Interpretation of the Experiment: Static Hypertext Version

In order to understand what the experiment means, we must write down the equation of the curve defined by projecting the wavefront curve at $t' = 1$ to the (standard) coordinates of the $X - Y$ plane of T. We do this by introducing certain coordinates for the $X - Y$ plane of T. We saw this in a Euclidean context in (3.2) of **Planes Intersecting Cones**. The "speed" of T' (as measured by T) is defined to be

$$\nu = \sqrt{\frac{A^2 + B^2}{C^2}}$$

We assume for this experiment that $\nu \neq 0$ otherwise, there is nothing to do.

We saw that the wavefront curve was an ellipse in the (stationary) frame of T by using the basis $\left[X', Y', T'\right]$ in T coordinates, given by

$$T' = \left(A, B, \frac{1}{\sqrt{1-\nu^2}}\right)$$

$$X' = \left(\frac{A}{\nu}, \frac{B}{\nu}, \frac{\nu}{\sqrt{1-\nu^2}}\right)$$

$$Y' = \sqrt{1-\nu^2}\left(-\frac{B}{\nu}, \frac{A}{\nu}, 0\right)$$

where these vectors satisfy the hyperbolic orthogonality relations:

$$\langle X', X' \rangle = \langle Y', Y' \rangle = 1 \text{ and } \langle T', T' \rangle = -1$$
$$\langle X', Y' \rangle = \langle Y', T' \rangle = \langle X', T' \rangle = 0$$

Then the points of the form:

$$T' + \cos(\theta) X' + \sin(\theta) Y', \quad 0 \le \theta < 2\pi$$

define the curve.

To project this ellipse, consider the following vectors in the $X - Y$ plane of T. U and V are the ***unit vectors*** in the $X - Y$ plane defined as:

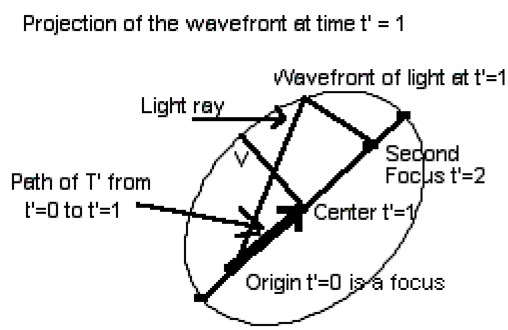
$$\begin{aligned} U &= \frac{1}{\sqrt{A^2 + B^2}}(A, B) = \frac{\sqrt{1-\nu^2}}{\nu}(A, B) \\ V &= \frac{1}{\sqrt{A^2 + B^2}}(-B, A) = \frac{\sqrt{1-\nu^2}}{\nu}(-B, A) \end{aligned} \tag{7.1}$$

From the definitions of X', Y' and T', it follows that the projections of the vectors of the form:

$T' + \cos(\theta) X' + \sin(\theta) Y', \quad 0 \le \theta < 2\pi$ into the $X - Y$ plane of T have the form:

$$\frac{v}{\sqrt{1-v^2}}U + \frac{\cos(\theta)}{\sqrt{1-v^2}}U + \sin(\theta)V \tag{7.2}$$

From this it follows that the ***projected*** curve is an ellipse with major axis on the line generated by U . This is illustrated below:



The center of the ellipse is the projection of T':

$$\frac{v}{\sqrt{1-v^2}}U$$

The endpoints on the major axis of the ellipse are:

$$\frac{v-1}{\sqrt{1-v^2}}U \text{ and } \frac{v+1}{\sqrt{1-v^2}}U$$

The length of the semi-major axis is thus

$$\frac{1}{\sqrt{1-v^2}}$$

while the length of the semi-minor axis is 1.

Recalling the standard form for an ellipse where $0 < c^2 < a^2$,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

we see that the foci of this ellipse must be at

$$(0,0) \text{ and } \frac{2v}{\sqrt{1-v^2}}U$$

because the distance from the center to the focus is easily seen, from these observations, to be

$$\frac{v}{\sqrt{1-v^2}}$$

The interpretation of this is the following. The projected ellipse is the projection to the $X-Y$ plane of the T observer of his measured spatial positions of the wavefront at the T' observer's time $t' = 1$. It is not a circle because this wavefront does not consist of simultaneous events for T (we assumed that $v \neq 0$).

