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Math 504
Gleason Spring 2005

Exercises 3.4

1. a) positive divisors of 6 are 1,2,3,6
   positive divisors of 15 are 1,3,5,15
   GCF = 3

   b) positive divisors of 12 are 1,2,3,4,6,12
   positive divisors of 54 are 1,2,3,6,9,27,54
   GCF = 6

   c) positive divisors of 21 are 1,3,7,21
   positive divisors of 63 are 1,3,7,9,21,63
   GCF = 21

   d) positive divisors of 28 = 1,2,4,7,14,28
   positive divisors of 54 = 1,2,3,6,9,27,54
   GCF = 2

   e) positive divisors of 17 are 1,17
   positive divisors of 36 are 1,2,3,4,6,9,12,18,36
   GCF = 1

3. a)

Equation	A	B	C
1) 62=4(14)+6	**62**	**14**	**6**
2) 14=2(6)+2	14	6	2
3) 6=3(2)+0	6	2	0

b) gcd(62,14) = gcd(14,6) = gcd(6,2) = gcd (2,0) = 2

c)

5. a) A = 15, B = 6 15 = 6(2)+3
   A = 6, B = 3 6 = 2(3) + 0
   gcd(15,6) = gcd(6,3) = gcd (3,0) = 3

   b) A = 54, B = 12 54 = 12(4) + 6
   A = 12, B = 6 12 = 4(2) + 0
   gcd(54,12) = gcd(12,6) = gcd (6,0) = 6

   c) A = 63, B = 21 63 = 21(3) + 0
   gcd(63,21) = gcd(21,0) = 21

   d) A = 54, B = 28 54 = 28(1) + 26
   A = 28, B = 26 28 = 26(1) + 2
   A = 26, B = 2 26 = 2(13) + 0
   gcd(54,28) = gcd(28,26) = gcd(26,2)= gcd(2,0) = 2

e) A = 36, B = 17 36 = 17(2) + 2
A = 17, B = 2 17 = 2(8) + 1
A = 2 B = 1 2 = 1(2) +0
gcd(36,17) = gcd(17,2) = gcd(2,1) = gcd(1,0) = 1

7.

Equation	Solve for remainders	substitute
62 = 4(14) + 6	62 – 4(14) = 6	
14 = 2 (6) + 2	14 – 2(62-4(14)) = 2	$2 = 9(14) + (-2)(62)$

11. List the sequential multiples of each number in the pair until you reach one that is in common.

a) multiples of 8 = 8,16,**24**,32,40,48……..
multiples of 12 = 12,**24**
LCM = 24

b) multiples of 6 = 6,12,18,24,**30**,36,42,48…….
Multiples of 15 = 15,**30**
LCM = 30

c) multiples of 18 = 18,36,**54**,72,80…..
multiples of 27 = 27, **54**
LCM = 54

d) multiples of 16 = 16,32,48,64,80,96,112,128,144,160,176,192,208,224,240,256,272,288,304,320,**336**,352…….
Multiples of 42 = 42,84,126,168,210,252,294,**336**
The LCM of 16 and 42 is 336.

12.

a	b	gcd(a,b)	lcm(a,b)	$a \cdot b$
8	12	4	24	96
6	15	3	30	90
18	27	9	54	486
16	42	2	336	672

The product of *a* and *b* is the same as the product of the lcm and gcf.

13. a) $6 \cdot 15 = 90$, and since the product of the lcm and gcf is also 90, the lcm will be $90 \div 3 = 30$.

b) $12 \cdot 54 = 648$, so lcm = $648 \div 6 = 108$

c) $21 \cdot 63 = 1323$, so lcm = $1323 \div 21 = 63$

d) $28 \cdot 54 = 1512$, so lcm = $1512 \div 2 = 756$

e) $17 \cdot 36 = 612$, so lcm = $612 \div 1 = 612$

16. Let A and B be positive integers. Suppose there are integers M and N so that 1 1=MA+NB.
    a. the gcd(A,B) must divide 1 and therefore must be equal to 1, because the only positive, integral divisor of 1 is 1.
    b. Let *d* be gcd(A,B). So, if $d|A$, then there is some A such that A = *dr* for some integer *r*. And if $d|B$, then there is some B such that B = *dt* for some integer *t*. (Theorem 7). So, substituting back into the original supposition,
    1 = M*(dr)* + N(*dt)*
    1 = d(Mr + Nt), so *d* divides 1 (Theorem 9) and also must be 1.

