The French Connection: Borda, Condorcet and the Mathematics of Voting Theory

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Those of us living in democracies today are accustomed to the use of voting in everything from televised competitions like "The Voice" to the Heisman Trophy in college football to US presidential elections. But we also know about the potential for the manipulation of both voter opinions and voting results; indeed, disillusionment with such abuses may help to explain the traditionally low turnout of college students and other young voters in US political elections.

Less commonly known, even among politicians, is the fact that there are deep fairness issues inherent in the very mechanics of voting whenever three or more candidates are involved—and that mathematics can help us to understand these. This fact is encapsulated in a somewhat startling theorem named for the economist and Nobel Prize laureate Kenneth Arrow (1921–2017), who first stated it in his 1951 doctoral thesis [Arrow 1951]. In essence, **Arrow's Impossibility Theorem** asserts that no matter how we decide to combine the individual preferences of a group of voters in order to select a single winner from a slate of three or more candidates, something can always go wrong with the election results in a way that can be characterized as either "irrational" or "unfair." In other words, when it comes to making a collective decision that involves more than two alternatives, there is no systematic way to approach voting that is both rational and fair!

The study of different methods for determining election results and the drawbacks of these methods actually dates back at least to the medieval period. Indeed, as Professor of Politics Iain McLean has remarked, "the theory of voting has in fact been discovered four times and lost three times" [McLean 1990, p. 99]. In this project, we explore the mathematical landscape of today's theory of voting from the viewpoint of works written in connection with its second discovery[2] in the late eighteenth century by two French mathematicians: Jean Charles, Chevalier de Borda (1733–1799) and Marie-Jean-Antoine-Nicolas de Caritat, Marquis de Condorcet (1743–1794). In particular, we examine excerpts from their "lost" texts:

- Borda's "Memoire sur les elections par scrutin" ("Memoir on elections by ballot"), published in 1784; and
- Condorcet's *Esssai sur L'Application de L'Analyse a la Probabilité des Décisions Rendues à la Pluralité des Voix (Essay on the Application of the Analysis of Probabilities to Decisions Rendered by a Plurality of Votes)*, published in 1785.

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[2] McLean's 1990 article examines the first of these discoveries, which was made by two medieval thinkers, Ramon Lull (c. l235–1315) and Nicolas of Cusa (1401–1464), within the context of ecclesiastical elections. More recently, McLean [2019] has written about the third discovery by Charles Dodgson (1832–1898), the British mathematician more widely known by his pen name Lewis Carroll, who was motivated to write on the topic as a result of certain election decisions made by the faculty at Christ Church, Oxford. The fourth discovery was, of course, due to Kenneth Arrow.

We begin in the next section by considering the historical context in which these original works were written, together with biographical sketches of the professional lives of their authors. For readers who may wish to learn more, the Appendix to this project provides additional biographical and historical detail related to both the period leading up to the start of the French Revolution in 1789 (a date that roughly corresponds to the publication of Borda's and Condorcet's works on voting theory) and during the Revolutionary period itself (by the end of which both men were dead).[3]

1. Who were Borda and Condorcet?[4]

Like all human beings, Borda and Condorcet were products of the place and time in which they lived, as were their texts on voting theory. Both men were born in France in the first half of the eighteenth century, Borda in 1733 and Condorcet in 1743. This situates their lives and works firmly within the period of the French Enlightenment,[5] and the political revolutions which that movement helped to spur in the latter part of the eighteenth century. In this section, we share a few highlights of their biographies.

A glance at their early lives reveals Borda and Condorcet shared several similarities. Both were of noble family backgrounds with long military traditions. Both received their initial mathematical training in the Jesuit tradition and early recognition for their mathematical talents. And both were elected to the prestigious French Academy of Sciences[6] while still in their twenties, Borda in 1756 and Condorcet in 1769. Importantly for the story we explore in this project, membership in the Academy was determined by election. The rules and regulations governing these elections were originally quite complicated. In the period immediately preceding the French Revolution, the usual process was for the current membership to produce, via voting, a list of ranked recommendations for new members that was forwarded to the King for his consideration; the King himself was not bound to accept the recommended ranking (or to even choose new members from the Academy's list), but he generally did so.[7] Such was the case in the membership elections of Borda and Condorcet.

[3] The Appendix could also be read in in place of Section 1, since all information in the latter appears in expanded form in the former.

[4] Biographical information for Borda is drawn from Fairclough 2018, Gillmour 1970–1990, Mascart 1919, Noguès n.d., O'Connor and Robertson 2003, Parker 1832, and Tiez 2017; and for Condorcet, from the 1911 *Encyclopedia Britannica*, Acton 2020, Fonseca n.d., Granger 1970–1990, Landes 2018, and O'Connor and Robertson 1996.

[5] While the starting date of the Enlightenment is debated by historians, its epicenter is universally recognized to be Paris. It was in the City of Lights that the artists, writers, politicians, scientists, and philosophers of the day began to regularly meet in intellectual circles, or *salons*, held in the homes of the (generally well-to-do) women (called *salonnières*) who organized and directed these gatherings. Among the *philosophes*, as these thinkers came to be called, human reason came to be seen as the primary source of authority with regard to our understanding of the universe in which we live, as well as a means to improve our living conditions as social beings inhabiting that universe. This in turn led to widespread criticism of existing political, religious and social institutions and the limitations they imposed on individual liberty and happiness.

[6] Founded in 1666, the Academy was one of the earliest learned societies in Europe; before the French Revolution, it was called the Royal Academy of Sciences.

[7] The Academy itself was expected to remain neutral in terms of politics, religion and social issues.

Looking beyond their early lives, there were also marked differences between Borda and Condorcet. Borda's mathematical and scientific interests were quite pragmatic. Upon completing his studies at the Jesuit College in La Flèche in 1748, he entered the French Army as a military mathematician. During this period of his military career, Borda wrote his first mathematical papers: a 1753 paper on geometry that brought Borda to the attention of the mathematician and *philosophe*, Jean le Rond d'Alembert[8] (1717–1783), and a paper on the theory of projectile motion that secured his 1756 election to the French Academy of Sciences. Soon thereafter, in 1758, Borda enrolled at the School of Engineering at Mézières, taking just one year to complete the two-year course of study. He then entered the French Navy, where he served throughout his life in a variety of leadership capacities.[9]

Although Condorcet's family had wanted him to pursue a military career, he rejected that path and instead transferred from the Jesuit College in Reims to the prestigious Collège de Navarre in Paris in 1758, and then studied at the equally prestigious Collège de Mazarin, also in Paris. At the age of 16, he successfully defended a dissertation on mathematical analysis (written in Latin) to a committee whose members included d'Alembert. Condorcet then established himself as an independent scholar, living in Paris on a small allowance provided by his mother,[10] where he continued his study of mathematics and wrote several works, including the *Essai sur le calcul intégral* [Condorcet, 1765] that helped to secure his 1769 election to the Academy of Sciences. Condorcet also fully immersed himself in Parisian Enlightenment society, attending the most important *salons* of the period and establishing friendships with the important *philosophes*, including d'Alembert[11] and the reform economist Jacques Turgot[12] (1727–1781). Condorcet's commitment to Enlightenment ideals also significantly influenced the direction of his academic work towards questions related to social issues. His interest in probability in the early 1780s, for example, was motivated in part by his vision of the role that probability could play in understanding and ameliorating social and economic conditions in France.

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[8] In addition to his work in mathematics, d'Alembert was a contributor to and co-editor (with *philosophe* Denis Diderot (1713–1784)) of the *Encyclopédie*, an important philosophical and literary project aimed at promoting the enlightenment of French society by way of educating its citizens.

[9] These included participation in maritime campaigns during the American Revolutionary War between 1777-1778, appointment as the Major General of the Naval Army in 1781, and an appointment as the French Inspector of Naval Shipbuilding in 1784. Borda also developed a number of utilitarian instruments, most notably the "repeating circle," a celestial navigational device that employed a rotating telescope and a system of repeated observations to greatly reduce measurement error. In 1778, he also published a pre-chronometer method for computing the longitude using lunar distances that was used by the Lewis and Clark expedition.

[10] Condorcet's father died in a military siege just days after his birth.

[11] Condorcet's friendship with d'Alembert was so close that, upon his death in 1783, d'Alembert left his property to Condorcet, thereby ending the financial difficulties that Condorcet faced up to that time. Soon thereafter, in 1786, Condorcet married the influential *salonnière* Sophie de Grouchy. The intellectual partnership between Condorcet and de Grouchy played a critical role in the social reform efforts in which Condorcet engaged during the last decade of his life. In contrast to Condorcet, Borda neither married nor participated in salon culture.

[12] It was Turgot who, while serving briefly as the national Minister of Finance under Louis XVI , appointed Condorcet as the General Inspector of the French Mint in 1775, a position that he held until 1790.

Borda and Condorcet differed not only in their choice of the mathematical problems that they pursued, but also with regard to their general views on the nature of science and mathematics itself.[13] Within the political realm, the differences in their ideological beliefs were even more pronounced. A staunch monarchist, Borda supported the traditionalist views of Turgot's political nemesis, Jacques Necker (1732–1804). Condorcet's alignment with Turgot's politics, on the other hand, placed him among the "modernists" who believed there was a need for constitutional restructuring in France. Taken together, their various ideological differences often placed them in opposite camps within the Academy of Sciences.

Of course, Borda's military and scientific travels often took him away from Paris and the day-to-day workings of the Academy. In contrast, Condorcet became intimately involved in the administrative work of the Academy of Sciences, serving first as its assistant secretary (1773–1776) and then as its Permanent Secretary (1776–1793). As Permanent Secretary, he was responsible for organizing the Academy's various elections, including a vote to allow the Academy's official printing house to publish his own *Essai* on voting theory [Condorcet 1785]. It was also in his capacity as Permanent Secretary that Condorcet arranged for publication of Borda's paper on voting theory (presented to the Academy in 1784) in the Academy's proceedings which were then in preparation (which happened to be for the year 1781).

Exactly five years after Condorcet announced his intention to publish his text on voting theory to the Academy, a landmark event that is often hailed as the official start of the French Revolution took place, when the Bastille prison was destroyed by a group of Parisians on July 14, 1789. Just a few days before this, a new National Constitutional Assembly began to function as a governing body, and, by September 1791, France became a constitutional monarchy with sovereignty residing effectively in that Assembly.

Simultaneously with the development of a new constitution, the Assembly began to institute reforms aimed at addressing a range of national concerns. Among these was a problem to which both Borda and Condorcet contributed their technical expertise: the development of a national system of weights and measures. Borda in particular was heavily involved in the coordination and conduct of the scientific work required to complete that effort, which culminated in the development of the (now nearly universal) base ten *metric system*. Perhaps not surprisingly in light of his political conservatism, his contributions to the development of the metric system essentially constituted the full extent of Borda's participation in revolutionary activities.

