Remainder Wheels and Group Theory

Lawrence Brenton

Lawrence Brenton (brenton@math.wayne.edu) was educated at the University of Pennsylvania, the University of Washington in Seattle, and the University of Bonn, Germany. He has served on the faculty of Wayne State University for 33 years, specializing in singularity theory in complex algebraic and analytic varieties. His teaching activities include directing the undergraduate research program for many years as well as mentoring graduate students in mathematics and mathematics education.

In the typical middle school mathematics curriculum a day or two is spent on "clock arithmetic." It is not entirely clear what it is that children are supposed to gain from this adventure. Perhaps they learn about military time—that's pretty cool—but I doubt that they acquire a fuller understanding of quotients and remainders, much less a meaningful introduction to number theory via the properties of the residue rings $\mathbf{Z}_n$ of integers mod n.

At the other end of the spectrum, many teachers' training programs now feature a "capstone" content area course. In the case of prospective mathematics teachers, the intent of this course is to pose to our college seniors the following question:

> *In order to prepare you for a career teaching school mathematics to children, we have required you to complete a rigorous curriculum in advanced calculus, linear and abstract algebra, real and complex analysis, geometry and topology, probability and statistics, discrete methods and applications. Why?*

This article gives a small example of an answer, based on my experience with such a capstone course at an urban research university.

Decimal clocks. Consider the problem, encountered perhaps in the fourth or fifth grade, of changing a fraction to a decimal by long division. It is hoped that most children at this level will acquire sufficient facility with this technique to be able to produce, for instance, the decimal representation $1/7 = .142857142857\ldots$. Several authors have suggested that the repeating nature of the decimal expansion can naturally be represented by writing the digits of the quotient on a six-hour clock, as in Figure 1. Already one question might occur to a bright and inquisitive student.

Question 1. *We know that the decimal will repeat, but is there a way to predict in advance how far we will have to go before the repetition occurs?*

Consider $2/7 = .\overline{285714}$. Will our bright and inquisitive scholar notice that these are exactly the same six digits, in the same order, only starting at a different place in the sequence? Indeed, with minimal encouragement a student might notice that all of the fractions 1/7, 2/7, 3/7, 4/7, 5/7, and 6/7 give the same six-hour clock, the only difference being the starting point: $3/7 = .\overline{\mathbf{4}28571}$, $4/7 = .\overline{\mathbf{5}71428}$, $5/7 = .\overline{\mathbf{7}14285}$,

$6/7 = \overline{\mathbf{8}57142}$. This jumps out at us if we label the hours of the clock in Figure 1 with both the fraction and the first decimal digit in its expansion (compare [**4**, Fig. 1]).

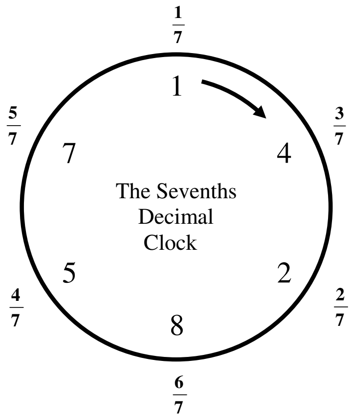

Figure 1. The decimal clock for denominator 7.

Let's try a different denominator.

$$
\begin{array}{rr}
1/13 = .\overline{\mathbf{0}76923} & 2/13 = .\overline{\mathbf{1}53846} \\
3/13 = .\overline{23\mathbf{0}769} & 5/13 = .\overline{3846\mathbf{1}5} \\
4/13 = .\overline{3\mathbf{0}7692} & 6/13 = .\overline{46\mathbf{1}538} \\
9/13 = .\overline{6923\mathbf{0}7} & 7/13 = .\overline{53846\mathbf{1}} \\
10/13 = .\overline{76923\mathbf{0}} & 8/13 = .\overline{6\mathbf{1}5384} \\
12/13 = .\overline{923\mathbf{0}76} & 11/13 = .\overline{846\mathbf{1}53}
\end{array}
$$

This time we have two sets, each representing half of the fractions $1/13, 2/13, \dots,$ $12/13$.

Question 2. *What is it about the fractions* $1/13$, $3/13$, $4/13$, $9/13$, $10/13$, *and* $12/13$ *that makes them "belong together," and similarly for the fractions* $2/13$, $5/13$, $6/13$, $7/13$, $8/13$, *and* $11/13$*?*

Why should prospective teachers study group theory? Many authors have analyzed the topic of repeating decimals from the point of view of number theory and the properties of modular arithmetic. An alternative approach is to invoke the language and concepts encountered in a beginning abstract algebra course.