17. Let *A* and *B* be positive integers. Suppose there are integers *M* and *N* so that 9 = *MA* + *MB.*
    a. the gcd(A,B) divides 9.
    b. let d = gcd(A,B) Then $d|A$, and $d|B$. If $d|A$, then there is some integer *r* such that A = *dr.* If $d|B$, then there is some integer *t* such that B = *dt.* (Thm 7)
    So, substituting back into the original supposition, 9 = M(*dr*) + N(*dt*).
    And 9 = *d(Mr + Nt)* and *d* divides 9, (Thm 9).

18. Theorem 21: If *p* is a prime number, and *p* divides the product *ab,* then either $p|a$ or $p|b$. Prove that if *p* is as prime number, and *p* divides the product $a_1 \cdot a_x \cdot a_3 ... a_n$, then *p* divides one of the a_i's.

By hypothesis, p is prime and p divides $a_1 \cdot a_2 \cdot a_3 ... a_n$. Show that this is true for the first case of a_1. Obviously $p|a_1 \to p|a_1$. (reflexive?) Assume that if $p|a_1 \cdot a_2 \cdot a_3 .... a_k$ then p divides one of the factors. Prove that $p|a_1 \cdot a_2 \cdot a_3 ... a_k \cdot a_{k+1}$. Let M = $a_1 \cdot a_2 \cdot a_3 ... a_k$, and N = a_{k+1}. By Theorem 21, *p* divides M or N.

Case 1 is $p|M$. By our assumption, *p* divides one of the *a's.* , so, this is true.
Case 2. is $p|N \to p|a_{k+1}$ and a_{k+1} is one of the a's. That's all folks.

Section 3.5 Exploratory excercises

#1. a) i. the product of $2 \cdot 5 \cdot 11 = 110$
ii the sum of $2 \cdot 5 \cdot 11 + 1 = 111$
iii prime factors are $3 \cdot 37$

b) i. the product of $2 \cdot 3 \cdot 5 = 30$	30 + 1 = 31	p.f. are 31
ii. $3 \cdot 7 \cdot 13 = 273$	273 + 1 = 274	p.f. are $2 \cdot 137$
iii. $5 \cdot 11 \cdot 17 = 935$	935 +1 = 936	p.f. are $2^3 \cdot 3^2 \cdot 13$

c) None of the original primes divide the number, none of the new primes equal old primes.

d) If p_1, p_2 and p_3 and p is a prime that divides $(p_1 \cdot p_2 \cdot p_3 + 1)$, then p is not equal to any one of the p_1, p_2 or p_3. Assume $p = p_1$

i. $p | p_1 \cdot p_2 \cdot p_3$ because p equals p_1 and is a factor.

ii. by hypothesis, p divides $p_1 \cdot p_2 \cdot p_3 + 1$. So, $p_1 \cdot p_2 \cdot p_3 + 1 - p_1 \cdot p_2 \cdot p_3 = 1$. Let M = $p_1 \cdot p_2 \cdot p_3 + 1$, and N = $p_1 \cdot p_2 \cdot p_3$. If $p|M$, then there is some number *s*, such that M = $p \cdot s$. If $p|N$, then there is some number *t* such that N = $p \cdot t$.

Iii By substitution, $ps - pt = 1$. $p(s - t) = 1$. Which means *p* must be 1, and 1 is not a prime number.

2.a) the negation of the statement there are an infinite number of primes is there is a finite number of primes.

b) Suppose there are a finite number of primes; *s* primes. Then we can find a complete list of them an write them as $p_1, p_2, p_{3.}...p_s$. Now let us put this list into the Prime Number Machine (that is, let us form the sum $p_1 \cdot p_2 \cdot p_3...p_s + 1$ and find the number of prime factors of this sum). Let p be one of the primes spewed out by the Prime Number Machine.

