircle. The second relates the area to the diameter and results from combining the first inequality with the isoperimetric inequality for polygons. Other proofs of inequality (1) and closely related inequalities appear for example in [**17**], [**19**], or [**28**].

Isodiametric Inequalities for Polygons. If a convex polygon has n sides, diameter d, perimeter L, and area A, then

$$L \le 2dn\sin(\pi/2n), \tag{1}$$

and

$$2A \le d^2 n\cos(\pi/n)\tan(\pi/2n). \tag{2}$$

Equality is achieved in (1) if n has an odd factor, and the regular n-gon is optimal if n is odd. In (2), equality is achieved when n is odd only by the regular n-gon.

As with the isoperimetric inequality, these inequalities for polygons are somewhat stronger than the classical isodiametric inequalities for convex curves in the plane, $L \le \pi d$ and $4A \le \pi d^2$. These small refinements will prove useful in our later investigations.

Later we return to the perimeter problem in the remaining case when n is a power of 2, but first we turn to the area problem and a hidden result of Reinhardt. Just after his proofs for the odd case, Reinhardt writes [**22**, p. 259]:

> It is clear that the final argument breaks down if n is an even number. In this case the problem remains unsolved.

He then investigates the various convex n-gons with fixed diameter that exhibit the maximal perimeter. This is a long section of his article, treating different cases that depend on the factorization of n. But after this, at the end of the paper, he abruptly returns to the isodiametric problems for the case of even n, showing that the regular n-gon never achieves the maximal area or perimeter when n is even and at least six. No hint of this result appears earlier in the article, so it seems possible that this epilogue was added after the original manuscript was prepared for publication. As a result, it is quite easy to overlook.

We can describe the essence of Reinhardt's argument with a diagram of a regular octagon in Figure 9a. The same process works for any regular n-gon when n is even and $n \ge 6$.

In the figure, v and w are opposite vertices in the regular octagon P with diameter d. Let P' be an octagon of the same diameter obtained by replacing v and w with v' and w' a short distance away, as shown. Then P and P' have the same area, but P' has

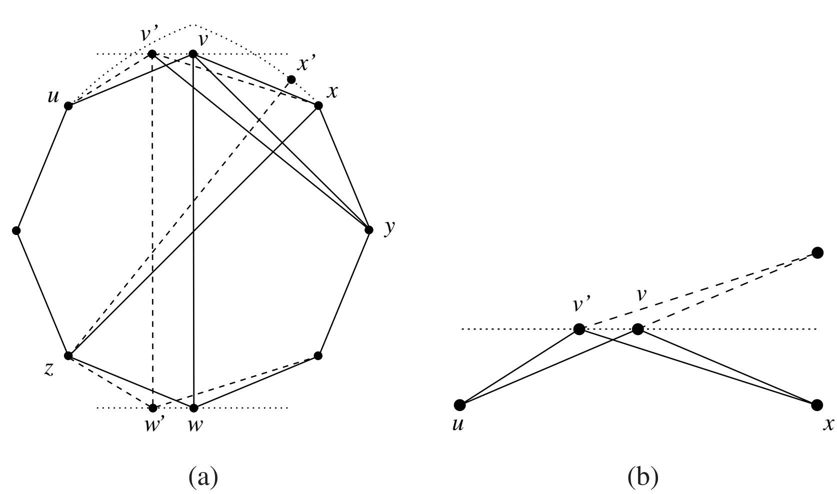


Figure 9.

larger perimeter. This is easy to see by imagining a mirror along the line connecting v and v', as in Figure 9b. In addition, the segment $v'y$ is not perpendicular to xz. Consequently, we can replace x with a vertex x' lying up the circular arc of radius d drawn about z in such a way that the line $x'z$ intersects $v'y$ in an angle closer to $\pi/2$, while still maintaining the same diameter. This makes another polygon P'' with diameter d and larger area.