[13] Borda considered himself a disciple of Georges-Louis Leclerc, count de Buffon (1707–1788), a fellow Academician whose views about mathematics and science often came into conflict with those of d'Alembert. As d'Alembert's *protégé*, Condorcet's general views about mathematics, science and its correct conduct thus frequently ran opposite to those of Borda and Buffon. See, for example, the discussion of the views held by Buffon and d'Alembert regarding rigor in mathematical proof in [Richards 2009].

In contrast, Condorcet became increasingly involved in the political events and social reforms[14] of the revolutionary period. When the National Constitutional Assembly was disbanded in September 1791 (its work in drafting a new constitution having been completed) and the Legislative Assembly took its place from October 1791 through August 1792, Condorcet was elected as a delegate to the new Assembly and served as its secretary. It was thus Condorcet who composed the Legislative Assembly's declaration that justified the suspension of the monarchy in 1792. In September of that year, the Legislative Assembly was replaced by the National Convention and the (First) Republic of France was declared. Condorcet was again elected as a delegate to the Convention, where he served as both secretary and vice-president, and also chaired the committee responsible for drafting the new republic's constitution. Regrettably, Condorcet's version of that constitution was rejected (in June 1793) in favor of a hastily-drafted "Constitution of Year I" that was put forward by a political group far more radical than the one with which Condorcet was associated. The Reign of Terror began soon thereafter. Between September 1793 and July 1794, Queen Marie-Antoinette and thousands of other French citizens from all social classes and professions were accused of counterrevolutionary actions and met their fate, as Louis XVI had somewhat earlier, at the blade of the guillotine.

Although Condorcet himself escaped the guillotine, he did not outlive the Reign of Terror. After openly and passionately accusing the Committee of Public Safety of using fear tactics to gain passage of their version of the constitution, a warrant for his arrest was issued by the Convention in July 1793. Condorcet became a fugitive, and hid in the Parisian home of Madame Vernet[15] for the next eight months.[16] Concerned about the peril in which he was placing his protector, Condorcet fled Paris on foot on March 25, 1794, but was denounced to local authorities by an innkeeper in the nearby village of Clamart-de-Vignoble. Placed under arrest, he was transferred to a guarded house located in the village of Bourg-la-Reine[17] on March 29, 1794. The following day, Condorcet was

[14] Condorcet was ahead of his time, for instance, in supporting the abolition of slavery, as well as equal political rights for Jews, Protestants and women. His first essay advocating the abolition of slavery, "Reflexions sur l'esclavage des nègres" ("Reflections on the enslavement of negros"), was published well before the revolution, in 1781. He was also a founding member of the Société des Amis des Noirs (Society of the Friends of Negros), established in 1788. In his 1790 essay "Sur l'admission des femme au droit de Cité" ("On the admission of women to the right of Citizenship"), Condorcet argued that the ability to reason, a human attribute shared by men and women of all races, justifies granting equal rights to all, and concluded that

> Either no member of the human race has any true rights, or else they all have the same ones; and anyone who votes against the rights of another, whatever his religion, colour or sex, automatically forfeits his own.

[15] Located on Paris' left bank on the Rue des Fossoyeurs (now 15 Rue Servandoni), Condorcet's refuge for these months is situated just behind the beautiful Église Saint Etienne de Mont, where the mathematician and philosopher Blaise Pascal (1623–1662) is buried.

[16] During this time, Condorcet wrote his most famous philosophical work, *Esquisse d'un tableau historique des progrès de l'esprit humain* (*Sketch of a historical picture of the progress of the human mind*), a declaration of his enlightenment belief in the power of rationality and perfectibility of human society.

[17] This house was located at what was then 49 Grande Rue (now 81 Ave Le Clerc). The town itself was known by a more revolutionary name at that time: Bourg-de-l'Égalité. Under its current name, Bourg-la-Reine is familiar to students of the history of mathematics as the birthplace of Évariste Galois (1811–1832), one of the founders of modern abstract algebra and a staunch supporter of the French republicans during the post-Napoléonic era, who died of injuries suffered in a duel at the age of 21.

found dead in his makeshift cell; whether he took his own life, died of natural causes, or was murdered is unknown to this day**.**

As for Borda, after retiring to his family estate during the Reign of Terror, he returned to Paris and resumed his work on the metric system. He died after a long illness on February 20, 1799, at age 65, just months before the *mètre des Archives*, a platinum bar selected as the first prototype of the then-new meter, was placed in the National Archive**.**

2. The Borda Count Method of Voting

The first method of voting that we consider is named in honor of Borda's 9-page 1784 paper, "Memoire sur les elections par scrutin" ("Memoir on elections by ballot"). Under the **Borda Count Method of Voting**, candidates are assigned points based on their rankings across all ballots; for instance, 1 point for each last place vote, 2 points for each second-to-last place vote, and so on. The candidate with the highest total number of points is then selected as the winner of the election. Borda himself called this voting method an "election by order of merit," and presented the following example of it.[18]

Suppose again 21 voters and three presented subjects *A, B, C* and let

A	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*B*	*B*	*B*	*B*	*B*	*B*	*B*	*C*	*C*	*C*	*C*	*C*	*C*
B	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*B*	*B*	*B*	*B*	*B*	*B*
C	*B*	*B*	*B*	*B*	*B*	*B*	*B*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*

be the 21 election ballots.

Exercise 1

Before reading more from Borda, take a look at these 21 ballots. Which candidate do you think should win this election and why? Is there another candidate who you think might have an equally strong claim to being the "fair" winner of this election? Why or why not?

Now let's go back to Borda's computations of the election results under his point system:

[18] All French-to-English translations in this project are due to its author. Throughout this project, we use the following symbol (and sans serif font) to set apart primary source excerpts:

One will have by what has been already said, the comparative value of votes by multiplying the first-place votes by 3, the second-place votes by 2, and the third by 1, which will give the following results.

$$\text{Votes for } A \begin{Bmatrix} \text{8 first-place votes, multiplied by } 3 \ = \ 24 \\ \text{13 third-place votes, multiplied by } 1 \ = \ 13 \end{Bmatrix} 37.$$

$$\text{Votes for B} \begin{Bmatrix} \text{7 first-place votes, multiplied by } 3 \ = \ 21 \\ \text{7 second-place votes, multiplied by } 2 \ = 14 \\ \text{7 third-place votes, multiplied by } 1 \ = \ 7 \end{Bmatrix} 42.$$

$$\text{Votes for C} \begin{Bmatrix} \text{6 first-place votes, multiplied by } 3 \ = \ 18 \\ \text{14 second-place votes, multiplied by } 2 \ = 28 \\ \text{1 third-place votes, multiplied by } 1 \ = \ 1 \end{Bmatrix} 47.$$

From which one can see the superiority of votes will be in favor of subject *C*, that the second place will be given to subject *B* & the last to subject *A.*[19]

Exercise 2

Verify that Borda's computations are correct for this particular election. Is the winner under the Borda Count Method of Voting the same as the candidate you selected as the election winner in Exercise 1? Are you surprised by this? Why or why not?

Borda continued his discussion of this example as follows.

It may be remarked that if one conducted the election in the usual manner, one would have had the following result:

8 votes for *A,*
7 votes for *B*,
6 votes for *C*,

That is to say that the plurality would have been for subject *A*, who is last in the opinions of the voters, & that the subject *C*, who is really the first, would have had fewer of the votes than each of the other two.[20]

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[19] Borda 1784, pp. 661–662.

[20] Borda 1784, pp. 661–662.

In this last excerpt, Borda was complaining about what is today called the **Plurality Method of Voting**, in which the candidate with the most first-place votes is selected as the winner. He first expressed this complaint (and illustrated it by way of this same example) at the very start of his paper in order to illustrate how the Plurality Method could "induce an error" in elections involving more than two candidates.

It's clear then that the subject *A* will have, in the collective opinion of the voters, a marked inferiority, as much with respect to *B* as with respect to *C*, because each of these latter [two], compared to the subject *A* has 13 votes, while the subject *A* only has 8; from this it clearly follows that the voice of the electors would exclude subject *A*.

Exercise 3

Look again at the 21 ballots in Borda's example (reproduced below for your convenience). Verify Borda's statement that the result of head-to-head comparisons with either of the other two candidates (*B* or *C*) to candidate *A* is 13 to 8. Do you agree with Borda that "from this it clearly follows that the voice of the electors would exclude subject *A*"? Explain why or why not. Suppose we decide to adopt this conclusion, and exclude candidate *A* from the election. Which of the other two candidates do you think should then win this election, and why?

A	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*B*	*B*	*B*	*B*	*B*	*B*	*B*	*C*	*C*	*C*	*C*	*C*	*C*
B	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*B*	*B*	*B*	*B*	*B*	*B*
C	*B*	*B*	*B*	*B*	*B*	*B*	*B*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*

Having noted that a candidate who loses to every other candidate in head-to-head comparisons (in this example, candidate *A*) could end up winning the overall election when only the first-place votes are considered in deciding the results of an election, here is the analogy that Borda provided to describe how such a situation could come about.

In reflecting on the reported example, one sees that the subject A only has the advantage in the results of the [plurality] election, because the two subjects *B* & C, who are superior to him, almost equally divided between them the votes of the 13 electors [who did not place *A* in first place]. One could compare them fairly exactly to two athletes, who, after having spent their forces one against the other, were then defeated by a third [athlete] more feeble than either of them.[21]

[21] Borda 1784, pp. 657–658.

Exercise 4

Comment on Borda's athlete analogy. How well does it seem to capture what went wrong in this particular example? Try to come up with another analogy that you think would describe it equally well or better than Borda's. How else might you describe this kind of election outcome? How often and why do you think this kind of situation happens in actual elections?

Borda went on in his paper to present a second method of voting which also assigns a point total to each candidate, but by way of a series of what Borda called "special elections" in which the candidates are "taken two by two." For example, in the case of three candidates *A, B,C,* there would be three such special elections (or pairwise comparisons), the results of which Borda represented with general variables as follows.