The first step in dispelling the mysteries of repeating decimals is to switch our attention from the digits of the quotient to the sequence of remainders in the indicated long division. Here is the complete calculation for $1/13$, with the remainder at each step in bold.

$$
\begin{array}{rl}
 & .076923 \\
13 & \overline{\big)\,1.000000} \\
 & \underline{0} \\
 & \mathbf{10}0 \\
 & \underline{91} \\
 & \mathbf{9}0 \\
 & \underline{78} \\
 & \mathbf{12}0 \\
 & \underline{117} \\
 & \mathbf{3}0 \\
 & \underline{26} \\
 & \mathbf{4}0 \\
 & \underline{39} \\
 & \mathbf{1}
\end{array}
$$

We know that the decimal expansion repeats after these six digits because we have returned to our original division problem $1 \div 13$. Thus it is really the sequence of remainders, 10, 9, 12, 3, 4, 1, that is important, not the decimal digits .076923 of the quotient. This also shows why the fractions $1/13$, $10/13$, $9/13$, $12/13$, $3/13$, and $4/13$ all have the same six repeating decimal digits.

If we label the decimal clocks with the remainders, as in Figure 2, instead of the digits of the quotients, the relation between the two sets is laid bare. To emphasize the change in viewpoint, we will now depict our results as "wheels" rather than "clocks." Since we are dealing with remainders when dividing by 13, we now compute in the ring $\mathbf{Z}_{13}$ of integers mod 13. Here the product of two remainders x and y is the remainder when xy is divided by 13. Then the second wheel is obtained simply by multiplying the first by 2: $2 \times 10 = 20 \equiv \mathbf{7} \pmod{13}$, $2 \times 9 = 18 \equiv \mathbf{5} \pmod{13}$, and so on. This makes perfect sense because $2/13$ is just 2 times $1/13$, and the factor 2 carries through the division mod 13 by the distributive property.

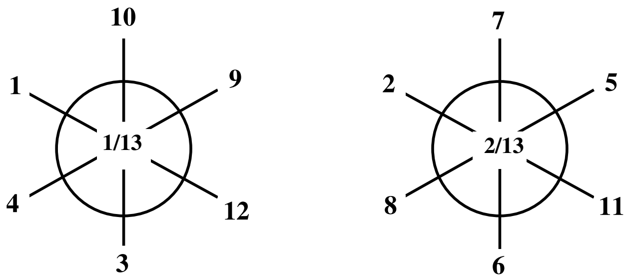

Figure 2. The two remainder wheels for denominator 13.

But why do the specific numbers 10, 9, 12, 3, 4, and 1 occur on the first wheel instead of some other numbers? Could we have predicted this without doing the division?

For the answer, we look again at the long division algorithm. At each step (so we teach our fourth graders) we "bring down a zero, divide by 13, and subtract to return the new remainder." But "bring down a zero" really means multiply by 10. That is, if r is the remainder at a particular step in the division, then the next remainder will be $10r \pmod{13}$. So starting with the first remainder, 10, the sequence of remainders in the decimal expansion of $1/13$ is $\mathbf{10} \pmod{13}$, $\mathbf{10^2} \pmod{13}$, $\mathbf{10^3} \pmod{13}$, $\ldots$.

This sequence repeats when we reach a power of 10 that returns to the starting point: $10^k \equiv 1 \pmod{13}$. In the language of group theory, the integer k is the *order* of 10 in the multiplicative group $\mathbf{Z}_{13}^*$ of non-zero elements of $\mathbf{Z}_{13}$, and the set $H = \{10, 10^2, 10^3, 10^4, 10^5, 10^6 \equiv 1\} \pmod{13}$ is the *cyclic subgroup* generated by 10. The set of elements of the second wheel, denoted $2H$ and containing all elements of $\mathbf{Z}_{13}^*$ of the form 2×10^j, is the *coset* of 2 modulo H.