We can summarize these results in the following theorem:

Theorem (Reinhardt). *Let n be an even integer greater than five. Among all convex polygons with n sides and diameter d, the regular n-gon has neither maximal area nor maximal perimeter.*

In the perimeter problem, the only new information surfaces when n is a power of 2, so with this result we can slightly strengthen our earlier statement: the regular n-gon has the optimal perimeter if *and only if* n is odd.

These results of Reinhardt for even n have since been rediscovered. In the perimeter problem, Tamvakis [**27**] showed in 1987 that the regular 2^m-gon is not optimal by constructing the polygons T_n that we describe in section 6. The area problem has seen much more interest. In 1956, Lenz [**19**] investigated the problem of determining the smallest number of convex sets of bounded diameter required to cover a given convex set in the plane. One bound he derived depends on the values of the maximal area of an n-gon with unit diameter for $n = 5$, 6, and 7. In the same year, he posed the question of determining the maximal area for even n as an unsolved problem in the Swiss journal *Elemente der Mathematik* [**18**], where Reinhardt's contribution for the even case is not mentioned. Two years later, Schäffer [**24**] supplied a very nice short proof that the regular n-gon is not optimal for even values of n (≥ 6), apparently without knowledge of Reinhardt's work. We omit the details of Schäffer's proof, but it's easy to describe the idea. Center an n-gon at the origin so that two vertices lie on the x-axis. Pull each of the vertices in the upper half-plane radially away from the origin by a short distance ϵ, and push the ones in the lower half-plane radially toward the origin by the same amount. The resulting polygon has larger area if $n \geq 6$, and the diameter is undisturbed if ϵ is small.

5. AREA CODE. So what are the optimal n-gons in the area problem when n is even? From Reinhardt and Schäffer we learn that the regular n-gon is not the answer, but we would like to know more. Can we explicitly construct polygons with area substantially closer to the bound given by the isodiametric inequality (2)?

Let's study the problem first for some small values of n. The case $n = 2$ doesn't look particularly interesting, since these aren't so much polygons as *digons* or perhaps *bigons*, so maybe we should just let bigons be bigons.[1] The case $n = 4$ is simple to analyze, for the area A of a convex quadrilateral depends only on the lengths d_1 and d_2 of its diagonals and their angle θ of intersection: $2A = d_1 d_2 \sin\theta$. Taking $d_1 = d_2 = d$, choosing $\theta = \pi/2$, and taking care to ensure that the intersection point is sufficiently central so that the diameter is d, we see there are infinitely many quadrilaterals with diameter d and maximal area $d^2/2$, including the square. This is, without a doubt, a four-gon conclusion.

For the case $n = 6$, Bieri proved in 1961 that the hexagon shown in Figure 10 has maximal area among all hexagons with fixed diameter that possess axes of symmetry [**4**]. Its area is about 3.92% larger than that of the regular hexagon. In 1975, Graham,

[1] Linguists undoubtedly prefer *digons*, since this is, etymologically speaking, purely Greek, while *bigons* mixes Latin and Greek roots. But digons just aren't as funny.

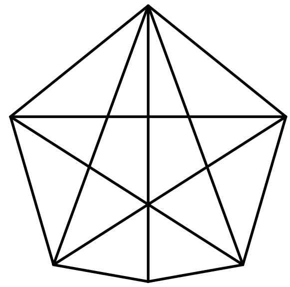
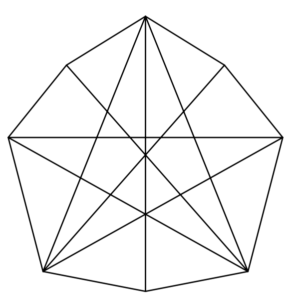


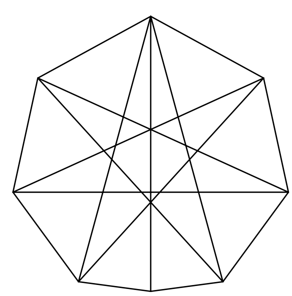
Figure 10. The optimal hexagon.