1.st election between A & B . . . $\begin{cases} a \text{ votes for } A, \\ b \text{ votes for } B, \end{cases}$

2.nd election between A & C . . . $\begin{cases} a' \text{ votes for } A, \\ c \text{ votes for } C, \end{cases}$

3.rd election between B & C . . . $\begin{cases} b' \text{ votes for B}, \\ c' \text{ votes for C}, \end{cases}$ . . . . [22]

Borda's plan was then to sum the points earned by each candidate across all that candidate's pairs (getting $a + a'$ for *A*, $b + b'$ for *B*, and $c + c'$ for *C*) and then use those sums to decide the election outcome. As it turns out, this method always gives the same overall election outcome as the basic Borda Count Method of Voting. Borda proved this fact by using some algebra and the observation that the total number of votes in each comparison adds up to the total number of voters in the election (so that $a + b = a' + c = b' + c'$). Following his algebraic demonstration that the two methods in his paper are, in fact, mathematically equivalent,[23] Borda concluded his discussion of the two methods he proposed by commenting:

> Besides [the fact that these two methods produce the same results], we will remark here that the second form of election of which we have spoken, would be awkward in practice, when a large number of candidates are presented, because

[22] Borda 1784, p. 662.

[23] Borda did not explicitly say why he offered up this second method. One reason may have been to address potential resistance to his own newly-proposed method (i.e., election by order of merit); after all, two-candidate elections are unproblematic (clearly, the one with the most votes wins), and on the surface his "special elections" method is based on a series of two-candidate elections (as in a series of wrestling matches).

the number of particular elections that it would be necessary to complete, will be very large. For this reason one should prefer the form of election by order of merit, which is much more expedient.[24]

Exercise 5

To better understand what it was that Borda found to be "awkward" about using head-to-head elections as part of a voting method, notice that adding a fourth candidate *D* will give us three additional pairs (*A* & *D*; *B* & *D*; *C* & *D*) to consider (in addition to the three pairs *A* & *B* ; *A* & *C*; *B* & *C*), for a total of 6 pairwise comparisons. Similarly, to get the number of pairs in an election with a fifth candidate *E*, we can take the total for 4 candidates (i.e., 6 pairs) and add in the number of pairs that involve *E*. Do this and record your answer in the following table; then complete the rest of the table. Do you agree or disagree with Borda that the use of pairwise comparisons is awkward? Why or why not?

Candidate Names	Number of Candidates	Number of Pairs	List of Candidate Pairs to Compare
A,*B*,*C*	3	3	*A* & B ; *A* & *C*; B & *C*
A,*B*,*C*,*D*	4	6	*A* & B ; *A* & *C* ; *A* & *D* ; B & *C ; B* & *D*; *C* & *D*
A,*B*,*C*,*D*,*E*	5		
	6		
	7		

In the next section of this project, we consider how Condorcet made use of pairwise "head-to-head" comparisons of candidates in a considerably different fashion than that proposed by Borda, and despite the computational "awkwardness" about which Borda complained.

[24] Borda 1784, pp. 663-664.

3. Condorcet, the Pairwise Comparison Method of Voting, two Fairness Criteria and a Paradox

Borda described the purpose of his brief (9-page) "Memoire" simply as an illustration that, in the case of elections with three or more candidates, the common belief that a plurality of (first-place) votes "always indicates the wish of the electorate" is mistaken [Borda 1784, p. 657]. In contrast, Condorcet began his 1875 book with a 191-page Discours Préliminaire, or Preface, in which he laid out the motivation behind and the general layout of his 304-page *Essai.* Voting theory itself occupied a small part of this massive book, which Condorcet conceived of as a broad investigation into the problem of how individuals within a society could be provided with sufficient assurance that the decisions rendered by groups of individuals (e.g., trial juries, legislators, or voters) are, in fact, correct. In the case of a jury trial, the notion of a "correct decision" is clear: guilty individuals should be judged to be guilty, while the innocent should go free. In the case of voters in a legislative assembly or a popular election, the notion of a "correct decision" was understood by Condorcet as the selection of the law or candidate that would best serve the common good. Naturally, the appraisals of the individuals voting on these decisions could be mistaken concerning the issue at hand, even in the best of circumstances. By quantifying the likelihood that each individual voter is correct, Condorcet was able to use the (relatively new) mathematical field of probability as a tool for analyzing collective decision-making procedures under various conditions (e.g., the number of jurors or voters, the competence of the individual decision makers), with the goal of maximizing the probability that the final outcome is correct.

In this project, we will set aside the probability aspects of Condorcet's text, and instead focus on three voting theory concepts that appear in his *Essai* and are now named in his honor: **Condorcet Paradox**, **Condorcet Candidate** and **Condorcet Fairness Criterion**. All three ideas are related to a voting method that involves head-to-head comparisons of each pair of candidates, a procedure that Borda also considered in connection to his (rejected) "special elections" method of voting. In the next few subsections, we examine excerpts from the Preface to Condorcet's *Essai* that justify the attachment of his name (and not Borda's) to these ideas.

3.1 Characterizing Fairness: The Condorcet and Majority Criteria

Condorcet's analysis of elections involving more than two candidates appeared relatively late in the Preface of his *Essai*, following several examples related to jury trials and legislative voting that illustrated his concern about the commonly-used Plurality Method of Voting. His primary reason for seeking an alternative to the Plurality Method of Voting was, in fact, quite similar to that presented by Borda:

> The method used in ordinary elections is defective. In effect, each voter is limited to naming the one that he prefers: thus in the example of three Candidates, someone that votes for *A* does not announce his view on the preference between *B*

& *C*, and similarly for the others. However, there may result from this manner of voting a decision that is actually contrary to the plurality [choice].[25]

Like Borda, Condorcet also illustrated how the Plurality Method of Voting could produce a result that was "actually contrary to the plurality [choice]" with a specific example.

Suppose, for example, 60 Voters, of whom 23 are in favor of *A*, 19 are in favor of *B*, & 18 are in favor of *C*; suppose next that the 23 voters for *A* would have unanimously decided that *C* is better than *B*; that the 19 Voters for *B* would have decided that *C* is better than *A*; and finally that of the 18 Voters for *C*, 16 would have decided that *B* is better than *A*, & only 2 that *A* is better than *B*.[26]

Let's pause here to try to organize the ballot information in this example so that it will be easier to compute the election results. A standard way to do this today is to construct a table, called a **Preference Schedule**, that collapses all identical ballots into a single column. For instance, in Condorcet's example, we know that 23 voters listed the three candidates in the following order of preference: *A*,*C*,*B*. We can record all 23 of these ballots in a single column of the table as follows:

Number of voters	23
First Place	*A*
Second Place	*C*
Third Place	*B*

Exercise 6

a. Complete the Preference Schedule for Condorcet's example by filling in the number of voters for each column. Verify that there are 60 votes in all.

Number of voters	23			
First Place	*A*	*B*	*C*	*C*
Second Place	*C*	*C*	*B*	*A*
Third Place	*B*	*A*	*A*	*B*

b. If we were to use the Plurality Method of Voting (in which the candidate with the most first place votes wins the election), which candidate would be selected as the winner of this election? Which candidates will come in second and third place overall? Do these results seem fair to you? Why or why not?

[25] Condorcet 1785, p. lviii.

[26] Condorcet 1785, p. lviii.

Now, let's go back to Condorcet's discussion of this same example.

One would have therefore,

1° 35 votes for the proposition *B* is better than *A*,
& 25 for the contradictory proposition.

2° 37 votes for the proposition *C* is better than *A*,
& 23 for the contradictory proposition.

3° 41 votes for the proposition *C* is better than *B*,
& 19 for the contradictory proposition.

We would therefore have the system of the three propositions that have the plurality, formed of the three propositions.

B is better than *A*,
C is better than *A*,
C is better than *B*,

which implies a vote in favor of *C*.[27]

Exercise 7

a. What did Condorcet mean by the phrase "contradictory proposition"?

b. Verify Condorcet's conclusion 1° by using your Preference Schedule from Exercise 6 to show that there were 35 voters in all who ranked *A* higher than *B* on their ballots, whereas the other 25 voters ranked *B* higher than *A.* Also verify Condorcet's conclusions 2° and 3°.

c. Do you agree with Condorcet's conclusion that the results of this election imply "a vote in favor of C"? Why or why not?

As you showed in Exercise 6(b), under the Plurality Method of Voting, candidate *A* would win this election, since *A* has more first-place votes than the other two candidates. According to Condorcet's analysis in the preceding excerpt, however, candidate *C should* be the winner, since *C* is preferred in every head-to-head comparison against the other candidates. Today, such a candidate is called a **Condorcet Candidate**, and the assertion that a Condorcet Candidate (when one exists) should win the election is known as the **Condorcet Fairness Criterion**.

[27] Condorcet 1785, p. lviii.

Exercise 8

Comment on whether you think the requirement in the Condorcet Fairness Criterion is or is not fair (and to whom). That is, if there is a candidate in an election who beats every other candidate in a head-to-head run-off, would it be unfair for that candidate to lose that election? Why or why not? Under what circumstances, if any, do you think it would be more fair to pick someone other than a Condorcet Candidate to be the overall winner of an election?

In the language of voting theory, Condorcet's example above (in which the Condorcet Candidate *C* would lose the election if we only counted first-place votes) proves that the **Plurality Method of Voting violates the Condorcet Fairness Criterion**. Of course, there are other elections in which the Plurality Method does choose the Condorcet Candidate as the winner.[28] (Pause briefly here to try to come up with your own example of this!) The key thing to notice about the language that we use today is that it suffices to find *one* election in which a particular voting method fails to meet the requirements of a particular fairness criterion in order to conclude that the method violates that fairness criterion. In other words, *one counterexample* is all it takes when it comes to proving *violation*. On the other hand, before we can say that a given method of voting *satisfies* a particular fairness criterion, we need to show that the requirements of the fairness criterion *hold in every possible election*. This might sound much harder to do!

Let's pause briefly in our reading of Condorcet's *Essai* to start thinking about how Borda's proposed voting method fares with regard to the particular idea of fairness captured in the Condorcet Fairness Criterion: does the Borda Count Method of Voting violate or satisfy the Condorcet Fairness Criterion.

Exercise 9

Let's look again at Borda's example of a three-candidate election (p. 5), for which the 21 ballots were:

A	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*B*	*B*	*B*	*B*	*B*	*B*	*B*	*C*	*C*	*C*	*C*	*C*	*C*
B	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*C*	*B*	*B*	*B*	*B*	*B*	*B*
C	*B*	*B*	*B*	*B*	*B*	*B*	*B*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*

a. Complete the Preference Schedule for this example by filling in the number of voters for each column. Be sure to account for all 21 votes.