1. What do we know about this prime? It is not equal to one of the primes.
2. This is a contraction because if it isn't equal to one of the first *s* primes, then it is a new prime, so there are more than *s* primes.

3.5 Exercises

2. a) 124 = $2 \cdot 62$

$62 = 2 \cdot 31$

Prime factors of 124 = $2^2 \cdot 31$

b) 225 = $5 \cdot 45$

$45 = 5 \cdot 9$

$9 = 3 \cdot 3$

Prime factors are $3^2 \cdot 5 \cdot 9$

c) 440 = $2 \cdot 220$

220 = $2 \cdot 110$

110 = $2 \cdot 55$

55 = $5 \cdot 11$

Prime factors are $2^3 \cdot 5 \cdot 11$

d) 1184 = $2 \cdot 592$

$592 = 2 \cdot 296$

$296 = 2 \cdot 148$

$148 = 2 \cdot 74$

$74 = 2 \cdot 37$

Prime factors are $2^4 \cdot 37$

11. GCF of $5^4 7^2$ and $2^3 5^3 7^2$
We are looking for common divisors, so I won't include the 2 because he isn't common. 5^3 will divide 5^4, but 5^4 will not divide 5^3, so I'll choose 5^3 as a common divisor. I will choose 7^2 as a common divisor. Their product is the GREATEST common divisor gcd is $5^3 \cdot 7^2$.

13a) gcd(96,144) by finding all common divisors
Divisors of 96 = 1,2,3,4,6,8,12,16,24,32,48,96
Divisors of 144 = 1,2,3,4,6,8,12,18,16,36,48,72,144
Common divisors are 1,2,3,4,6,8,12,16,48 so the gcd is 48.

b) A = 144, B = 96 144 = 96(1) = 48
A = 96, B = 48 96 = 48(2) +0
Gcd(144,96) = gcd(96,48) = gcd(48,0) = 48

c) $96 = 2 \cdot 48$

$48 = 2 \cdot 24$

$24 = 2 \cdot 12$

$12 = 2 \cdot 6$

$6 = 2 \cdot 3$

Prime factors are $2^5 \cdot 3$

$144 = 2 \cdot 72$

$72 = 2 \cdot 36$

$36 = 2 \cdot 18$

$18 = 2 \cdot 9$

$9 = 3 \cdot 3$

Prime factors are $2^4 \cdot 3^2$

Gcd(144,96) = $2^4 \cdot 3 = 48$

18. a) Find all positive integers less than 30 that have exactly three positive divisors.
4,9,25
b) They are all squares. Furthermore, the number that is squared is prime.
c) If the number has only three factors, two of them have to be one and itself. That leaves room for only one more factor, say *p*, so it must be prime, or the factors of *p* would generate more factors, then there would be more than 3.

19) a) Find all positive integers less than 30 that have exactly four divisors.
6,10,14,15,21,22,26,(I have a bit of a problem here, because 8 and 27 also have exactly four divisors, but not in their prime factorization.)

b) the other two factors besides one and itself have to be prime.

c) Show that if *n* has exactly four factors, then *n* is a product of two distinct primes. Since two of the factors are one and itself, the other two factors, say, *p* and *q* have to be distinct. Otherwise if p and q were the same number, then there would be only 3 factors. Or if one of the numbers were not prime, then it would have another factor that also divides *n*, giving *n* more than four factors.

Show that if *n* is the product of two distinct primes then it has exactly four factors. If *n* is the product of two distinct primes, then they are 1,*p*,*q*, and *n*. If there were one other factor, *a*, then it would have to have a number, *k*, to multiply by to get *n*. so $a \cdot k = n = p \cdot q$. Well, $a \cdot k \neq p \cdot q$ because *p* and *q* are prime. (By theorem 23) unless *a* and *k* are equal to *p* and *q*, in which case, there are four factors.