motivated by Lenz's question, proved that this hexagon is in fact optimal among all hexagons with fixed diameter [**15**]. We define the *skeleton* of a polygon with diameter d as the collection of its vertices, together with all the line segments of maximal length connecting any two of its vertices. For example, the skeleton of a regular hexagon is an asterisk ($\ast$), while the skeleton of the Bieri-Graham hexagon has a pinwheel shape: a five-pointed star, with an extra line segment (Figure 10). Graham conjectured that the optimal n-gon in general has a similar skeleton when n is even: a circuit of length $n-1$, together with a single additional edge connecting to the remaining vertex.

More recently, Audet, Hansen, Messine, and Xiong [**1**] verified Graham's conjecture for the case $n = 8$, showing that the optimal octagon has a skeleton of this form. It is interesting that other arrangements can also produce octagons with area larger than that of the regular octagon. Figure 11 shows two octagons with markedly different skeletons: the first has area about 0.52% larger than the regular octagon; the second is the optimal octagon, about 2.79% larger than the regular one.



(a) Improved area.



(b) Optimal area.

Figure 11. Better octagons.

In hope of gaining some insight into the general case, we can try to construct some improved n-gons for larger values of n. We assume a skeleton with the pinwheel shape of Graham's conjecture, and we further assume the presence of an axis of symmetry, like we see in the optimal hexagon and octagon. In general, we can describe such a $2m$-gon by selecting $m-2$ parameters. Figure 12 shows the strategy for the case $m = 4$: choosing α_1 determines the angle at the top of the pinwheel, and α_2 is the next angle we meet as we visit the vertices in order in tracing out the skeleton. After this, the angle at v_3 is determined by the constraint on the length of the horizontal line segment connecting v_4 and v_5.

We can then write down a lengthy formula for the area in terms of these $m-2$ angles and search for good solutions by using some numerical optimization software

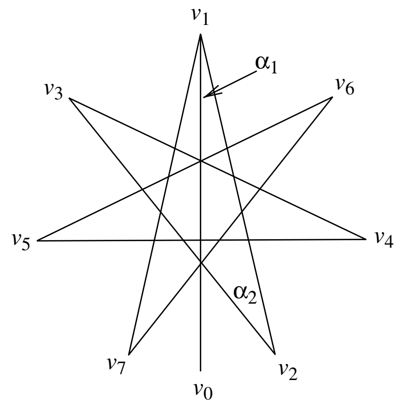


Figure 12. Constructing a symmetric $2m$-gon given $m-2$ angles.

(see [**20**] for more details). Figure 13 depicts improved polygons created by following this strategy for even n in the range $10 \le n \le 20$, with the assistance of Mathematica. Table 1 summarizes the improvement obtained in the area in each case, choosing $d = 2$ so that the limiting area is simply π. In the table, $A(P_n)$ denotes the area of the regular n-gon, $A(Q_n)$ is the area of the polygon constructed, M_n is the upper bound from the isodiametric inequality (2), and the last column shows the percentage increase in the area of Q_n over P_n.