Number of voters				
First Place	*A*	*A*	*B*	*C*
Second Place	*B*	*C*	*C*	*B*
Third Place	*C*	*B*	*A*	*A*

[28] There are also elections in which there is no Condorcet Candidate at all, and we'll see examples of this below. In such cases, however, the Condorcet Fairness Criterion simply doesn't apply. After all, a non-existent candidate can neither win nor lose an election.

b. Use the Preference Schedule from part (a) to determine who is the winner of each one-on-one comparison: *A* versus *B* ; *A* versus *C* ; *B* versus *C*.

c. Based on part (b), which of the three candidates is a Condorcet Candidate for this election? [Someone is, so if you didn't find one, go back and recount the ballots in the one-on-one comparisons!]

d. Recall (from the first excerpt on page 6) that the winner of this election under the Borda Count Method of Voting is candidate *C*. Is *C* also the Condorcet Candidate? That is, does this example show that the Borda Count Method of Voting violates the Condorcet Fairness Principle? [Remember: one example is never enough to show that a method of voting satisfies a fairness criterion, but is sufficient to establish violation.]

e. Do you think that what happens for the Condorcet Candidate in this example will always happen when the Borda Count method is used to select the winner? That is, could there be some elections in which the Condorcet Candidate wins under the Borda Count method and other elections in which the Condorcet Candidate loses under the Borda Count method? Give some reasons for your response (but don't worry about coming up with a complete proof).

Let's return now to our reading of Condorcet' *Essai.* Having used the example presented above to exhibit one inherent flaw in the Plurality Method of Voting (e.g., it violates the Condorcet Fairness Criterion), Condorcet noted that there is also one circumstance in which the Plurality Method of Voting could suffice to select the correct choice among candidates.

> One sees therefore already that one should reject the form of election generally adopted: if one wishes to conserve it, it will be possible [to do so] only in the case where one is not required to elect on the spot, & where one could require that only someone who would have gathered more than half of the votes would be looked upon as elected.[29]

Today, a candidate with a majority (i.e., over 50%) of the first-place votes is known as a **Majority Candidate,** and the assertion that a Majority Candidate (when one exists) should win the election is known as the **Majority Fairness Criterion**. In the excerpt that we have just read, Condorcet has (implicitly) claimed that a Majority Candidate is also necessarily a Condorcet Candidate.[30] Let's pause again in our reading to think about this claim, and the Majority Fairness Criterion more generally.

[29] Condorcet 1785, p. lviii.

[30] Note that the converse is not true. In Condorcet's first election example (represented in Exercise 6), the Condorcet Candidate *C* did not have a majority of first-place votes, and thus was not a Majority Candidate.

Exercise 10

Comment on whether you think the requirement in the Majority Fairness Criterion is or is not fair (and to whom). That is, if there is a candidate in an election who has over 50% of the first-place votes, would it be unfair for that candidate to lose that election? Why or why not? Under what circumstances, if any, do you think it would be more fair to pick someone other than a Majority Candidate to be the winner?

Exercise 11

Suppose *M* is a Majority Candidate in a particular election, so that *M* has over half of the first-place votes.

a. Use full sentences to explain why each of the following are true, no matter how many other candidates there are in that election or how those candidates are placed on the ballots.

   i. *M* is also a Condorcet Candidate.

   ii. *M* will necessarily win the election under the Plurality Method of Voting.

   In other words, the **Plurality Method of Voting satisfies the Majority Fairness Criterion**.

b. Why are the results in part a not a contradiction to the fact that Plurality Method of Voting violates the Condorcet Fairness Criterion? Again, use full sentences to explain.

Exercise 12

Let's now start thinking about whether the Borda Count Method of Voting violates or satisfies the Majority Fairness Criterion by considering the following election involving 4 candidates and 26 voters.

Number of voters	14	8	4
First Place	*A*	*B*	*D*
Second Place	*B*	*D*	*A*
Third Place	*C*	*C*	*C*
Fourth Place	*D*	*A*	*B*

a. Pause to look over the Preference Schedule. Who do you think should win this election, and why?

b. Explain why *A* is a Majority Candidate.

c. Rank the four candidates using the Borda Count Method of Voting, assigning 4 points for each first-place vote, 3 for each second-place vote, 2 for each third-place vote and 1 for each fourth-place vote.

d. Does this example show that the Borda Count Method of Voting violates[31] the Majority Fairness Criterion?

e. Do you think that what happens for the Majority Candidate in this example will always happen when the Borda Count Method is used to select the winner? That is, could there be some elections in which the Majority Candidate wins under the Borda Count Method and other elections in which the Majority Candidate loses under the Borda Count Method? Give some reasons for your response (but don't worry about coming up with a complete proof).

3.2 The Pairwise Comparison Method of Voting, and a Voting Paradox

Let's go back now to reading Condorcet, to see what he had to say about what should be done to replace the flawed Plurality Method of Voting.

> So one should in general substitute for this [plurality] form [of voting] that in which each Voter, [by] expressing the order according to which he places the Candidates, would pronounce at the same time on the preference that he would accord respectively to each.[32]

In other words, an election should be decided based on the outcomes of the head-to-head comparisons of every pair of candidates, which in turn can readily be determined from voters' ranked ordering of all candidates. We've already seen, in Exercise 9, how to determine the outcome of each pairwise comparison from the Preference Schedule for an election. (Don't worry if you feel you need more practice with this—there are more such exercises coming soon!) Importantly, Condorcet's use of head-to-head comparisons differed from the way such comparisons featured in Borda's "special elections" method. In particular, Condorcet tallied the actual number of votes cast for each candidate in the pair only to determine which of the two candidates in the pair wins that particular comparison: did the voters find that A is better than B, or that B is better than A?[33] Once these results are known for all pairs, choosing the winner is (generally) straightforward: simply count the number of pairs each candidate won, and whoever has the highest such number is declared the overall winner. This method of choosing an election winner based on how many pairwise

[31] Remember: one example is never enough to show that a method of voting satisfies a fairness criterion, but is sufficient to establish violation.

[32] Condorcet 1785, p. lviii.

[33] In contrast, recall that Borda's method of special elections summed the numbers of votes cast for each candidate in all of the various pairs in a way that rendered that method equivalent to the basic Borda Count Method of Voting. Given this equivalency together with the dramatic increase that occurs in the number of pairs as the number of candidates increases, Borda ultimately rejected the use of head-to-head comparisons in this manner as an avoidable inconvenience. This is perhaps why a Condorcet Candidate isn't instead called a Borda candidate!

comparisons each candidate wins is known today by the descriptive name **Pairwise Comparison Method of Voting**.

Exercise 13

Determine the election outcomes (winner, second-place, etc.) for the election with the following Preference Schedule under the Pairwise Comparison Method of Voting. (Remember from Exercise 5 that there are 6 pairs in consider in all.) Is there either a Majority Candidate or a Condorcet Candidate in this election? Use full sentences to explain how you know.

Number of voters	13	9	7	5
First Place	*C*	*A*	*D*	*D*
Second Place	*D*	*B*	*A*	*C*
Third Place	*B*	*D*	*B*	*A*
Fourth Place	*A*	*C*	*C*	*B*

Exercise 14

Use full sentences to explain why each of the following claims (implicitly) made by Condorcet in the last excerpt we read from his *Essai* are true.

a. The **Pairwise Comparison Method of Voting satisfies the Condorcet Fairness Criterion**. That is, if *C* is a Condorcet Candidate in an election, then *C* will necessarily win that election under the Pairwise Comparison Method of Voting.

b. The **Pairwise Comparison Method of Voting satisfies the Majority Fairness Criterion**. That is, if *M* is a Majority Candidate in an election, then *M* will necessarily win that election under the Pairwise Comparison Method of Voting. (*Hint: See Exercise 11a..)*

Although Exercise 14 shows us that the Pairwise Comparison Method of Voting has strengths with regard to fairness criteria, this method does require us to compute the results of every possible pair of candidates. As you saw in Exercise 5, the number of such pairs rises quite quickly as the number of candidates increases. Condorcet was well aware of this fact as well, as he hinted here:

> One would draw from this order [in which candidates are ranked by each voter] the three propositions which should form each opinion, if there are three Candidates; the six propositions that should form each opinion, if there are four Candidates; the ten, if there are five, &c. in comparing the votes in favor of each of these propositions or their contraries.[34]

[34] Condorcet 1785, p. lx.

Bonus Exercise

In the following excerpt from Part 2 of his *Essai*, Condorcet gave general formulas for computing both the number of different individual preference ballots that are possible, and the number of different pairwise comparisons that must be considered in an election involving n candidates. Explain why the formulas that he gave are correct.

1.° . . . Therefore for n Candidates, one will have $n(n-1)\ldots 2$ possible opinions [that a Voter can give] . . . .

2.° Each Voter having thus given his opinion by indicating the candidates' order of worth, if one compares them two by two, one will have in each opinion $\frac{n(n-1)}{2}$ propositions to consider separately . . . .[35]

Condorcet was also well aware that another (perhaps more troubling) complication can sometimes arise when using the Pairwise Comparison Method of Voting. He again illustrated this by way of an example.

Let us suppose indeed that in the example already chosen, where one has 23 votes for *A*, 19 for *B*, 18 for *C*, the 23 votes for *A* are for the proposition *B* is better than *C*; this proposition [*B* is better than *C*] will have a plurality of 42 votes against 18.

Let us next suppose that of the 19 votes in favor of *B*, there are 17 for *B* is better than *C*, & 2 for the contradictory proposition; this proposition *C* is better than *A* will have a plurality of 35 votes against 25. We suppose finally that of the 18 votes for *C*, 10 are for the proposition *A* is better than *B*, & 8 for the contradictory proposition, we will have a plurality of 33 votes against 27 in favor of the proposition *A* is better than *B*.[36]

Notice that Condorcet's presentation of this example mixes together the information about the individual ballots with the results of the pairwise comparison scores that result from that ballot information. Let's try separating those out to better understand his example.

[35] Condorcet 1785, pp. 125–126.

[36] Condorcet 1785, p. lxj.

Exercise 15

a. Use the information in the first sentence of each of the three paragraphs of the preceding excerpt to construct the Preference Schedule for this election.

b. Use the Preference Schedule in part a to verify Condorcet's values for the scores in each of the three pairwise comparison (*B* & *C* ; *A* & *C* ; *A* & *B).* These values are found in the second sentence of each of the three paragraphs of the preceding excerpt.

c. Comment on what you notice about these election results. Is there a Condorcet Candidate for this election? Is it possible to declare an overall winner here? If so, who should the winner be and why? If not, why not?

Here is Condorcet's statement of the outcome for this particular election.