Let's try $n = 41$. The cyclic group generated by 10 in the group $\mathbf{Z}_{41}^*$ is $H = \{10, 10^2, 10^3, 10^4, 10^5 \equiv 1\} = \{10, 18, 16, 37, 1\}$, and the cosets are $2H = \{20, 36, 32, 33, 2\}$, $3H = \{30, 13, 7, 29, 3\}$, $4H = \{40, 31, 23, 25, 4\}$, $5H = \{9, 8, 39, 21, 5\}$, $6H = \{19, \mathbf{26}, 14, 17, 6\}$, $11H = \{28, 34, 12, 38, 11\}$, and $15H = \{27, 24, 35, 22, 15\}$. Thus, for example, if we expand $26/41$ as a decimal, it follows that the expression will repeat after five steps. Furthermore, by locating 26 in the coset $6H$ we know that the remainders will be exactly the five elements in that coset, in the order 26, 14, 17, 6, 19, then back to 26.

The group $\mathbf{Z}_n^*$. We have already used some ideas from group theory: group, subgroup, order of an element, coset. Can we learn more by delving deeper? The examples so far call attention to the multiplicative groups $\mathbf{Z}_p^*$ of non-zero integers mod p, where p is prime ($p = 7$, 13, and 41 in our examples). What happens when the denominator is not prime?

Suppose we ask the students (fifth graders or collegians) to write down all the reduced proper fractions with denominator 39: 1/39, 2/39, ~~3/39~~, 4/39, 5/39, ~~6/39~~, 7/39, . . . , ~~13/39~~, 14/39, ...

Obviously we must cross out all those fractions whose numerators have a common factor with 39 (3/39 is "really" 1/13, so we already dealt with it when we studied fractions with denominator 13). The set of numerators that remain is $G = \{1, 2, 4, 5, 7, 8, 10, 11, 14, 16, 17, 19, 20, 22, 23, 25, 28, 29, 31, 32, 34, 35, 37, 38\}$.

But this is not just any old set of numbers. They form a *group* under the operation multiplication mod 39—namely, the multiplicative group $\mathbf{Z}_{39}^*$ of units in the ring $\mathbf{Z}_{39}$. Indeed, if students remember a few key facts about greatest common divisors, it is easy to show that the set of residues x relatively prime to n is closed under multiplication and that every element has an inverse. We can calculate the order of this group by appealing to the Euler φ function: if $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ for distinct primes p_i, then $\varphi(n) = |\mathbf{Z}_n^*| = (p_1 - 1)p_1^{\alpha_1 - 1} \times (p_2 - 1)p_2^{\alpha_2 - 1} \times \cdots \times (p_k - 1)p_k^{\alpha_k - 1}$.

My experience with this population of students (prospective school mathematics teachers) is that formulas of this sort are often of little lasting interest to them. Here, motivation for the study of these concepts is provided by starting with a concrete and elementary—even trivial—fifth grade question, "How many proper fractions are there, in reduced form, with denominator 39?" For $n = 39$, the formula predicts that there should be 24 such fractions, as indeed there are.

Now let's expand one of these fractions $x/39$ as a decimal. What are the possibilities for the period of the expansion before it repeats? Group theory gives a key insight.

Lagrange's Theorem. *If G is a finite group and if H is a subgroup of G, then the order of H divides the order of G.*

As in our other examples above, the period of the decimal expansion of $1/39$ is the order of the cyclic subgroup generated by 10 in the group $\mathbf{Z}_{39}^*$. The order of this group is $\varphi(39) = 24$. Therefore the period of the decimal expansion of $x/39$ will be one of the divisors of 24: 1, 2, 3, 4, 6, 8, 12, or 24. (If a little more knowledge of number theory is assumed, 24 can be ruled out since $\mathbf{Z}_n^*$ is not cyclic if n has more than one odd prime divisor.)

In fact, the order of 10 is 6. Since $10^6 \equiv 1 \pmod{39}$, the full subgroup H generated by 10 in the group $\mathbf{Z}_{39}^*$ is $\{10, 10^2, 10^3, 10^4, 10^5, 10^6\} \pmod{39} = \{10, 22, 25, 16, 4, 1\}$. And this, indeed, is the sequence of remainders of the long division $1 \div 39$, as is easily confirmed directly. Further, since there are six elements in H, each coset (remainder wheel) will also have six elements and there will be four of them (including H) to make up the full collection of 24 fractions (4 is the *index* of H in $\mathbf{Z}_{39}^*$).

