

H.W. ASSIGNMENT # 5 / 3.4 & 3.5

MA 504 / DISCRETE MATHEMATICS / DR. J. GLEASON

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(10)

3.4 PAGES 132  $\Rightarrow$  134 EXERCISES 1, 3, 5, 7, 11, 12, 13, 16, 17, 18

3.5 PAGES 140  $\Rightarrow$  142 EXPLORATORY 1 & 2; EXERCISES 2, 11, 13, 18, 19

(7)

TOTAL 17

3.4 (1) (A) LIST ALL + DIVISORS (FACTORS) OF EA. NUMBER,

(B) LIST ALL COMMON DIVISORS (FACTORS) OF EA. PAIR, <sup>1<sup>ST</sup> SET  $\cap$  2<sup>ND</sup> SET</sup>

(C) STATE GCD (GCF) OF EA. PAIR. <sup>GREATEST MEMBER IN SET FROM (B)</sup>

(a) (6, 15) (A)  $6 \Rightarrow \{1, 2, 3, 6\}$  (B)  $CF(6, 15) = \{1, 3\}$  (C)  $GCF(6, 15) = \{3\}$   
 $15 \Rightarrow \{1, 3, 5, 15\}$

(b) (12, 54) (A)  $12 \Rightarrow \{1, 2, 3, 4, 6, 12\}$  (B)  $CF(12, 54) = \{1, 2, 3, 6\}$  (C)  $GCF(12, 54) = \{6\}$   
 $54 \Rightarrow \{1, 2, 3, 6, 9, 18, 27, 54\}$

(c) (21, 63) (A)  $21 \Rightarrow \{1, 3, 7, 21\}$  (B)  $CF(21, 63) = \{1, 3, 7, 21\}$  (C)  $GCF(21, 63) = \{21\}$   
 $63 \Rightarrow \{1, 3, 7, 9, 21, 63\}$

(d) (28, 54) (A)  $28 \Rightarrow \{1, 2, 4, 7, 14, 28\}$  (B)  $CF(28, 54) = \{1, 2\}$  (C)  $GCF(28, 54) = \{2\}$   
 $54 \Rightarrow \{1, 2, 3, 6, 9, 18, 27, 54\}$

(e) (17, 36) (A)  $17 \Rightarrow \{1, 17\}$  (B)  $CF(17, 36) = \{1\}$  (C)  $GCF(17, 36) = \{1\}$   
 $36 \Rightarrow \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

3a) FIND  $GCF(14, 62) = 2$

EQUATION	A	B	R	(GCF = B WHEN R = 0)
① $62 = 4(14) + 6$	62	14	6	
② $14 = 2(6) + 2$	14	6	2	
③ $6 = 3(2) + 0$	6	2	0	

(b) IF  $A = BQ + R$ , THEN  $GCF(A, B) = GCF(B, R)$

NOW LET  $A = 62$  &  $B = 14$ ,  $62 = 14(4) + 6$ , SO  $GCF(62, 14) = GCF(14, 6)$   
 NOW LET  $A = 14$  &  $B = 6$ ,  $14 = 6(2) + 2$ , NOW  $GCF(14, 6) = GCF(6, 2)$  CON IT



3.4

11 INFORMAL PROCEDURE FOR FINDING LCM(a,b).

(A) LIST THE FIRST 6 OR 8 MULTIPLES (OLD-FASHION TIMES TABLES!) OF "a", THEN LIST THE FIRST 6 OR 8 MULTIPLES (AGAIN, OLD-FASHION TIME TABLES!) OF "b" (B) THEN PLAY THE "MATCH-GAME" (FIND THE INTERSECTION OF THE TWO SETS) AND MAKE A LIST OF "COMMON MULTIPLES" (C) THE SMALLEST (LEAST) NUMBER IN THIS LIST (THE INTERSECTION) IS LCM(a,b).