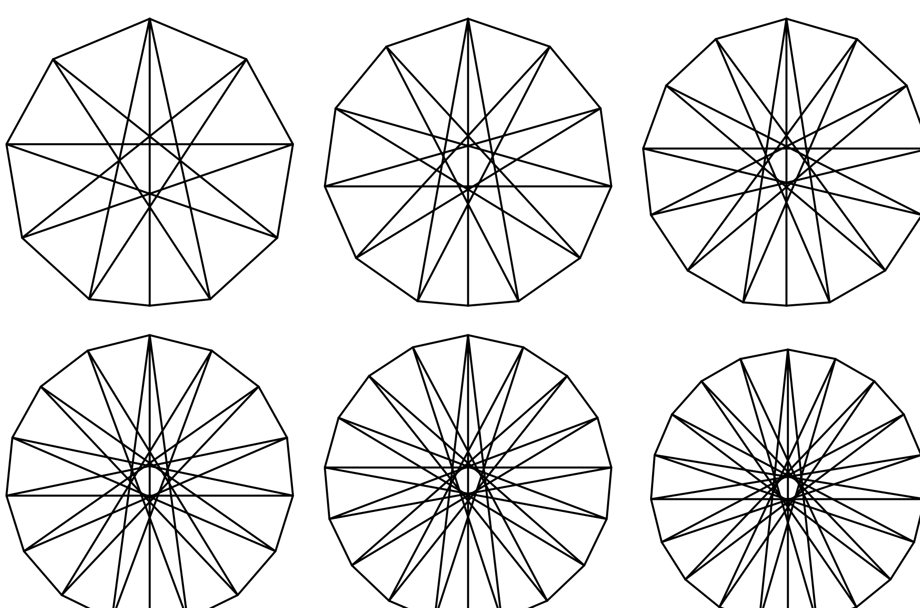


Figure 13. Improved polygons Q_n for $n = 10$ through $n = 20$.

It is not surprising that the percentage area gained in the improved polygons diminishes with n. After all,

$$M_n = \pi - \frac{5\pi^3}{12n^2} + \frac{\pi^5}{120n^4} + O\left(\frac{1}{n^6}\right),$$

Table 1. Areas of the improved polygons.

n	$A(P_n)$	$A(Q_n)$	M_n	Gain
6	2.5981	2.6999	2.7846	3.92%
8	2.8284	2.9075	2.9403	2.79%
10	2.9389	2.9965	3.0127	1.96%
12	3.0000	3.0429	3.0520	1.43%
14	3.0372	3.0701	3.0757	1.08%
16	3.0615	3.0874	3.0912	0.85%
18	3.0782	3.0992	3.1017	0.68%
20	3.0902	3.1074	3.1093	0.56%

and, for even n,

$$A(P_n) = \pi - \frac{2\pi^3}{3n^2} + \frac{2\pi^5}{15n^4} + O\left(\frac{1}{n^6}\right),$$

so $M_n - A(P_n) \sim \pi^3/4n^2$. Still, it is interesting that $A(Q_n)$ appears to approach the upper bound significantly faster than $A(P_n)$ does as n increases. In fact, a least-squares fit for n satisfying $10 \leq n \leq 20$ suggests that the quantity $M_n - A(Q_n)$ behaves like $20.2/n^{3.10}$, so maybe studying these figures can point us to a more general quantitative improvement in the isodiametric area problem.

The shapes in Figure 13 all appear to be strikingly symmetric—in each shape, the angles at the points of the star inside the polygon look very similar, especially for larger n. In fact, though, the angles $\alpha_2, \alpha_3, \dots, \alpha_{m-2}$ vary a bit, so that the lengths of the polygon's edges decrease if we trace the boundary from north to south. Nevertheless, it looks like we can get a reasonable first approximation by choosing all the angles of the star to have the same value. The star has $n-1$ points, so this makes for $\pi/(n-1)$ at each point. Now the convex hull of such a star is simply a regular $(n-1)$-gon, and adding the handle of the pinwheel splits one edge, pulling its midpoint outward by a short distance. We call the shape we obtain here R_n. Then a short calculation reveals that its area is given by

$$\begin{aligned} A(R_n) &= 2\tan\left(\frac{\pi}{2n-2}\right)\left((n-2)\cos\left(\frac{\pi}{n-1}\right) + 2\cos\left(\frac{\pi}{2n-2}\right) - 1\right) \\ &= \pi - \frac{5\pi^3}{12n^2} - \frac{7\pi^3}{12n^3} + O\left(\frac{1}{n^4}\right), \end{aligned}$$

implying that $M_n - A(R_n) \sim 7\pi^3/12n^3$, which seems to confirm our earlier estimate.