> The system that obtains the plurality [in this election] will therefore be composed of three propositions,
>
> *A* is better than *B,*
> *B* is better than *C,*
> *C* is better than *A.*
>
> This system is ... one of the two [possible systems] that implies contradiction.[37]

Did you notice the contradictory nature of these conclusions too? If we use the symbol ">" to represent "is better than" (or "beats"),[38] then the three propositions in the final system become:

$$A > B,$$
$$B > C,$$
$$C > A.$$

But wait! Given $A > B$ and $B > C$, we would usually expect to have $A > C$, by what mathematicians call the **transitive property**.[39] That is, if *A* beats *B* and *B* beats *C*, we fully expect that *A* would also beat *C*. When we fill out our individual preference ballots as a voter, that will certainly be the case; indeed, the very nature of the act of ranking the candidates in integer order (i.e., 1st, 2nd, 3rd, etc.), together with the transitivity nature of integer order, forces us to act rationally here (or, at least, to try to do so). But, as we see in this particular example, it's possible for rational (or transitive) rankings at the individual level to lead to irrational (or intransitive) decisions at the group level. This particular

[37] Condorcet 1785, p. lxj. (A very slight adjustment has been made to the primary source here.)

[38] Condorcet used the symbols < and > at some points in his *Essai*, but more often he wrote out the relation between two candidates in words. You should use whichever representation (symbolic or verbal) you find easiest.

[39] The idea behind the name "transitivity" for this property is that *B* serves as a transit from *A* to *C*.

feature of the Pairwise Comparison Method of Voting is today known as the **Condorcet Paradox:** the final election results of the head-to-head comparisons *can* be non-transitive.[40]

Exercise 16

At the end of the last excerpt that we read from his *Essai*, Condorcet mentioned that there are two possible contradictory systems (or non-transitive election results) that can come up in an election involving three candidates A, B, C. One of these two was the final outcome in his example:

$A > B$	(A beats B)
$B > C$	(B beats C)
$C > A$	(C beats A)

a. What is the other contradictory (non-transitive) system?

b. Show that there are only two contradictory systems in elections involving three candidates by writing out all eight possible systems that can result from the outcomes of their pairwise comparisons. For each of the six non-contradictory systems, also indicate the rank order for the overall outcomes (e.g., who is the winner of the election and who comes in second and third place overall).

Condorcet himself was not fazed by the fact that the final results of the election could be non-transitive. Indeed, he did not even view this as an impediment to the viability of the Pairwise Comparison Method of Voting. Instead, he devoted several pages of his *Essai* to a discussion of how to proceed with voting in order to produce the winner with the highest probability of being the correct choice in the event that election results do result in one of these "contradictory systems." The next two exercises take a look at examples that he provided to show how this might even be fairly straightforward to do.

Exercise 17

Below are the six propositions representing the final outcome for an election involving four candidates A, B, C, D. Where does the Condorcet Paradox show up in this election? Are you able to still (fairly) select a winner for this election? Why or why not?

1. *A* is better than *B,*
2. *A* is better than *C,*
3. *A* is better than *D,*
4. *B* is better than *C,*
5. *D* is better than *B,*
6. *C* is better than *D.*[41]

[40] An opinion of the form "$A > B$, $B > C$, $C > A$" is also known as a Condorcet cycle.

[41] Condorcet 1785, p. clxxj.

Exercise 18

Below are the ten propositions representing the final outcome for an election involving five candidates *A*, *B*, *C*, *D*, *E*. Notice that we could easily select the top two finishers in this election, but run into problems with the final rankings of the other three candidates due to the Condorcet Paradox. What might be done to try to resolve this difficulty, or do you think it's not possible to do so fairly? Use full sentences to explain your response.

1. *A* is better than *B,*
2. *A* is better than *C,*
3. *A* is better than *D,*
4. *A* is better than *E,*
5. *B* is better than *C,*
6. *B* is better than *D,*
7. *B* is better than *E,*
8. *C* is better than *D.*
9. *E* is better than *C,*
10. *D* is better than *E.* [42]

3.3 How Fair is the Borda Count Method of Voting?

We've now seen that the Pairwise Comparison Method of Voting (Condorcet's recommended replacement for the Plurality Method of Voting) satisfies the two fairness criteria we've defined thus far (Condorcet and Majority), but at the risk of a non-transitive group outcome (in which case some additional provision is needed to choose an election winner) and at some computational cost (at least in elections with lots of candidates). Of course, the Borda Count Method of Voting (Borda's proposed alternative to the Plurality Method of Voting) also selects the Condorcet and Majority Candidates as the overall winner in at least some cases (see, for example, Exercises 9 and 12), and, because it is based on integer point totals (which always fall in transitive order), avoids both the non-transitivity that can occur with Condorcet's method, as well as the potentially tedious computations of pairwise comparison. Naturally, Condorcet was well aware of Borda's suggested alternative, which he described as follows:

> A celebrated Geometer, who has observed before us the drawback of ordinary elections, has proposed a method, which consists of having each Voter give the order in which he ranks the candidates; of then giving to each first-place vote, the value of unity, for example; to each second-place vote a value less than unity; a value still smaller for each third-place vote, & so on, & of then choosing the candidate for whom the sum of these values, taken across all the voters, will be the largest.[43]

[42] Condorcet 1785, p. clxxj.

[43] Condorcet 1785, p. clxxvij.

Exercise 19

Notice that, in his description of the Borda Count Method of Voting, Condorcet assigned 1 point for every first-place vote, then successively smaller (fractional or zero) values for second-place, third-place and so on. Let's try out this strategy for assigning points with Borda's original example, for which we found (in Exercise 9) the following Preference Schedule:

Number of voters	1	7	7	6
First Place	*A*	*A*	*B*	*C*
Second Place	*B*	*C*	*C*	*B*
Third Place	*C*	*B*	*A*	*A*

a. Determine the winner under the Borda Count Method of Voting when we assign 1 point for each first-place vote, 1/2 point for each second-place vote and 1/3 point for each third-place vote. Who are the second- and third-place finishers overall?

b. Repeat part (a) using the following assignment of points: 1 point for each first-place vote, 1/10 point for each second-place vote and 0 point for each third-place vote.

c. Repeat part (a), now using the following assignment of points: 1 point for each first-place vote, 1/7 point for each second-place vote and 0 points for each third-place vote.

d. Comment on what you notice about the effect of these different point assignments on the final overall rankings in this election. Does this make the Borda Count Method of Voting seem more or less fair to you, and why?

e. Although using fractions may seem an unwelcome complication, we could easily clear all fractions to get integer point values that give the same election results.[44] Condorcet's description of Borda's method does, however, make one thing clear: although the point values must be assigned in a descending order (from first place on down), there is no reason why those values must decrease by 1 point each time. We could, for instance, decide to assign 7 points for each first-place vote, 3 for each second-place vote and 1 for each third-place vote. Why might we want to vary how many points are assigned to these different places in this way? Give at least two different reasons.

[44] To do this, simply multiply the assigned points by the least common denominator of the various fractions involved.

In part a, for instance, note that 1, 1/2 and 1/3 have the least common denominator of 6. If you make each first-place vote worth $6 \times 1 = 6$ points; each second-place vote worth $6 \times 1/2 = 3$ points; and each third-place vote worth $6 \times 1/3 = 2$ points, then recompute the totals for each candidate, you will find that those totals are also 6 times what they were in your original computations. The *order* of the final point values won't change as a result, however, so that we get the same final election outcome. Similarly, in parts (b) and (c), we could multiply by 10 and 7 respectively, and assign (10,1,0) and (7,1,0) points respectively for each (1st,2nd,3rd)-place vote. Notice that assigning 0 points for last place always makes the computation easier!

Condorcet continued his discussion of Borda's method with the following remarks, before giving an example to illustrate what he viewed as the main drawback of the Borda Count Method of Voting.

This method has the advantage of being very simple, & one can without doubt, by setting the law of decreases for these values, avoid many of the difficulties that the ordinary method has, of giving as the decision of the plurality one that is really contrary to it: but this method is not strictly protected from that drawback. [45]

You likely agree with Condorcet's comment about the simplicity of the Borda Count Method of Voting. In Exercise 19, you've also explored the effect of "setting the law of decreases" on election outcomes. As for the drawback that Condorcet mentioned in this connection, you may have already detected this yourself while working through the various exercises involving Borda's method. If not, then Exercise 20 presents one more example of the Borda Count Method of Voting which makes this particular drawback clear.

Exercise 20

In this exercise, we return to the question of how the Borda Count Method of Voting fares with regard to the Condorcet Fairness Criterion, which you first explored in Exercise 9.

Here is the Preference Schedule for the example provided by Condorcet to show that the **Borda Count Method of Voting violates the Condorcet Fairness Criterion**.

Number of voters	30	1	29	10	10	1
First Place	*A*	*A*	*B*	*B*	*C*	*C*
Second Place	*B*	*C*	*A*	*C*	*A*	*B*
Third Place	*C*	*B*	*C*	*A*	*B*	*A*

a. Show that *A* is a Condorcet Candidate in this election. (Note that you don't need to consider the comparison of *B* versus *C* to do this!)

b. Determine the election winner under the Borda Count Method of Voting using the assignment of 3 points for each first-place vote, 2 points for each second-place vote and 1 point for each third-place vote. Explain why the election result proves that the Borda Count Method of Voting violates the Condorcet Fairness Criterion.

c. In the last excerpt we read from his *Essai*, Condorcet stated that we could, "by setting the law of decreases for these values, avoid many of the difficulties that the ordinary method has." By this, he seems to have meant that we could arrange for the Condorcet Candidate to win under Borda's simply by choosing the right point assignments. Try doing this for this

[45] Condorcet 1785, p. clxxvij.

election by making up at least three different point assignments of your own, and finding the election winner for each.

d. Comment on what you noticed in part (c). Were you successful in finding a point assignment that chose the Condorcet Candidate *A* as the election winner under the Borda Count Method of Voting? If so, how much can you vary the distribution of points in that system (and in what way) and still keep *A* as the winner? If not, are you convinced that there is no such system? In either case, justify your response as completely as you can.

Exercise 21

In this exercise, we return to the question of how the Borda Count Method of Voting fares with regard to the Condorcet Fairness Criterion, which you first explored in Exercise 12.

Consider the following Preference Schedule for an election with 3 candidates and 100 voters.

Number of voters	51	49
First Place	*A*	*B*
Second Place	*B*	*C*
Third Place	*C*	*A*

a. Explain why *A* is a Majority Candidate in this election.

b. Determine the election winner under the Borda Count Method of Voting using the assignment of 3 points for each first-place vote, 2 points for each second-place vote and 1 point for each third-place vote. Explain why the election result proves that the **Borda Count Method of Voting violates the Majority Fairness Criterion**.

c. Comment on what you notice about the Preference Schedule that allows this violation to occur. How is it that *A* had over half of the first-place votes, but yet lost the election? Do you think the candidate who ends up winning the election under the Borda Count Method of Voting was nevertheless a fair choice? Explain why or why not.