The major axis of this ellipse contains, between the focus at the origin and the second focus, the line of spatial positions of the world-line of T' between his time $t' = 0$ and time $t' = 2$. In particular, he moves from the origin (first focus) to the center in 1 unit of his time, which is $\frac{1}{\sqrt{1-v^2}}$ units of the T observer's time. His projection must therefore move to the second focus at his time $t' = 2$.

The rays of light, emitted at various angles from the emission event, project to segments connecting the first to the second focus via a point on the boundary of the projected ellipse. The questions, of course, are: Why should the sum of the lengths be constant? And what does this constant represent? And the answer will now be fairly easy to see.

Consider a ray of light that bounces from the emission to the reception event. Describe its itinerary in the reference frame of observer T. It starts from the origin event

$$I = (x_0, y_0, t_0) = (0, 0, 0)$$

and arrives at the mirror at event,

$$J = (x_1, y_1, t_1)$$

Next, it reflects to the reception event:

$$K = (x_2, y_2, t_2)$$

In the first leg of its itinerary, it lies in the light cone, so

$$(x_1 - x_0)^2 + (y_1 - y_0)^2 = (t_1 - t_0)^2$$

Since the square roots of both sides are equal, we conclude that the spatial $(X-Y)$ projection of that path has length the absolute value: $|t_1 - t_0|$. This is simply $t_1 - t_0$ since $t_1 > t_0$.

In the second leg of its itinerary, it also lies in a light cone, so

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = (t_2 - t_1)^2$$

We conclude for the same reason that the spatial projection of that path has length $t_2 - t_1$.

Therefore, the sum of the spatial lengths is $t_2 - t_0$. This is equal to the **time** that observer T measures from the emission event to the reception event. That is, of course, the time coordinate in the frame of T of the event $2T'$. But from the definition of T' in the previous section

$$T' = \left(A, B, \frac{1}{\sqrt{1-v^2}} \right)$$

we see that this is:

$$\frac{2}{\sqrt{1-v^2}}$$

And this, of course, is the length of the major axis of the projected ellipse. That is a physical interpretation of why the sum of the lengths of the paths from a point on the ellipse to its foci is constant. It is the time T measures between the emission and reception event, which is the same for all rays of light.

Next, we consider the focus-directrix property of the ellipse. We showed in **Planes Intersecting Cones** that the intersection of the plane of simultaneity for T': $t' = 1$ with the plane of simultaneity for T: $t = 0$, is a line that we call λ.

$$A \cdot x + B \cdot y + 1 = 0$$

We represented the tangent plane, the plane of simultaneity for T': $t' = 1$ as the graph of a function of *x* and *y*:

$$f(x,y) = t = \frac{A \cdot x + B \cdot y + 1}{\sqrt{A^2 + B^2 + 1}}$$

In the present context, the function $f(x,y) = t$ has the following interpretation. With each point (x,y) there is a unique point on the plane of simultaneity for T': $t' = 1$ that projects to it. $f(x,y) = t$ is the time coordinate that standard observer T ascribes to it. The line λ is the set of events with $t = 0$ and $t' = 1$.

Clearly, λ is perpendicular to the line generated by U, the major axis of the ellipse. If we let (x,y) be an arbitrary point of the $t = 0$ plane, there is a unique number $\mu(x,y)$ and point $(x_0, y_0) \in \lambda$ such that

$$(x,y) = (x_0, y_0) + \mu(x,y) \cdot \vec{U}$$

The number $|\mu(x,y)|$ is the distance from (x,y) to the line λ.

Now we saw that if in the Euclidean context, ϕ is the dihedral angle between the planes $t = 0$ and the graph of $f(x,y) = t$ (that is, the angle formed by intersecting these planes with a plane perpendicular to λ) then for each (x,y)

$$f(x,y) = \tan(\phi) \cdot \mu(x,y)$$

We noted that $0 < \phi < \frac{\pi}{4}$.

For a point (x,y) in the ellipse, $f(x,y) = t$ is by the interpretation of the experiment above: $t_1 - t_0$ the time of arrival as measured by T of a light ray emitted by T'. This number in turn is the length of the segment as measured by T connecting the focus at the origin to the point (x,y) on the ellipse. Therefore the ratio of the length of the segment connecting the focus to the point (x,y) in the ellipse to the distance from (x,y) to the line λ is the constant $\tan(\phi) = \varepsilon$. We usually call ε the eccentricity of the ellipse.