Could R_n ever be optimal? The quadrilateral R_4 is indeed one of the infinitely many optimal arrangements for this value, but R_n is certainly not the best for other small numbers, since the examples constructed earlier are all better. But what happens for larger n? The answer, as one might expect, is no, and this can be proved with a more complicated construction. The details appear in [**20**], but we can describe the main idea here.

In the first approximation to the examples of Figure 13, we chose all the angles of the star to be the same. In fact, however, the angle at the top of the star is significantly larger than $\pi/(n-1)$, and the sequence $\alpha_2, \dots, \alpha_{m-2}$ shows an oscillating pattern, with the even indices exhibiting values somewhat larger than the odd indices, and this wobbling attenuates as one reads down the sequence. As a second approximation, we can try to incorporate more of this structure into the strategy and optimize over three free parameters, call them α, β, and γ: set $\alpha_1 = \alpha$, $\alpha_2 = \beta + \gamma$, $\alpha_3 = \beta - \gamma$, and $\alpha_k = \beta$ for all the rest. Optimizing the area here lets us construct a polygon S_n with area larger than that of R_n when $n \geq 6$. In fact, analyzing the area of S_n lets us reduce the $7/12$ from the $1/n^3$ term of $A(R_n)$ to a number slightly smaller than $7/15$. Table 2 shows the areas of the polygons R_n and S_n for some small n, along with the area of the best known polygons Q_n from Figure 13.

Table 2. Areas of the improved polygons.

n	$A(R_n)$	$A(S_n)$	$A(Q_n)$
10	2.99303	2.99612	2.99655
12	3.04079	3.04259	3.04292
14	3.06875	3.07001	3.07012
16	3.08651	3.08735	3.08745
18	3.09849	3.09911	3.09915
20	3.10695	3.10740	3.10744

6. SECURING THE PERIMETER. Reinhardt's theorem tells us the optimal value of the perimeter for a convex n-gon of fixed diameter in every case except when n is a power of two. Here, we know that the regular n-gon is not optimal, and Reinhardt showed that the upper bound in the inequality (1) is never attained in this case. Can we construct optimal polygons, or at least improved ones, for these stubborn cases? The solution for $n = 4$ turns out to be easy to describe: our old friend R_4, shown in Figure 14, is the unique solution.

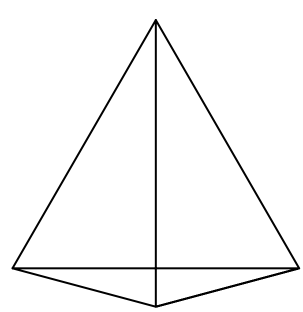


Figure 14. The quadrilateral R_4.

Theorem. *The unique convex quadrilateral with fixed diameter d and maximal perimeter is R_4, whose perimeter is*

$$L(R_4) = 2d\left(1 + \sqrt{2 - \sqrt{3}}\right).$$

Proof. Let Q be a quadrilateral with fixed diameter d. We can assume that the diagonals of Q both have length d and that they meet at right angles. Let r and s denote the lengths of the two longest line segments connecting the point of intersection of the diagonals with vertices of Q, labeled so that $r \geq s$. Because Q has diameter d, we must have $r^2 + s^2 \leq d^2$. But increasing either r or s alone increases the perimeter, as we can verify using a geometric argument like the one of Figure 9b, so we can assume that $r^2 + s^2 = d^2$. Accordingly, r must lie between $(\sqrt{2}/2)d$ and $(\sqrt{3}/2)d$, and we can check that the perimeter is increasing in this range. The quadrilateral R_4 corresponds to the choice $r = (\sqrt{3}/2)d$. ■

For higher powers of two, we might hope that that the polygon R_n would come to our aid again. Let $\mathcal{M}_n$ denote the upper bound on the perimeter in the isodiametric inequality (1). A straightforward computation shows that

$$\mathcal{M}_n - L(P_n) \sim \frac{\pi^3 d}{8n^2},$$

whereas

$$\mathcal{M}_n - L(R_n) \sim \frac{5\pi^3 d}{96n^3},$$

so in general the R_n are indeed better. Once again, however, they are not optimal when $n > 4$. To see this, we construct one more family of polygons.