Exercise 22

We've now seen that the Borda Count Method of Voting violates both the Majority and Condorcet Fairness Criteria, whereas the Plurality Method of Voting violates only the Condorcet Fairness Criterion. Yet Borda's proposed the Borda Count Method of Voting as a means to prevent a "Condorcet loser" (i.e., a candidate who loses to all others in head-to-head comparisons) from winning the election. (See Exercise 2 for a reminder of his example.) How convincing do you find Borda's reason for abandoning Plurality in favor of Borda Count as a voting method? What other strengths do you think the Borda Count Method of Voting has relative to the Plurality Method of Voting in terms of fairness? For instance, does it matter that the Borda Count Method of Voting takes into account all of the voters' rankings, and not just their first-place choices? Why or why not? Are the relative strengths of the Borda Count Method of Voting enough to use it (and in what types of elections), in spite of its weaknesses? Again, why or why not?

3.4 Can We Avoid the Condorcet Paradox?

Condorcet did think there is a way to avoid the Condorcet Paradox—but not via a mechanical fix to the voting method itself. The mathematical analysis that he carried out in his 1785 *Essai* fully convinced him that, despite its imperfections, using the Pairwise Comparison Method of Voting in elections involving three or more candidates is the best option for maximizing the probability that the final outcome of such elections would be the correct one. Furthermore, as we noted earlier, the correctness of the final outcome for Condorcet was to be judged relative to the common good: not just assuring individual voters that their preferences are being fairly considered, but choosing the candidate whose election would genuinely serve the best interests of society-at-large.

Although written as a text on probability, Condorcet's *Essai*, was thus ultimately aimed at non-mathematical goals that he highly cherished, and whose accomplishment he felt would require changes to society itself. For instance, Condorcet closed his discussion of what we today call the Condorcet Paradox with the following observations.

> Finally, let us observe that these contradictory systems cannot arise without indicating uncertainty in [the] opinions [of the voters], . . .
>
> Moreover, the Voters must be enlightened, and all the more enlightened, as the questions which they decide are more complicated; otherwise we will find a form of decision which will preserve the fear of a false decision, but which at the same time making any decision almost impossible, will only be a means of perpetuating abuses and bad laws. Thus the form of the assemblies which decide the strength of men, is much less important for their happiness than the enlightenment of those who compose them: & the progress of reason will contribute more to the good of the Peoples than the forms of political constitutions.[46]

And, later in Part 2 of his *Essai*, he noted that

> if the Voters are not very enlightened, it will be often possible to avoid a decision contradicting the truth, by choosing a form which almost removes the hope of having a decision, which is condemned to preserve abuses & prejudices.[47]

[46] Condorcet 1785, p. lxx.

[47] Condorcet 1785, p. 135.

Exercise 23

In the *Inside Higher Education* opinion piece "How Higher Education Empowers Student Voters," Political Science Professor Hahrie Han wrote the following. Comment on how Han's view relates to Condorcet's comments on "enlightened voters."

> Colleges and universities are also distinctly positioned to educate young people in nonpartisan ways and empower them to make judgments about candidates and issues. They can help students learn to collaborate across differences of race, gender identity and sexual orientation, ethnic background and nation of origin, political affiliation, urban/rural divides, and economic background. College classes can challenge students to put together information from multiple sources and judge various forms of evidence. They can also teach them to analyze social problems, consider possible solutions and take action. They can model ways that Americans can debate issues passionately while tolerating alternative points of view.[48]

As the quintessential Enlightenment thinker, Condorcet was an avowed optimist who believed strongly in the possibility of individual enlightenment and social progress. Indeed, he carried out much of the analysis in his *Essai* under assumptions about the circumstances in which voting is carried out that might be considered today to be quite naive.

> [W]e will first suppose that the assemblies are composed of Voters who are equally right-minded and with equal insight: we suppose that none of the Voters has any influence on the voices of the others, and that all opine in good faith.[49]

Despite his optimism, Condorcet was nevertheless well-acquainted with the realities of voting (both within the Academy of Sciences and in the political arena). He also dedicated later sections of his *Essai* to an examination of voting under non-ideal conditions such as

> inequality of insight of right-mindedness of the voters, the supposition that the probability of their votes is not constant, the influence that some could have over the others, the bad faith of some.[50]

He further commented that

> these latter researches were necessary in order to be able to apply the theory in practice.[51]

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[48] Han 2009.

[49] Condorcet 1785, p. xxj.

[50] Condorcet 1785, p. xxiij.

[51] Condorcet 1785, p. xxiij.

Exercise 24

For this exercise, reflect on the last three quotes from Condorcet's *Essai* given above.

a. Comment on why you think Condorcet decided to carry out most of his analysis of voting under such ideal assumptions about the conditions of voting. For example, in what ways (if any) might these assumptions have simplified the mathematical analyses that he wished to do?

b. Of the assumptions that Condorcet initially made (first quote above), which do you think are the most unrealistic? To what extent do you think that "these latter researches" in which Condorcet dropped one or more of those ideal conditions "were necessary in order to be able to apply the theory in practice," both in his time and today? Do you think this will always be the case, or is it possible that ideal voting conditions of the kind described by Condorcet might exist in the future? Explain why or why not.

4. Voting in (and after) the Revolution: Can Elimination Help?

As noted earlier, Borda and Condorcet wrote their works on voting theory not long before the start of the French Revolution in 1789. Both works were, however, essentially forgotten for centuries, until Kenneth Arrow's ground-breaking work drew attention to the application of mathematics to group decision making in the mid-twentieth century.

Clearly, however, elections of all kinds were being held in France during the revolution. And, within those elections, the use of ballots to select from a slate of three or more candidates was not unknown. For example, delegates to the First National Convention that governed France between 1792 and 1795 were chosen by some local assemblies through a multi-stage balloting process similar to what is known today as the **Plurality with Elimination Method of Voting:** if a candidate receives a majority of the first-place votes, then that individual is declared the election winner; otherwise, the candidate with the least number of first-place votes is eliminated from the Preference Schedule, and the process is repeated until a Majority Candidate emerges in some round. Suppose, for example, that we wish to find the Plurality with Elimination winner for the election in Borda's original example; here's that Preference Schedule (from Exercise 6):

Number of voters	1	7	7	6
First Place	*A*	*A*	*B*	*C*
Second Place	*B*	*C*	*C*	*B*
Third Place	*C*	*B*	*A*	*A*

Note first that, since there are 21 voters in all, a majority will require at least 11 first-place votes (since 50% of 21 = 10.5). Since *A* starts with 8 first-place votes, *B* with 7 first-place votes, and *C* with only 6 first-place votes, no one has a majority in Round 1. We thus eliminate Candidate *C* (who has the least first-place votes) and shift other candidates up in the schedule where necessary to fill any vacancies:

Number of voters	1	7	7	6
First Place	*A*	*A*	*B*	~~*C*~~
Second Place	*B*	~~*C*~~	~~*C*~~	*B*
Third Place	~~*C*~~	*B*	*A*	*A*

Number of voters	1	7	7	6
First Place	*A*	*A*	*B*	*B*
Second Place	*B*	*B*	*A*	*A*

Or, more simply,

Number of voters	8	13
First Place	*A*	*B*
Second Place	*B*	*A*

From the Round 2 Preference Schedule, we see that candidate *B* does now have the necessary majority of the first-place votes (11 or more), and thus *B* wins the election under the Plurality with Elimination Method of Voting (with *A* coming in second and *C* coming in third). But remember (from Exercise 9) that *C* is the Condorcet Candidate in this election! This example thus proves that the **Plurality with Elimination Method of Voting violates the Condorcet Fairness Criterion.** On the other hand, the first step of this voting procedure guarantees that whenever there is a Majority Candidate, that candidate wins the election (in Round 1); thus, the **Plurality with Elimination Method of Voting satisfies the Majority Fairness Criterion.**

Exercise 25

For each of the following Preference Schedules, determine the final ranking for all candidates under Plurality with Elimination Method of Voting.

a.

Number of voters	6	5	4	2
First Place	*A*	*B*	*C*	*C*
Second Place	*C*	*C*	*B*	*A*
Third Place	*B*	*A*	*A*	*B*

b.

Number of voters	30	1	29	10	10	1
First Place	*A*	*A*	*B*	*B*	*C*	*C*
Second Place	*B*	*C*	*A*	*C*	*A*	*B*
Third Place	*C*	*B*	*C*	*A*	*B*	*A*

c.

Number of voters	14	8	4
First Place	*A*	*B*	*D*
Second Place	*B*	*D*	*A*
Third Place	*C*	*C*	*C*
Fourth Place	*D*	*A*	*B*

d.

Number of voters	13	9	7	5
First Place	*C*	*A*	*D*	*D*
Second Place	*D*	*B*	*A*	*C*
Third Place	*B*	*D*	*B*	*A*
Fourth Place	*A*	*C*	*C*	*B*

Exercise 26

We've seen that both the (simple) Plurality and the (more complicated) Plurality with Elimination Methods of Voting satisfy the Majority Fairness Criterion, but violate the Condorcet Fairness Criterion. Given this, what do you see as the advantages (if any) of the Plurality with Elimination Method of Voting over the basic Plurality Method of Voting? Do either of these methods have a relative advantage over the other with regard to being "fair"? Explain why or why not.

5. Isn't There More to Being Fair?

The texts written by Borda and Condorcet essentially focused on the two fairness criteria we have studied so far: Majority and Condorcet. Perhaps you and your classmates have wondered about other notions of fairness along the way, and perhaps you have even tried to define and discuss criteria to capture those notions. In this section, we take a look at two other Fairness Criteria that are now a standard part of the mathematical study of voting theory, both of which consider what should happen in situations where a recount is needed for some reason, and for which the original preference ballots of some voters end up being changed in certain well-defined ways.

For the first of these two fairness criteria, the ballot change is due to a "change of heart" on the part of a candidate. We imagine that the ballots have been processed under some particular voting method, so that we know the winner of that election under that method. Then, for some reason, one (or more) of the other (non-winning) candidates drops out of the election, and the ballots are then re-processed using the same method of voting. Under these circumstances, we expect the same candidate to again win the race—after all, the candidate(s) who dropped out are now irrelevant alternatives. This is precisely what the **Independence of Irrelevant Alternatives Fairness Criterion** says: having someone other than the winner drop out of the election shouldn't change the election winner. As it happens, violations of this criteria are more widespread than you might think!

Exercise 27

Before looking at specific election examples, pause to think about how it might be possible for election results to change in a recount after one of the non-winning candidates drops out. Do this for each of the four voting methods we've seen in this project (Plurality, Plurality with Elimination, Borda Count and Pairwise Comparison). Are there any of these methods that you think will be immune to violations of the Independence of Irrelevant Alternatives Fairness Criterion? If so, which one(s) and why?