(a) 8: {8, 16, 24, 32, 40, 48, 56, 64, 72, ...} (b) {24, 48, 72, 96, ...}  
 (A) 12: {12, 24, 36, 48, 60, 72, 84, ...} (c) LCM(8,12) = {24}

(a) 6: {6, 12, 18, 24, 30, 36, 42, 48, 54, 60, ...} (b) {30, 60, 90, 120, ...}  
 15: {15, 30, 45, 60, 75, ...} (c) LCM(6,15) = {30}

(a) 18: {18, 36, 54, 72, 90, 108, 126, ...} (b) {54, 108, 162, 216, ...}  
 27: {27, 54, 81, 108, 135, 162} (c) LCM(18,27) = {54}

(a) 16: {16, 32, 48, 64, 80, 96, 112, 128, ...} (b) {336, 672, 1008, 1344, ...}  
 42: {42, 84, 126, 168, 210, 252, 294, 336, ...} (c) LCM(16,42) = {336}

(12)	<u>a</u>	<u>b</u>	<u>GCF(a,b)</u>	<u>LCM(a,b)</u>	<u>a · b</u>
	8	12	4	x 24	= 96 ✓
	6	15	3	x 30	= 90 ✓
	18	27	9	x 54	= 486 ✓
	16	42	2	x 336	= 672 ✓

CONJECTURE: THE PRODUCT OF THE GCF(a,b) AND THE LCM(a,b) EQUALS THE PRODUCT OF a AND b.  $GCF(a,b) \times LCM(a,b) = a \cdot b$

3.4

13 FIND THE LCM'S

- a)  $GCF(6, 15) = 3$
- b)  $GCF(12, 9) = 6$
- c)  $GCF(21, 63) = 21$
- d)  $GCF(28, 54) = 2$
- e)  $GCF(17, 36) = 1$

$a \cdot b$   
 90  
 648  
 1323  
 1512  
 612

$LCM(a, b) = \frac{a \cdot b}{GCF(a, b)}$

30  
 108  
 63  
 756  
 612

MENTAL MATH/  
 CALC.  
 EXERCISES

FROM CONJECTURE IN EX#12  $GCF(a, b) \times LCM(a, b) = a \cdot b$   
 $\therefore LCM(a, b) = \frac{a \cdot b}{GCF(a, b)}$

16 LET A AND B BE + INTEG. SUPPOSE THERE ARE INTEG.

M AND N SO THAT  $1 = MA + NB$

EXAMPLES: EX#4  $GCF(252, 180) = (-2)(252) + (3)(180) = 36$

EX#5  $GCF(36, 14) = (2)(36) + (-5)(14) = 2$

PG. 129 (3e)  $GCF(36, 17) = (-8)(36) + (17)(17) = 1$

THM #20: FOR POSITIVE INTEG. a AND b, WE CAN FIND INTEG. M AND N SUCH THAT  $GCF(a, b) = M(A) + N(B)$

a)  $\therefore GCF(A, B) = 1$

b) ASSUME INTEG. C BE A FACTOR > 1 OF BOTH A AND B.

$C | A$  AND  $C | B$  DEF OF FACTOR

$\therefore C | AM$  AND  $C | BN$  THM #9, PG. 104  
 FOR INTEGERS M & N RESPECTIVELY

AND  $C | (AM + BN)$  THM #7, PG. 104

$1 = MA + NB$  GIVEN

$\therefore C | 1$ , & INTEG. k EXISTS  $\rightarrow 1 = ck$ . SUBSTITUTION  
 CONTRADICTION TO MY ORIGINAL HYPOTHESIS THAT INTEGER

$C > 1$ . BECAUSE ONLY + INTEG. TO  $\div 1$  IS 1,  $\therefore C$  MUST = 1,  
 $\therefore GCF(A, B) = 1$ .

3.4

(17) LET A AND B BE POSITIVE INTEGERS. SUPPOSE THERE ARE INTEGERS M AND N SO THAT  $9 = MA + NB$ .

(a) I CAN SAY  $GCF(A, B) | 9$ . IN OTHER WORDS,  $9 = k(GCF(A, B))$ , WHERE k IS AN INTEGER.

(b) ASSUME INTEGER c IS GCF(A, B).