Given a positive integer n (≥ 3), write $n = 3q + r$ with $r = 0$, 1, or 2. Subdivide r of the bounding arcs of a Reuleaux triangle into $q + 1$ subarcs of equal size, and subdivide the remaining $3 - r$ arcs into q subarcs of equal size. Let T_n denote the polygon with n sides obtained as the convex hull of the endpoints of these subdivisions. Thus $T_4 = R_4$, Figures 8a and 8c show T_6 and T_9, while T_8 divides two arcs into three pieces and one into two segments. If n is divisible by 3, then certainly T_n is optimal. Otherwise, it's easy to check that

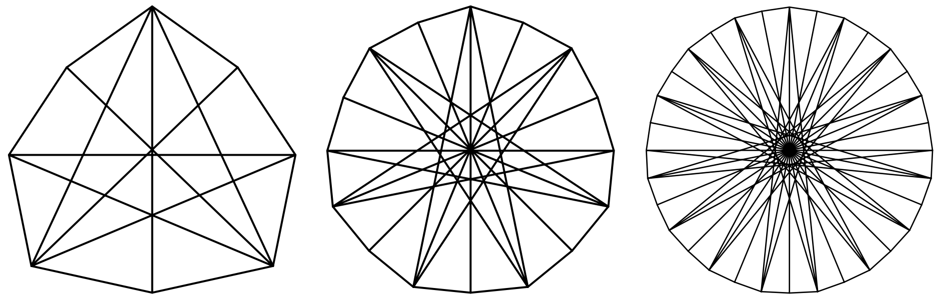
$$\mathcal{M}_n - L(T_n) \sim \frac{\pi^3 d}{4n^4},$$

meaning that these polygons are already better than the R_n. Tamvakis [**27**] introduced these polygons, asking whether T_n is optimal for all powers of two.

We can answer this question. The octagon Q_8 from Figure 11b, which is optimal in the area problem, already has larger perimeter than T_8 ($L(Q_8) = 3.11924\ldots$ and $L(T_8) = 3.11905\ldots$ when $d = 1$), and if we adjust the angles of the pinwheel slightly to maximize the perimeter rather than the area, we can build an octagon with still larger perimeter (3.11959...). But we can do even better! By optimizing the shape of Figure 11a for the perimeter, we can make the octagon shown in Figure 15a. Its perimeter is 3.12114..., which is more than 99.99% of the upper bound. A similar construction shows that T_{16} isn't optimal either. This time, the optimized pinwheel falls just short, but the hexadecagon shown in Figure 15b is better, achieving more than 99.9998% of the theoretical maximum. Likewise, Figure 15c illustrates a polygon with thirty-two sides (a triacontakaidigon, if you prefer) with perimeter larger than T_{32}. In general, by selecting parameters carefully, we can create a polygon V_n that satisfies

$$\mathcal{M}_n - L(V_n) \sim \frac{\pi^5 d}{16n^5}$$

(see [**20**] for more details).



(a) $n = 8$ (b) $n = 16$ (c) $n = 32$

Figure 15. Improved perimeters.

This brings to mind a question raised in the introduction: Is it possible to find a polygon with fixed diameter having both maximal area *and* maximal perimeter? We can answer this now, at least in part: the regular polygon optimizes both quantities when the number of sides n is odd, and R_4 is the unique solution for quadrilaterals. Also, it's easy to check that the perimeter of the Bieri-Graham hexagon is smaller than that of the optimal hexagon in the perimeter problem, by about 0.194%, and our calculations on octagons rule out the case $n = 8$. What about larger even values of n?