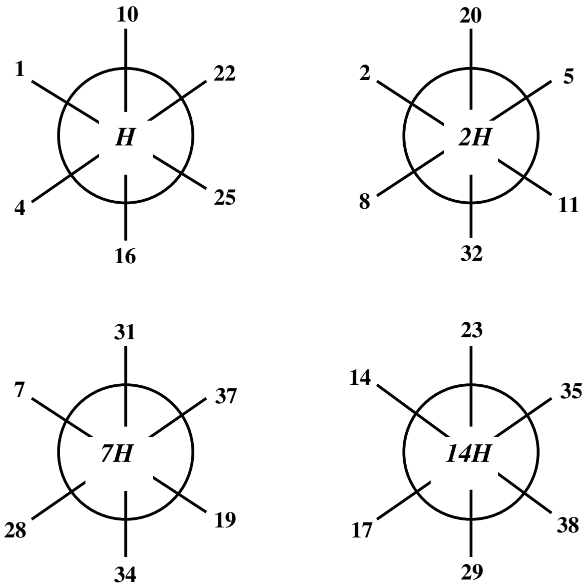

Figure 3. The cosets of the cyclic subgroup H generated by 10 in $\mathbf{Z}_{39}^*$.

What did the students learn? The motivation for this experiment was to give students an opportunity to take what they had learned in their abstract algebra course and to apply it in a familiar and concrete setting. The hope was that students would be able to use group theory to come to a deeper understanding of the astonishingly subtle and powerful ideas that are embedded in the familiar decimal numbering system and in the arithmetical algorithms that largely constitute the elementary school mathematics curriculum.

In practice, the opposite occurred. I found that by taking the time to explore a concrete and familiar example, the college students finally were able to come to grips with some of the material that they had encountered, sometimes without much real understanding, in their abstract algebra class.

Typically a first undergraduate course in abstract algebra starts with the elements of group theory, perhaps after a review of sets, functions, relations, and the integers. Almost all students understand the definition and some examples of groups. Subgroups seem natural enough, and even homomorphisms can be plowed through without our losing too great a portion of our audience.

But far too many students begin to fall by the wayside when the course turns to the concept of the *factor group* (quotient group) G/H. This idea, however, is quite easily understood in the examples considered in this paper. A factor group just means that the remainder wheels themselves can be made into a group. As easy as that sounds, I still found it useful actually to cut four wheels out of cardboard, label the spokes appropriately, and put them in bag. And even so, some students had to get used to the idea that the answer to the question "What's in the bag?" is *four wheels* rather than 24 numbers. See [**1**] and [**2**] for some speculation on why so many students balk at the concept of a set of sets.

To impose a group structure on the set of wheels displayed in Figure 3, it is natural and intuitive just to multiply each of the numbers on one wheel by each of the numbers on another, reduce mod 39, and see which wheel the product is on. Figure 4 shows the induced product $2H \otimes 14H$ in this factor group.

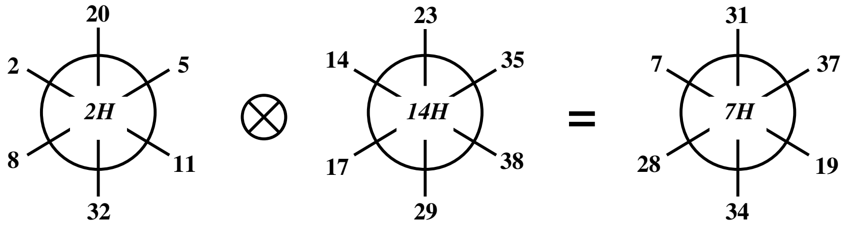

Figure 4. The product of two cosets in the factor group $\mathbf{Z}_{39}^{*}/H$.

Does this really work? This exercise provides a nice example of the power and utility of mathematical abstraction and generalization. To confirm that multiplication of these two particular remainder wheels is well defined we must perform 36 multiplications mod 39 and verify that the answer is always on the same wheel (*7H* in this example). This gets boring pretty quickly. Furthermore, students can readily understand that no matter how many examples they check, they still cannot be certain that it will *always* work.

On the other hand, a formal proof is quite easy. The basic idea is simply that $a \times 10^r \times b \times 10^s \equiv ab \times 10^{r+s} \pmod{n}$, which requires nothing more than commutativity and associativity of modular arithmetic. Furthermore, it is easy to modify the proof for any Abelian group G and subgroup H. Perhaps students can be led from this point to a consideration of normal subgroups in the non-Abelian case.

With this particular population of students, I have been encouraged by the results of this "bottom up" approach to the study of subgroups and factor groups. My experience with this challenging topic has been that the "top down" approach of "definition, theorem, proof" leaves all too many students on the outside looking in.