The calculation in **Planes Intersecting Cones** actually gave a value for ε. The intersection of the directrix λ with the extension of the major axis is the point

$$-\frac{1}{\sqrt{A^2 + B^2}} \cdot \vec{U} = -\frac{\sqrt{1-v^2}}{v} \cdot \vec{U}$$

And the value of the eccentricity ε is the speed of T' as measured by T

$$\sqrt{\frac{A^2 + B^2}{A^2 + B^2 + 1}} = v$$

Now, we might ask how general this construction is. It is clear that for any particular direction of motion of T' with respect to T, we will obtain among the projected ellipses one representative of the similarity class of ellipses for each speed v. And every similarity class will be represented, simply by choosing the representative with semi-minor axis of length 1, and semi-major axis of length

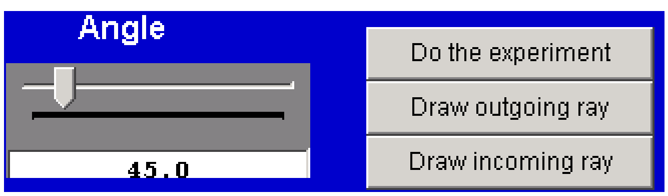
$$\frac{1}{\sqrt{1-v^2}}$$

Therefore this physical argument is completely general.

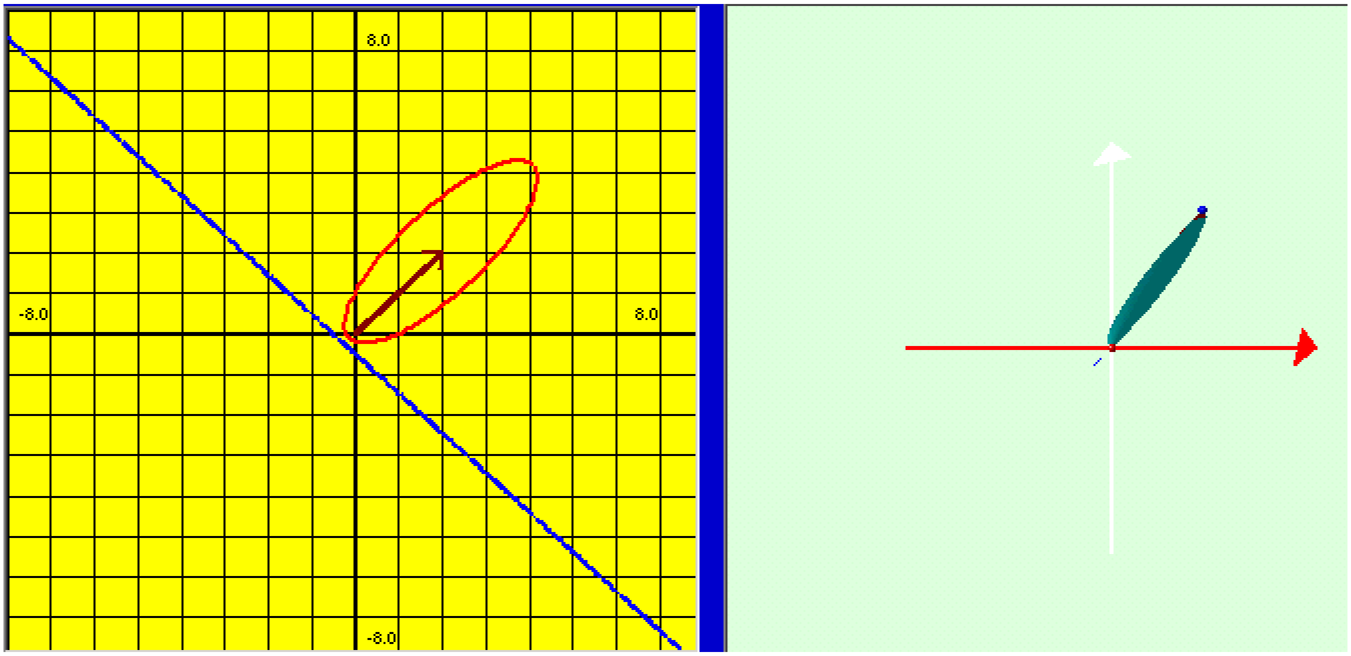
What happens if T' is stationary with respect to $T\,(v = 0)$? In that case, this construction breaks down when we attempt to build the T' observer's orthonormal basis. But in that case, there is no need to build such a basis. The foci collapse to the center, the ellipse is a circle, and everything happens as it does in the frame of T'.