Exercise 28

Use the following examples to think further about how our four voting methods fare with respect to the Independence of Irrelevant Alternatives Fairness Criterion.

Within each example, you'll need to apply the specified voting method twice: once for the Preference Schedule as given, and again after removing the specified dropout candidate and shifting other candidates up on a ballot (but without changing the order of the remaining candidates!) as needed to fill vacancies.

After finding the original winner and the recount winner, complete one of the following:

- If the winner **did** change in the recount election (which means the method in question violates this fairness criterion), what was it about the Preference Schedule that seemed to have allowed the violation to occur? Also comment on your assessment of which of the two is more fair in terms of being the voters' choice: the original election winner or the final recount winner.
- If the winner **did not** change in the recount election, remember that we can't yet say that the method in question satisfies this fairness criterion (since "satisfaction" requires that the conditions of the criterion hold in every possible election). In this case, comment on what the results of this particular example suggest to you about whether the method in question might fail to meet the requirements of the Independence of Irrelevant Alternatives Fairness Criterion in some other election. Continue to discuss this voting method with your classmates as needed to either come up with an example that proves a violation occurs, or to give a convincing argument that the method in question satisfies this fairness criterion.

Once you are done with this exercise, you should have complete lists of the methods that satisfy and of those that violate the Independence of Irrelevant Alternatives Fairness Criterion, and complete justifications for each of those lists.

a. Plurality; Dropout candidate = C

Number of voters	48	49	3
First Place	A	B	C
Second Place	C	A	B
Third Place	B	C	A

b. Plurality with Elimination; Dropout candidate = A

Number of voters	3	49	48
First Place	A	B	C
Second Place	C	C	A
Third Place	B	A	B

c. Borda Count; Dropout candidate = *B*
NOTE: You found the winner of this election under the Borda Count Method of Voting in Exercise 12.

Number of voters	14	8	4
First Place	*A*	*B*	*D*
Second Place	*B*	*D*	*A*
Third Place	*C*	*C*	*C*
Fourth Place	*D*	*A*	*B*

d. Pairwise Comparison; Dropout candidate = *D*

Number of voters	9	6	11	7	3
First Place	*A*	*A*	*B*	*C*	*D*
Second Place	*D*	*D*	*C*	*A*	*C*
Third Place	*B*	*C*	*A*	*D*	*B*
Fourth Place	*C*	*B*	*D*	*B*	*A*

Notice that the conditions of the Independence of Irrelevant Alternatives Fairness Criterion assume that, in the recount election, none of the voters changed their individual preferences. The final fairness criterion that we consider in this project keeps all the candidates in the election, but allows voters to change their mind before the recount–but only in such a way that the original winner of the election moves up on the ballot. For instance, in Exercise 25(a), you showed that Candidate *A* is the election winner under the Plurality Method of Voting for the following Preference Schedule:

Number of voters	6	5	4	2
First Place	*A*	*B*	*C*	*C*
Second Place	*C*	*A*	*B*	*A*
Third Place	*B*	*C*	*A*	*B*

Let’s suppose, though, that the 2 voters in the 4th column change their minds before the recount, and reverse the order of *A* and *C*:

Number of voters	6	5	4	2
First Place	*A*	*B*	*C*	*A*
Second Place	*C*	*C*	*B*	*C*
Third Place	*B*	*C*	*A*	*B*

Although the previous winner *A* has only moved up on those two ballots, in the Plurality with Elimination recount, we now begin by eliminating Candidate *C* (who now has the smallest number of first-place votes); this gives the following Round 2 Preference Schedule:

Number of voters	8	9
First Place	*A*	*B*
Second Place	*B*	*A*

But since the voters prefer candidate *B* over candidate *A* (albeit by a slim margin), the previous winner *A* now loses the recount. This example thus lets us conclude that the **Plurality with Elimination Method of Voting violates the Monotonicity[52] Fairness Criterion**.

Exercise 29

Before looking at specific election examples, pause to think about whether the election results for any of our other three voting methods might be susceptible to change in a recount after some of the voters shift the original winning candidate upwards on their ballots. Do you think that any of these methods are immune to violations of the Monotonicity Fairness Criterion? If so, which one(s) and why? For those methods that you don't think will be immune to a violation, why not?

Exercise 30

Use the following examples to think further about how our other three voting methods fare with respect to the Monotonicity Fairness Criterion.

For each of these examples, you have already found the winner of the original election in Exercise 28. You'll thus need to apply the specified voting method only once, for the recount election, after making the indicated ballot change (each of which moves the original winner up on some of voters' ballots).

After finding the original winner and the recount winner, complete one of the following:

- <u>If the winner **did** change in the recount election</u> (which means the method in question violates this fairness criterion), what was it about the preference schedule that seemed to have allowed the violation to occur? Also comment on your assessment of which of the two is more fair in terms of being the voters' choice: the original election winner or the final recount winner.
- <u>If the winner did **not** change in the recount election</u>, remember that we can't yet say that the method in question satisfies this fairness criterion (since "satisfaction" requires that the conditions of the criterion hold in every possible election). In this case, comment on what the results of this particular example suggest to you about whether the method in question might yet violate the Independence of Irrelevant Alternatives Fairness Criterion in some other election. Continue to discuss this voting method and other election examples with your classmates to either come up

[52] The word "monotonicity" in the name of this fairness criterion is related to the word "monotone," which comes from the Latin and can be translated as "one direction." The idea here is that the winner can only move in the upwards direction on any voter's ballot. A violation occurs when the winner changes to a loser as a result of that upward mobility.

with an example that proves violation occurs, or to give a convincing argument that the method in question satisfies this fairness criterion.

Once you are done with this exercise, you should have complete lists of the methods that satisfy and of those that violate the Monotonicity Fairness Criterion, and complete justifications for each of those lists.

a. Plurality; Ballot change: switch *B* and *C* in the 1st column.

Number of voters	48	49	3
First Place	*A*	*B*	*C*
Second Place	*C*	*A*	*B*
Third Place	*B*	*C*	*A*

b. Borda Count; Ballot change: switch *A* and *C* in the 2nd column.

Number of voters	14	8	4
First Place	*A*	*B*	*D*
Second Place	*B*	*D*	*A*
Third Place	*C*	*C*	*C*
Fourth Place	*D*	*A*	*B*

c. Pairwise Comparison; Ballot change: switch *A* and *C* in the 2nd column.

Number of voters	9	6	11	7	3
First Place	*A*	*A*	*B*	*C*	*D*
Second Place	*D*	*D*	*C*	*A*	*C*
Third Place	*B*	*C*	*A*	*D*	*B*
Fourth Place	*C*	*B*	*D*	*B*	*A*

Exercise 31

The following partially-completed table summarizes the relationship between the various Methods of Voting and Fairness Criteria that we have considered in this project. Complete the rest of this table, based on your work in Exercises 28 and 30. Then comment on which method(s) you think are most and least fair overall, and why.

Summary Table: Methods of Voting versus Fairness Criteria (for Exercise 31)

	Majority	Condorcet	Independence of Irrelevant Alternatives	Monotonicity
Plurality	Satisfies	VIOLATES		VIOLATES
Plurality with Elimination	Satisfies	VIOLATES		
Borda Count	VIOLATES	VIOLATES		
Pairwise Comparison	Satisfies	Satisfies		

6. And the Winner Is . . .

The use of ranked preference ballots that would require a more elaborate algorithm for determining the election outcome, such as those proposed by Borda and Condorcet, never did find its way into the popular political elections of the late eighteenth century. There was one institutional setting in which the men's work did garner attention, however: the Academy of Sciences. Indeed, the Academy has been described as "a living laboratory where[in] to test, in the small, the virtues and the vices of different voting rules" [Barberà et al. 2020, p. 2].

One of the methods tested within this laboratory was the Borda Count Method of Voting, which was adopted in 1795 for the Academy's membership elections at the recommendation of French statesman, author and historian Pierre Daunou (1761–1840). During the time that Borda's method was in use, Napoléon Bonaparte[53] (who was then First Consul of France, and later its self-proclaimed Emperor) was elected as a member of the National Institute that replaced all the learned Academies (including the Academy of Sciences) in France between 1795 and 1815. Napoléon also served as president of the National Institute for a brief time, during which his sole intervention in its running appears to have been his March 1800 request for a review of the balloting system—the very same Borda Count system by which he himself had been elected to the Institute. As chance would have it, that review was conducted by Daunou—the very same person whose recommendation of Borda's method led to its adoption in 1795. Surprisingly, Daunou reversed directions in his report (presented in July 1801 and published in 1803), and instead expressed a preference for Condorcet's

[53] Although not a scientist himself, Napoléon was a strong supporter of mathematics and science. He is famously quoted, for example, as having said "The advancement and perfection of mathematics are intimately connected with the prosperity of the State." Napoléon wrote these words on August 1, 1812, in a letter to the great French mathematician Pierre-Simon de Laplace (1749–1827), in which he thanked Laplace for providing him with the copy of his 1812 *Théorie analytique de probabilités (Analytic probability theory).*

pairwise comparison method over the point system of Borda.[54] Here is an excerpt from his remarks on the two methods.

> Indeed, the results of this addition [in Borda's method] with regard to two subjects are variable according to the number of their other competitors, and according to the various ranks that these others obtain before, after or between the . . . two [given] subjects; while the comparisons between two candidates [in Condorcet's method], or the preference given to one over the other, is, in the thought of each elector as in the general thought, a simple, constant, determined comparison, independent of all other relations.[55]

Exercise 32

a. Based on your work in this project (and especially Exercise 19), do you think that Daunou was correct in saying that the results of the Borda Count Method of Voting "are variable according to the number of their other competitors, and according to the various ranks that these others obtain before, after or between?" Why or why not?

b. With regard to his comments about the Pairwise Comparison Method of Voting, do you agree with Daunou that "the preference given to one [candidate] over the other, is . . . a simple, constant, determined comparison, independent of all other relations"? Why or why not?

c. Comment on how Daunou's comparison of the voting methods proposed by Borda and Condorcet relates to his evaluation of the relative fairness of each voting method. Do you agree or disagree with his evaluation? Why or why not?

Interestingly, despite the strong criticism of Borda's method in Daunou's report, the National Institute continued to use the Borda Count Method of Voting for some (but not all) of its elections. It seems, therefore, that one conclusion that the Institute did take away from that report was the idea that different voting methods are more or less suitable for different types of elections.

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[54] In fact, Daunou's report was quite of critical not only of the Borda Count Method of Voting, but also of Borda's presentation of that method in his 1784 "Memoir." For instance, after recounting Condorcet's observations on the defects of Borda's proposed method, he wrote [Daunou 1803, p. 46]:

> But it suffices to attentively read the memoir of Borda himself, to be convinced that his method is based on inaccurate, incomplete observations, which lead to very false results.