THEN  $c | A$  AND  $c | B$   
 $\therefore c | AM$  AND  $c | BN$   
AND  $c | (AM + BN)$

$9 \Rightarrow MA + NB$

$\therefore c | 9$

AND FINALLY  $GCF(A, B) | 9$

DEF OF FACTOR  
THM #9, Pg. 104  
THM #7, Pg. 104  
GIVEN  
SUBSTITUTION  
SUBSTITUTION

(18) PROVE BY INDUCTION: IF p IS A PRIME NUMBER, AND p DIVIDES THE PRODUCT  $a_1 \cdot a_2 \cdot a_3 \cdot \dots \cdot a_n$ , THEN p DIVIDES ONE OF THE  $a_i$ 'S.

BASIS STEP:  $p | a_1 \rightarrow p | a_1$ . IF p DIVIDES A PRODUCT CONSISTING OF A SINGLE FACTOR  $a_1$ , THEN p DIVIDES THAT SINGLE FACTOR  $a_1$ .

INDUCTIVE STEP: IF  $p | a_1 \cdot a_2 \cdot a_3 \cdot \dots \cdot a_r$ , THEN p | ONE OF THE  $a_i$ 'S IMPLIES, IF  $p | a_1 \cdot a_2 \cdot a_3 \cdot \dots \cdot a_r \cdot a_{r+1}$  THEN p | ONE OF THE  $a_i$ 'S.

SUPPOSE  $p | a_1 \cdot a_2 \cdot a_3 \cdot \dots \cdot a_r \cdot a_{r+1}$

THE DIVIDEND,  $a_1 \cdot a_2 \cdot a_3 \cdot \dots \cdot a_r \cdot a_{r+1}$  CAN BE REPRESENTED AS THE PRODUCT OF TWO FACTORS, WITH  $(a_1 \cdot a_2 \cdot a_3 \cdot \dots \cdot a_r)$  THE FIRST FACTOR AND  $(a_{r+1})$  THE SECOND FACTOR. SINCE p DIVIDES THE PRODUCT OF THESE TWO FACTORS  $(p | (a_1 \cdot a_2 \cdot a_3 \cdot \dots \cdot a_r)(a_{r+1}))$ ,

$\rightarrow$  CONT

3.4  
18 CONT

APPLYING THEOREM 21,  $p$  DIVIDES FIRST FACTOR  
OR  $p \mid$  SECOND FACTOR.

$\therefore$  SINCE  $p \mid (a_1 \cdot a_2 \cdot a_3 \cdot \dots \cdot a_r)(a_{r+1})$ ,  $p \mid (a_1 \cdot a_2 \cdot a_3 \cdot \dots \cdot a_r)$  OR  
 $p \mid a_{r+1}$ .

CASE I

IN CASE I,  $p \mid a_1 \cdot a_2 \cdot a_3 \cdot \dots \cdot a_r$ , THE INDUCTIVE  
HYPOTHESIS STATES THAT  $p$  MUST DIVIDE ONE OF  
THE FACTORS  $a_i$  FOR  $i = 1, 2, 3, 4, \dots, r$ .  $\therefore p$   
DIVIDES ONE OF THE FACTORS  $a_i$  FOR  $i = 1, 2, 3, \dots,$   
 $r, r+1$ .

3.5  
EXPLORATORY (1)

(1) (a-1) COMPUTE PRODUCT  $2 \cdot 5 \cdot 11 = \underline{110}$

(-2)  $+ 1 = \underline{111}$

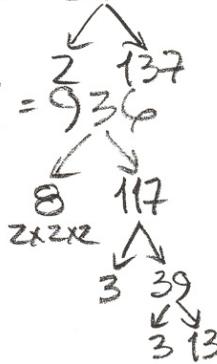
(-3)  $111 = \underline{3 \cdot 37}$   
 $\swarrow \searrow$   
 3 37

YIELDS NEW PRIMES 3 AND 37,

(b-1)  $2 \cdot 3 \cdot 5 = 30, + 1 = \underline{31}$

(b-2)  $3 \cdot 7 \cdot 13 = 273, + 1 = \underline{274}$      2 AND 137

(b-3)  $5 \cdot 11 \cdot 17 = 935, + 1 = \underline{936}$      2, 3, AND 13



(c)     INPUT                      OUTPUT

$2, 5, 11$                       (3) 37

$2, 3, 5$                        $\longrightarrow$  31

$3, 7, 13$                       (2) 137

$5, 11, 17$                       (2) 3, 13

CONJECTURE: THE OUTPUT SET OF PRIMES IS UNIQUE FROM THE INPUT SET OF PRIMES,  $\bigcap_{\text{put}} \bigcap_{\text{put}} = \emptyset$ .

