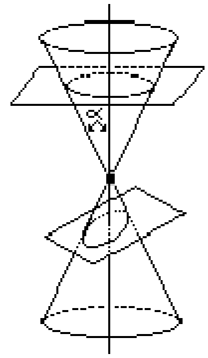
Introduction: Static Hypertext Version

A Physical Interpretation of the Focus-Locus and Focus-Directrix Properties of Ellipses

The conic sections called ellipses have a number of definitions. The simplest is perhaps the one from which their name derives, the usual "plane-slicing-cone" construction.



CIRCLE AND ELLIPSE

This construction in the Euclidean geometry of $\mathbb{R}^3$ is especially easy to visualize. Let us use a "standard" cone in Euclidean (x, y, t) space given by the equation

$$x^2 + y^2 - t^2 = 0$$

Let us parameterize the construction in the following way. Suppose P is an arbitrary point in the interior of the cone. Then there is a unique hyperboloid

$$x^2 + y^2 - t^2 = -k^2,\ k > 0$$

that contains P. Let T_P be the plane tangent to that hyperboloid at P. Finally, let E_P be the intersection of that plane with the cone. Thus, the procedure: $P \to E_P$ associates an ellipse with each interior point of the cone. We will show that P is the center of E_P, and determine its semi-major and semi-minor axes as well as its foci.

Now two interesting properties of these ellipses lead to new, and seemingly independent characterizations of these metric curves. We describe the first as the *focus-locus property*. Each ellipse has a pair of "foci" and the sum of distances from each point on the ellipse to the foci is constant. Richard Feynman used this latter description to construct yet another characterization of an ellipse, and to use it to derive Kepler's first law of planetary motion from Newtonian gravitation [Goodstein and Goodstein, 1996].

We describe the second property as the *focus-directrix property*. There is a line in the plane of the ellipse called its directrix. An ellipse (that is not a circle) is the locus of points, whose distance from a certain focus has constant ratio (strictly between 0 and 1) with the distance to the directrix. That ratio is called the *eccentricity* of the ellipse.

It is natural to ask what the relation is between these constructions. Given an ellipse E_P, what are its foci? And what are the dimensions of its major and minor axes? These questions are easy to answer, as we will show on the next page, but the straightforward calculation gives no insight into why we might expect that the sum of the distances from any point in the ellipse to the two foci to be constant. If we ask what the directrix is, we will find it easier to answer both questions not for E_P itself but for a new ellipse that is closely related to E_P.

We will use another characterization of conic sections in this microworld to form a link between "plane-slicing-cone" definition and the focus-locus and focus-directrix characterizations of ellipses. The latter two characterizations are properties of similarity classes of ellipses, and we will establish those properties for all similarity classes. It happens that the ellipses E_p can also be characterized as **"conic intersections,"** that is, the intersections of pairs of light cones in 2+1 spacetime. On this way of viewing them, there emerges a simple physical interpretation of the focus-locus and focus-directrix properties of ellipses. This latter view of conics leads to an interesting physical experiment that will show the way. Now, by 2+1 spacetime, we mean $\mathbb{R}^3$ equipped with a certain non-degenerate hyperbolic inner product, and not a Euclidean one.

These geometries are different, and so we must use some care when speaking about metric invariants. Which metric do we mean? When we speak of ellipses in 2+1 spacetime, we will always be referring to objects embedded in a plane that inherits Euclidean structure from the hyperbolic metric. In relativistic terms, all of the vectors in these planes are "space-like". The analogous constructions that yield hyperbolae must use planes that always contain a "time-like" subspace of dimension 1, hence do not inherit Euclidean structure.

Thus, we will consider simultaneously two geometric structures for $\mathbb{R}^3$: Euclidean structure, and hyperbolic 2+1 space structure. special relativity provides the lexicon that gives us a smooth transition between the points of view. This peripatetic strategy has the obvious pedagogic virtue of stimulating cross-disciplinary thinking. But it also faces the pitfalls that such non-linear approaches usually do. While the facts of hyperbolic geometry and linear algebra that we need are fairly elementary, the physical intuition required to apply them to this problem depends strongly on the experience and the imagination of the reader.

And if this story has a subplot, it is that we wish to cultivate for students of mathematics, the visual and physical intuition on which special relativity is based. The section titled: **Light Rays, Clocks, and Rulers: A Visual Primer in Special Relativity** is our modest attempt to do that. Throughout the story, however, we have tried to recruit 3 dimensional Graphics in a variety of ways to support this visual intuition, and so we recommend that beginners experiment with the interactions if that is at all practicable.

We do not wish to slight hyperbolae, for which there are analogous "focus-locus" and "focus-directrix" characterizations. There is a relativistic experiment similar to the one we will discuss here that provides motivation and insight into that property. In order to keep this story within bounds, we have reluctantly decided not to include it here, but to leave it to the initiative and imagination of the reader to extend these ideas to include them.

Henri Poincare [Poincare, H., 1905] was one of the first to point out the role of groups of physical motions in determining what part of geometry is "invention" (or "convention") and what part is an expression of the biological and psychological inheritance that constrains the way we can think about the world. This is a theme that Jean Piaget [Piaget, J., 1971] developed throughout his life, and has much to say about the epigenetic basis of mathematical knowledge. The Lorentz group (and the Poincare group) of special relativity describe a highly non-intuitive constraint on the way that we can view the physical world, insofar as they describe experiments (about the propagation of light) that do not fall into the domain of ordinary human experience. But they also describe a geometric view of nature [Whitehead, A.N., 1919] that cannot be dismissed as "invention". This geometric and dynamic view of nature weaves "time" and "space" into a whole in which each loses its individual identity, as Minkowski observed.

Now it may seem surprising, on the face of it, that such an esoteric view, rooted as it is in a sophisticated physical interpretation of the natural phenonema, can have something useful to say about the elementary geometry of conic sections. But when physical interpretation can build a bridge between mathematical concepts, both the mathematics and the physics are enriched thereby. In this case, physics gives a synthetic interpretation of the "focus-locus" definition of conic sections, and attaches a straightforward (if somewhat surprising) meaning to the sum of distances to foci that appear in the associated construction of ellipses.

In the next section, **Planes Intersecting Cones**, we will develop some of the details of the $P \rightarrow E_p$ construction described above, in a special case that that will be the basis of our general strategy to follow later. After that, in **Hyperbolic Geometry of 2+1 Spacetime**, we will show that conic sections are also "conic intersections," that is, the intersection of light cones, in order to set the stage for the physical point of view (special relativity). Next follows a short discussion and an experiment in special relativity: **Light Rays, Clocks and Rulers: A Primer in Special Relativity**. With these preliminaries aside, we will describe an experiment: **A Thought Experiment** that will, we hope, give a simple way to think about the focus-locus property of ellipses while connecting it to the plane-slicing-cone view. The final section, **Interpretation of the Experiment**, brings all of the facts together.