Description of the Exploration in the Dynamic Version

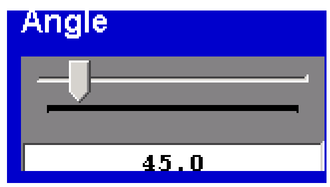
The exploration for this page lets you check the similarity properties: focus-locus and focus-directrix. You establish the velocity $T' = \left(A, B, \frac{1}{\sqrt{1-v^2}}\right)$ of T' in the frame of T by dragging the arrow to an appropriate location that determines $(A, B) = \vec{U} \cdot \sqrt{A^2 + B^2}$. Then use the control panel to press **Do the experiment**.



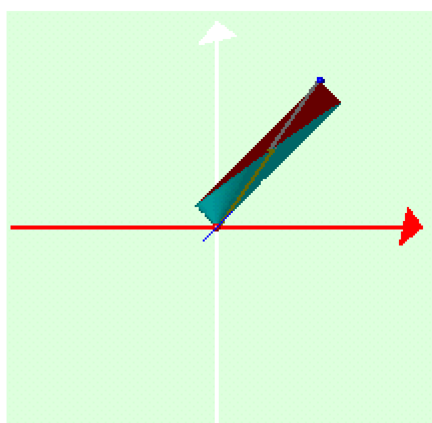
You will see on the right the 3 dimensional emission and reception phases of the experiment. On the left, you see the plane of the projection with the point $(A, B) = \vec{U} \cdot \sqrt{A^2 + B^2}$ indicated by the arrow emanating from the origin, a focus of the light red ellipse. You will also see the blue directrix for the ellipse assuming that v is not too small.



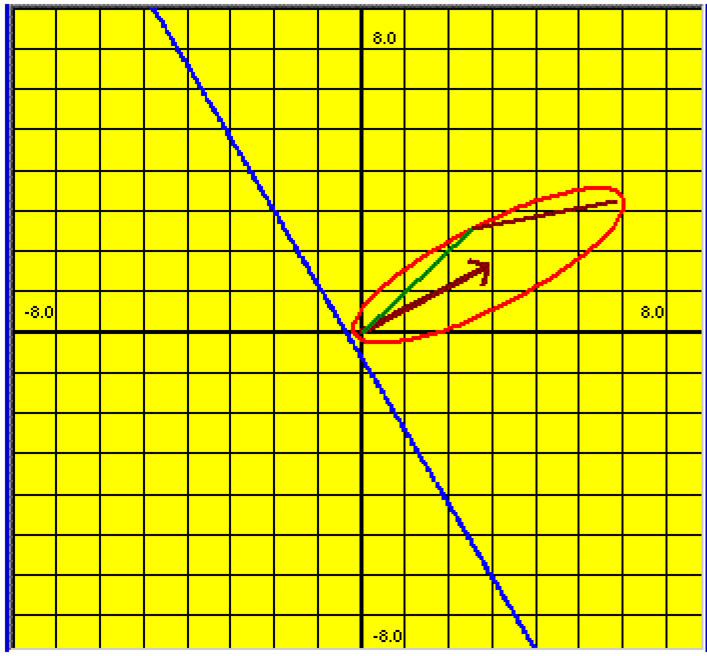
Next, select an angle for the ray of light



and press successively, **Draw outgoing ray**, and **Draw incoming ray**. You will see the rays of light drawn along the light cones this time. The T displacement of each one is a measure of its length.



The left 2D Graph window actually shows the projections.



And the time durations (as measured by T) are reported in the text field:

Time: 3.6212109
Time: 3.3970987

As you choose different angles, you will see that, of course, they add to the same number.

Finally, you may check the eccentricity for each light ray. For that, press the button:

Check eccentricity

You will see information in the text field:

For the point: [2.5605828,2.5605828]
Distance to Focus: 3.6212109
Distance to Directrix: 3.7778529
Ratio: 0.9585367

Change the angle, and you will see new information:

For the point: [5.9302410,3.4238262]
Distance to Focus: 6.8476525
Distance to Directrix: 7.1438601
Ratio: 0.9585367

Notice that the ratios (the eccentricity) remain the same.