Little appears to be known about this, but we can take care of one substantial case. If $n = 2p$, with p an odd prime, then it follows from Reinhardt's paper that there is a *unique* convex n-gon with maximal perimeter. To construct it, we simply split each edge of the regular p-gon, just as we did in Figure 8a for $p = 3$. The result follows by verifying that the area of R_{2p} is always larger than that of the polygon with optimal perimeter. For other even values of n it's harder to check, since we don't know the optimal area, and there is more than one polygon that achieves the maximal perimeter. It seems reasonable that $n = 4$ is the only case where a single polygon with an even number of sides is optimal in both isodiametric problems. Also, since Reuleaux polygons arise naturally in the perimeter problem, it's worth noting here that the Reuleaux triangle has the *minimal* area among all sets of constant width in the plane. This is the Blaschke-Lebesgue theorem (see, for instance, [**16**]).

With all the constraints given to you in your charge to design a new coin for circulation in the U.S., you might well decide to recommend a regular n-gon with n odd. The vending industry, of course, prefers a shape with constant width, but a regular polygon with an odd number of sides probably has a tolerable variation in its width, at least when n is sizable. Still, you're likely to need at least seven or nine sides. On the other hand, with the triskaidekaphilic theme of many of the country's other symbols, maybe $n = 13$ is the right choice.

7. SHOW ME THE MONEY. The predicament described in the introduction is not entirely fictional. In February 2005, a bill entitled "The Presidential \$1 Coin Act of 2005" was introduced in the House of Representatives [**21**]. It would direct the U.S. Mint to begin issuing a new one-dollar coin beginning in 2007. The design would feature the Statue of Liberty on the reverse, and the name and likeness of a former U.S. president on the obverse. The featured chief executive would begin with George Washington and proceed chronologically through the presidents, four per year, just as the design on the reverse of the quarter now features each state in turn. (History buffs might be able to guess the only president to get two coins in the series.)

The bill passed the House handily in April 2005 (422 to 6) and was referred to the Senate. There it received a favorable report from the Committee on Banking, Housing, and Urban Affairs, and as of September 2005 it sat on the Senate legislative calendar. But with fully seventy-two senators cosponsoring the legislation, its fate seemed hardly in doubt.

United States Federal Code [**10**] prescribes the exact diameter of the $1 coin (1.043 inches), mandates a golden color and a "distinctive edge" for it, and requires "tactile and visual features that make the denomination of the coin readily discernible." Further, the new legislation explicitly stipulates that the phrases E PLURIBUS UNUM and IN GOD WE TRUST, as well as the year of issuance and any mint mark, be inscribed on the edge of the coin, in order to maximize space for sculptors and to promote a new "golden age" of coinage in the U.S. It also directs the Treasury Department to meet regularly with vending machine manufacturers, transit officials, armored car operators, banks, car wash operators (really), and other groups to help gauge demand and facilitate the adoption of the new coin.

But nothing in federal law, or in the proposed legislation, requires that the dollar coin be round.

Added in proof. The Presidential $1 Coin Act passed the Senate unanimously in November 2005 and was signed into law by President Bush on 22 December 2005.

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Music of the Spheres

Today was a rainbow day
Horizon leaned the right way
Tonight in moonlight
We sleep beneath the stars we sight
Oh, it's amazing
By the sea, star-gazing
My guiding inspiration all through the years
Has been music and the music of the spheres

Tides are getting strong
I get my guitar to sing you a song
The notes that bring peace
Are numbers from Ancient Greece
Oh, hear a pattern
In the path of Saturn
I marvel at the harmonies that caress our ears
Sweet music and the music of the spheres

I end my simple tune
While stars twinkle ‘round the moon
Like vibrating strings
We resonate with all these things
Oh, chords and notes,
Words and hopes,
Spinning in the myst'ry of why we cry our tears
For music and the music of the spheres

—Submitted by Lawrence Lesser, University of Texas at El Paso