(d-1) SINCE WE ARE ASSUMING  $p = p_1$ , THEN  $p | p_1$ , B/C  $p_1 = (1) p$  AND BY THM #9, SINCE  $p | p_1$ , THEN  $p | p_1 (p_2 \cdot p_3 \dots p_n)$

(d-2) BY HYPOTHESIS,  $p | p_1 \cdot p_2 \cdot p_3 + 1$   
 AND FROM (d-1)

$\therefore$   $p | p_1 \cdot p_2 \cdot p_3 + 1$   
 $\therefore$   $p | p_1 \cdot p_2 \cdot p_3$   
 $\therefore$   $p | (p_1 \cdot p_2 \cdot p_3) (-1)$  BY THM #9  
 $\therefore$   $p | [(-1)(p_1 \cdot p_2 \cdot p_3) + (p_1 \cdot p_2 \cdot p_3) + 1]$  THM #7  
 $\therefore$   $p | 1$  CANCELLATION

(d-3) THIS,  $p = 1$ , IS A CONTRADICTION, B/C BY DEF. ALL PRIMES  $> 1$ .

3.5 EXPLORATORY

(2a) THERE IS A FINITE NUMBER OF PRIMES.  
(NOT INFINITE)

(2b) (1) THIS SPEWED OUT (OUTPUT PRIME) PRIME IS NOT ONE OF THE  $\leq$  INPUT PRIMES,  $P_1, P_2, P_3, \dots, P_s$ .

(2b2) THE LIST OF INPUT PRIMES,  $P_1, P_2, P_3, \dots, P_s$  IS SUPPOSED TO BE THE COMPLETE LIST OF FINITE PRIMES,  $\therefore$  A CONTRADICTION.

EXERCISE

(2a) 124

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    124
   /  \
  4    31
 /  \
2   2

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$2 \times 2 \times 31 = 2^2 \times 31$

(b) 225

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    225
   /  \
  5    45
     /  \
    5    9
       /  \
      3   3

```

$3 \times 3 \times 5 \times 5 = 3^2 \times 5^2$

(c) 440

```

    440
   /  \
  10   44
 /  \ /  \
2  5 4  11
     /  \
    2  2

```

$2 \times 2 \times 2 \times 5 \times 11 = 2^3 \times 5 \times 11$

(d) 1184