More fun facts. The results of this paper are merely the starting point. Many researchers have pursued the subject of repeating decimals in interesting directions. For instance, Ecker [**4**] and Kalman [**5**] raise the question of which primes p have period $p-1$. Leavitt [**6**] examines *Midy's Theorem*—the peculiar fact that for fractions with an even period, the sum of the first half of the repeating digits and the last half add up to a string of nines (for example, $5/13 = .\overline{384615}$, and $384 + 615 = 999$). In 2004, this result was expanded by Ginsberg [**3**]. In a different direction, Schiller [**7**] analyses the relative frequencies of the digits that appear in the expansion of a repeating decimal and proves that, "insofar as possible," every digit occurs equally often. Also, it is clear that there is nothing special about the number 10 in these discussions, and all the results generalize naturally to arithmetic in other bases.

Here are a couple of intriguing notions that students might be willing to investigate on their own.

The fractions $1/41 = .\overline{02439}$ and $1/271 = .\overline{00369}$ each have period 5. In fact, 41 and 271 are the only two primes with this property. Now multiply them together: $41 \times 271 = 11111$.

Well, that's a pretty cool number (its square is also cool). Maybe the kids, fifth graders or college students, will wonder next what the factors of 111111 are, and what fractions $1/n$ have period 6 (there are 53 such numbers; three are 7, 13, and 39, as we have seen).

Now take all the numbers on the remainder wheel for $1/41$ and add them together: $10 + 18 + 16 + 37 + 1 = 82$, a multiple of 41. Try the $2/41$ remainder wheel: $20 + 36 + 32 + 33 + 2 = 123$, also a multiple of 41.

In the classroom, the students made this observation without prompting. Looking at the remainder wheels, some students felt it was natural to take all the numbers they saw, add them together, and see if anything happened. I actually discouraged them from trying this, on the grounds that we are studying the *multiplicative group* $\mathbf{Z}_n^*$. But the students, ring theorists at heart, would have none of it and were eventually able to formulate and prove the following original (to them) theorem ($1/37$ is a good example to begin with, in leading students to this result).

Theorem. *Let n be a positive integer not divisible by 2, 3, or 5, and suppose that the decimal expansion of $1/n$ has period k. Then n is a factor of the integer $111\ldots11$ (k 1's). Furthermore, the sum of the partial remainders in the indicated long division of every reduced proper fraction x/n is a multiple of n.*

Acknowledgments. I thank the referees and the editor for many useful suggestions regarding the content and exposition of this paper, and for calling my attention to related articles by other authors.

References

1. M. Asiala, E. Dubinsky, D. Mathews, S. Morics, and A. Okra, Development of students' understanding of cosets, normality, and quotient groups, *Journal of Mathematical Behavior* **16** (1997) 241–309.
2. L. Brenton and T. Edwards, Sets of sets: a cognitive obstacle in forming the quotient group concept, *College Math. J.* **15** (2003) 31–38.
3. B. Ginsberg, Midy's (nearly) secret theorem—an extension after 165 years, *College Math. J.* **35** (2204) 26–30.
4. M. Ecker, The alluring lore of cyclic numbers, *College Math. J.* **14** (1983) 105–109.
5. D. Kalman, Fractions with cycling digit patterns, *College Math. J.* **27** (1998) 109–115.
6. W. Leavitt, A theorem on repeating decimals, *Amer. Math. Monthly* **74** (1967) 669–673.
7. J. Schiller, A theorem in the decimal representation of rationals, *Amer. Math. Monthly* **66** (1959) 797–798.

"It Was Only a Sign Error"

From David Cox (dac@cs.amherst.edu), Amherst College:

According to an article titled "Corps of Engineers goofs on anti-flood mapping" in the November 18, 2007, issue of the *Springfield Republican*, on June 20, 2007, the Army Corps of Engineers released flood-risk maps for parts of New Orleans that claimed that the drainage canals would reduce flooding during a major storm by 5.5 feet. However, in a report dated November 7, the 5.5 feet estimate was changed to 6 inches: " 'We've made some corrections,' the engineer, Ed Link, told reporters on Friday. Link said mistakes were made in the calculations for two sub-basins that include Lakeview and nearby neighborhoods. In one, a minus sign was used instead of a plus sign."

The article also mentions that this discrepancy was buried in the appendices to the November 7 report, and came to light only when a reporter for a local New Orleans television station asked the engineer about it.

Cox adds, "There are two important points here. First, sign errors can have serious consequences; and second, quantitative literacy is important for everyone, including journalists."