```

    1184
   /  \
  16   74
 /  \ /  \
4  4 2  37
 /  \
2  2  2  2

```

$2 \times 2 \times 2 \times 2 \times 2 \times 37 = 2^5 \times 37$

(11)  $GCF(5^4 7^2, 2^3 5^3 7^3) = 5^3 \cdot 7^2 = 6125$

EXPLANATION: LET  $d$  BE A COMMON FACTOR OF  $5^4 7^2$  AND  $2^3 5^3 7^3$ . THE FACT THAT  $d$  DIVIDES  $5^4 7^2$  REQUIRES THAT THE PRIME FACTORS OF  $d$  BE 5'S AND 7'S. FURTHER, THERE ARE NO MORE THAN 4-5'S AND 2-7'S. SIMILARLY, SINCE  $d$  DIVIDES  $2^3 5^3 7^3$ , THE PRIME DIVISORS OF  $d$  CAN ONLY HAVE 2'S, 5'S, AND 7'S, AND THERE CAN BE NO MORE THAN 3-2'S, 3-5'S, AND 3-7'S. PUTTING THESE TWO OBSERVATIONS TOGETHER,  $d$  MUST LOOK LIKE  $5^k 7^r$ , WHERE  $k \leq 3$  AND  $r \leq 2$ . SINCE  $5^3 \cdot 7^2$  IS THE GREATEST OF THESE COMMON FACTORS, THE  $GCF = 5^3 \cdot 7^2$ .

### 3.5 (13)

FIND THE GCF (96, 144) IN FOUR WAYS:

① FIND COMMON FACTORS AND CHOOSE GREATEST.

A = 96  $\Rightarrow$  {1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96}

B = 144  $\Rightarrow$  {1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72, 144}

$A \cap B = \{1, 2, 3, 4, 6, 8, 12, 16, 24, 48\}$ ,  $\therefore$  GCF(96, 144) = {48}

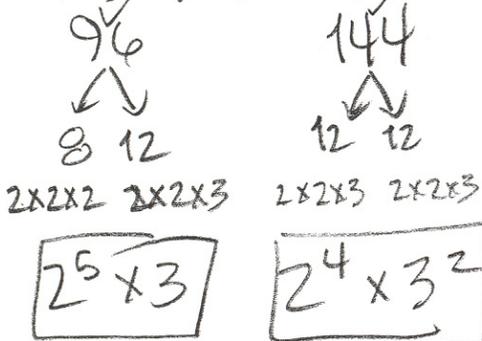
② APPLY EUCLIDEAN ALGORITHM

<u>EQUATION</u>	<u>A</u>	<u>B</u>	<u>R</u>
$144 = 96(1) + 48$	144	96	48
$96 = 48(2) + 0$	96	<span style="border: 1px solid black; padding: 2px;">48</span>	0

$\therefore$  GCF(144, 96) = 48

GCF = B<sub>1</sub>  
WHEN R = 0

③ FIND PRIME FACTORIZATION AND APPLY NEW REASONING:



STEP #1: WRITE DOWN COMMON BASES

STEP #2: SELECT LESSER EXPONENT FOR EACH CHOSEN BASE.

GCF(96, 144) =  $2^4 \times 3^1 = 2^4 \times 3 = 48$

OPTIONAL

Use TI-84+ SILVER EDITION  
GCF(144, 96) = 48



3.5 (18)

(a) All + INT. < 30 @ EXACTLY 3 + DIVISORS

- $2^2 = 4 \Rightarrow \{1, 2, 4\}$
  - $3^2 = 9 \Rightarrow \{1, 3, 9\}$
  - $5^2 = 25 \Rightarrow \{1, 5, 25\}$
  - $7^2 = 49 \Rightarrow \{1, 7, 49\}$
  - $11^2 = 121 \Rightarrow \{1, 11, 121\}$
  - $13^2 = 169 \Rightarrow \{1, 13, 169\}$
- 1   d   d<sup>2</sup>

ADDITIONAL REFLECTION

(b) CONJECTURE: p IS A POSITIVE INTEGER @ EXACTLY THREE (3) POSITIVE DIVISORS IF AND ONLY IF p IS A PERFECT SQUARE OF A PRIME INTEGER.

p IS A POSITIVE INTEGER @ EXACTLY (3) POSITIVE DIVISORS GIVEN PRIME NUMBERS HAVE EXACTLY (2) DIVISORS, 1 AND ITSELF DEF OF PRIME

∴ p IS NOT A PRIME NUMBER, B/C (3) DIVISORS, 1 AND THE NUMBER ITSELF (p) ARE FACTORS OF p MULT. IDENTITY

∴ THE 3<sup>RD</sup> FACTOR OF p MUST BE A PRIME NO. (d) FUND. THM. OF ARITHMETIC

∴ THE THREE FACTORS OF p MUST BE 1, d, d<sup>2</sup>, @ p = d<sup>2</sup>.

NOW LET p = PERFECT SQUARE OF PRIME NUMBER (d)

∴ p = d<sup>2</sup> AND THE DIVISORS OF p ARE 1, d, AND d<sup>2</sup>

∴ p HAS EXACTLY THREE POSITIVE DIVISORS.

(19) @ All + INT. < 30 @ EXACTLY 4 + DIVISORS

- 6 ⇒ {1, 2, 3, 6}
- \* 8 ⇒ {1, 2, 4, 8}
- 10 ⇒ {1, 2, 5, 10}
- 14 ⇒ {1, 2, 7, 14}
- 15 ⇒ {1, 3, 5, 15}
- 21 ⇒ {1, 3, 7, 21}
- 22 ⇒ {1, 2, 11, 22}
- 26 ⇒ {1, 2, 13, 26}
- \* 27 ⇒ {1, 3, 9, 27}

4 & 9 NOT PRIMES!!!

(b) CONJECTURE #1 IF m IS A PRODUCT OF TWO DISTINCT PRIMES, THEN m HAS EXACTLY FOUR POSITIVE FACTORS.

(THE CONVERSE: IF m HAS EXACTLY FOUR POSITIVE FACTORS, THEN m IS A PRODUCT OF TWO DISTINCT PRIMES IS NOT TRUE, B/C OF 8 ⇒ {1, 2, 4, 8} AND 27 ⇒ {1, 3, 9, 27})

### 3.5 PROOF OF CONJECTURE #1

19a) LET  $m = pq$  WHERE  $p$  AND  $q$  ARE PRIMES @  $p \neq q$ .  $\therefore 1, p, q,$  AND  $m$  ARE FOUR FACTORS OF  $m$ . LET  $a =$  SOME OTHER FACTOR OF  $m$ . THEN  $a|m$  AND  $m = ak$  @  $k =$  INTEGER.  $\therefore pq = ak =$  (PRIME FACTORIZATION  $a$ ) (PRIME FACTORIZATION OF  $k$ ). BY FUNDAMENTAL THM OF ARITHMETIC,  $a$  AND  $k$  MUST BE PRIMES @ ONE BEING  $p$  AND THE OTHER BEING  $q$ .  $\therefore$  THERE ARE EXACTLY FOUR POSITIVE FACTORS OF  $m \Rightarrow 1, p, q, m$ .

PLEASE SEE CONJECTURE #2 AND ITS RESPECTIVE PROOF ON PAGE -12-.

### 3.5 19b CONJECTURE #2:

A NUMBER  $M$  HAS EXACTLY FOUR POSITIVE FACTORS IF AND ONLY IF IT IS EITHER A PRODUCT OF TWO DISTINCT PRIMES, OR IT IS A PERFECT CUBE OF A PRIME.

PROOF CONJECTURE #2:  $P \Leftrightarrow Q \wedge R$   
4 FACTORS' PRIMES PERFECT CUBE

SHOW: IF  $M$  HAS EXACTLY FOUR POSITIVE FACTORS, THEN  $M$  IS A PRODUCT OF TWO DISTINCT PRIMES OR IT IS A PERFECT CUBE OF A PRIME ( $n$ ). SINCE BOTH 1 AND  $M$  ARE FACTORS OF ANY NUMBER  $M$ , THEN THE PRIME FACTORIZATION CONSISTS OF EITHER TWO FACTORS  $p$  AND  $q$ , WITH  $pq = M$  OR ONE FACTOR CUBED,  $n^3 = M$  MEANING THE OTHER TWO FACTORS ARE  $n$  AND  $n^2$ .

IN THE FIRST SITUATION IF  $p = q$ , THEN  $M$  WOULD ONLY HAVE THREE POSITIVE FACTORS, BY PROOF IN PROBLEM 8,  $\therefore p$  AND  $q$  MUST BE DISTINCT AND THE FOUR FACTORS ARE 1,  $p$ ,  $q$ ,  $M$ . IN THE SECOND SITUATION, THE THREE FACTORS ARE 1,  $n$ ,  $n^2$ ,  $n^3$  WITH  $n^3 = M$ .

SHOW: IF  $M$  IS A PRODUCT OF TWO DISTINCT PRIMES, THEN  $M$  HAS EXACTLY FOUR FACTORS (ALREADY PROVEN AS MY CONJECTURE #1) OR  $M$  IS A PERFECT CUBE OF A PRIME ( $n$ ). SUPPOSE  $M = n^3$ , THEN BY THE FUNDAMENTAL THEOREM OF ARITHMETIC, ITS FACTORS MUST BE 1,  $n$ ,  $n^2$ ,  $n^3$ , EXACTLY FOUR FACTORS.