By the time that Napoléon requested Daunou's review, both Borda and Condorcet were dead, so that neither could have issued a rebuttal to Daunou's remarks on their ideas about voting theory.

[55] Daunou 1803, p. 48.

Exercise 33

In this project, we've considered four different methods of voting:

Plurality, Plurality with Elimination, Borda Count and Pairwise Comparison.

a. Research how different countries or large organizations conduct their various elections. Try to find at least one real-life example that uses each of the four methods. In each case, do you think that the voting method used is appropriate to the situation? If so, why? If not, what method do you think would be more appropriate and why?

b. Which of the four methods that we've studied (if any) do you think would be most appropriate to use for the following elections within the context of a college or university? Justify your choice in each case. Assume that the election is done by way of preference ballots, and that there are at least three candidates in each case.

   i. President of the Associated Student Government; voters = all full-time students

   ii. President of the Faculty Senate; voters = all faculty senators (who are in turn elected by the faculty within their respective academic units)

   iii. Students' Choice for Teacher of the Year; voters = all registered students

   iv. Name of the school mascot; voters = all faculty, students and staff

7. Epilogue

This brings our exploration of the French connection to the mathematics of voting theory to an end. In closing, we offer a few final exercises as a means to check your understanding of the various methods and ideas we've encountered along the way—and some words of advice.

Voting is a privilege, one that is enjoyed by too few people in today's world, and which can be too easily lost. Those of us who enjoy this privilege also bear the responsibility of helping to fulfill Condorcet's vision of enlightened voters committed to the common good of society-at-large. Be sure to do your bit by becoming educated about the issues, identifying and setting aside your prejudices (we all have them, after all), and, most importantly, getting out there to vote.

Exercise 34

Consider the following Preference Schedule. If the Plurality Method of Voting is used to choose the election winner, will the results illustrate a violation of the Condorcet Fairness Criterion?

Number of voters	8	7	6
First Place	A	B	C
Second Place	B	C	A
Third Place	C	A	B

Exercise 35

Consider the following Preference Schedule.

Number of voters	5	8	2	4
First Place	*A*	*B*	*D*	*D*
Second Place	*C*	*C*	*B*	*E*
Third Place	*E*	*D*	*E*	*A*
Fourth Place	*D*	*A*	*C*	*B*
Fifth Place	*B*	*E*	*A*	*C*

a. Rank the candidates using the Plurality Method of Voting.

b. Rank the candidates using the Plurality with Elimination Method of Voting.

c. Rank the candidates using the Borda Count Method of Voting.

d. Rank the candidates using the Pairwise Comparison Method of Voting.

e. Do the results of any of these elections illustrate a violation of the Majority Fairness Criterion? Explain why or why not.

f. Do the results of any of these elections illustrate a violation of the Condorcet Fairness Criterion? Explain why or why not.

Exercise 36

An election is held between 4 candidates, *A, B, C, D*. The Preference Schedule reveals that *C* is a Condorcet Candidate, while A is the winner under the Plurality Method of Voting.

Determine which of the following statements must be true, which must be false, and which we cannot decide without additional information. In each case, use a full sentence to explain.

Statement I. *C* is a Majority Candidate.

Statement II. *C* will win the election under the Borda Count Method of Voting.

Statement III. *A* will win the election under the Plurality with Elimination Method of Voting.

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APPENDIX[1]

This Appendix provides additional biographical and historical detail related to the period leading up to the start of the French Revolution in 1789 (a date that roughly corresponds to the publication of Borda's and Condorcet's works on voting theory) and during the Revolutionary period itself (by the end of which both men were dead). As an expanded version of Section 1 ("Who were Borda and Condorcet?") of the project "The French Connection: Borda, Condorcet and the Mathematics of Voting Theory," it can be read instead of or in addition to that section.

The Pre-Revolutionary Period

Like all human beings, Borda and Condorcet were products of the place and time in which they lived, as were their texts on voting theory. Both men were born in France in the first half of the eighteenth century, Borda in 1733 and Condorcet in 1743. This situates their lives and works firmly within the period of the French Enlightenment, and the political revolutions which that movement helped to spur in the latter part of the eighteenth century.

While the starting date of the Enlightenment is debated by historians, its epicenter is universally recognized to be Paris. It was in the City of Lights that the artists, writers, politicians, scientists, and philosophers of the day began to regularly meet in intellectual circles, or *salons*, held in the homes of the (generally well-to-do) women, or *salonnières*, who organized and directed these gatherings.[57] Among the *philosophes*, as these thinkers came to be called, science and its successes over the preceding two centuries served as an important model of how knowledge can be generated through human reason and the evidence of our senses. Indeed, the name of the Enlightenment period (*Siècle des lumières* in French) itself grew out of the metaphor of bringing light to the dark ages that predated the scientific revolution. During the Enlightenment, human reason came to be seen not only as a means to shed light on our understanding of the universe in which we live, but also as a means to improve our living conditions as social beings inhabiting that universe. This belief in the power of human reason as the primary source of authority led in turn to widespread criticism of existing political, religious and social institutions and the limitations they imposed on individual liberty and happiness. One hears clear echoes of these concerns, for example, in the Declaration of Independence of the United States, whose writers were strongly influenced by Enlightenment ideals.

Unsurprisingly, one institution that escaped the criticism of the Enlightenment thinkers was the French Royal Academy of Sciences.[58] Founded in 1666 as one of the earliest learned societies in Europe, the Academy was initially organized around six sections (mathematics, mechanics, astronomy, chemistry, botany, and anatomy); members of the Academy belonged to one particular section based on their recognized accomplishments within that field of study. Importantly for the

[1] The material in this Appendix is a slightly edited version of the historical sections of the onscreen article (also written by the project author) that shares the same title as this project.

[57] To learn more about the Enlightenment, Parisian *salon* culture and the lives of the leading *salonnières*, see the references Brewer 2020, Bristow 2017, Haag 2020 and Lougee 2020.

[58] After the French Revolution, this was known simply as the French Academy of Sciences.

mathematics we examine in this project, membership in the Academy was determined by election. The rules and regulations governing these elections were originally quite complicated, and Academy commissions were appointed to study the problem of reaching majority consensus in Academy elections as early as 1699 [Urken 2009]. In the period immediately preceding the French Revolution, the usual process was for the current membership to produce, via voting, a list of ranked recommendations for new members that was forwarded to the King for his consideration; the King himself was not bound to accept the recommended ranking, or to even choose new members from the Academy's list. The Academy itself, on the other hand, was expected to remain neutral in terms of politics, religion and social issues.

Both our authors were elected to the Academy while still in their twenties, Borda in 1756 and Condorcet in 1769, based on the quality of their early works in mathematics. They also both began their mathematical educations under the Jesuits, Borda at the college in La Flèche and Condorcet first privately and then at the college in Reims. Despite the explicitly religious nature of the Society of Jesus itself, the goals behind the educational opportunities that it provided extended to the secular in that the Society sought to provide boys with the skills necessary to "take their place in that world even more strongly" and "be in a position to serve the common good" [Grendler 2014, p. 9]. Many of the boys who attended Jesuit colleges were of lower- and middle-class backgrounds, and the education they received was important simply for the sake of earning a living. Others—including both Borda and Condorcet—came from families with connections to the nobility[59] who were interested in seeing their sons trained for their future careers as well as to assume leadership positions within society.

In keeping with the job-oriented training that a Jesuit education sought to provide its students, mathematics was often taught in Jesuit colleges with an emphasis on its practical applications. The Jesuit college at La Flèche offered a course of study that was especially well-suited for the training of military engineers. Upon completing his studies at La Flèche in 1748, Borda entered the French army as a military mathematician. During this period of his military career, he wrote his first mathematical papers: a 1753 paper on geometry that brought Borda to the attention of the mathematician and *philosophe* Jean le Rond d'Alembert[60] (1717–1783), and a paper on the theory

[59] Recall that Cordorcet's actual name was Marie-Jean-Antoine-Nicolas de Caritat. The Caritats were an ancient family that had long resided in the town of Condorcet (located not far from Marseilles), from which he derived his title "Marquis de Condorcet." This was a courtesy title (versus a title associated with land ownership) which had no legal significance, but it served to indicate that Condorcet's parents were members of the nobility. Borda came from a family of lesser untitled nobility that resided in the town of Dax, in the southwest region of France. His father held the rank of Lord (in French: *Seigneur*), while Borda himself held the rank of Knight (in French: *Chevalier*).

[60] In addition to his work in mathematics and physics, d'Alembert was a contributor to and co-editor (with the *philosophe* Denis Diderot (1713–1784)) of the *Encyclopédie, Encyclopédie, ou dictionnaire raisonné des sciences, des arts et des métiers* (*Encyclopedia, or a Systematic Dictionary of the Sciences, Arts, and Crafts*), better known simply as the *Encyclopédie*,. Throughout his life, his destiny was shaped by Parisian *salon* culture and the influential women at its heart. The illegimate son of the chevalier Louis-Camus Destouches (1668–1726) and the runaway-nun-turned-*salonnière* Claudine Alexandrine Guérin de Tencin (1685–1749), d'Alembert's given name, Jean le Rond, is that of the Parisian church of Saint Jean le Rond, on whose steps he was abandoned as a child. His father later placed d'Alembert in the care of a glazier's wife and paid for his upbringing and education. As an adult, he became the close friend of another famous *salonnière,* Jeanne Julie Éléonore de Lespinasse (1732–1776), who cared for him during a serious illness. He also benefitted from the patronage of yet a third celebrated *salonnière*, Marie Anne de Vichy-Chamrond, Marquise du Deffand (1697–1780), whose influence was instrumental in gaining d'Alembert's 1752 election to the French Academy, a highly prestigious literary group whose 40 members are known as "the immortals" (in French: *les immortels*). He served as the French Academy's Permanent Secretary from 1772–1783, and held that position at the

of projectile motion that secured his 1756 election to the Academy. Soon thereafter, in 1758, Borda enrolled at the School of Engineering at Mézières, taking just one year to complete the two-year course of study. He then entered the French navy, where he served in a variety of leadership capacities. These included participation in maritime campaigns during the American Revolutionary War between 1777–1778, appointment as the Major General of the Naval Army in 1781, and an appointment as the French Inspector of Naval Shipbuilding in 1784. In 1782, Borda was briefly held by the English after being captured while returning to France from Martinique, but he was soon released on his word of honor, perhaps due to his reputation as a scientist.

Ironically, Borda's family had wanted him to pursue a career as a magistrate, while Condorcet's family had wanted him to pursue a military career. In fact, both families had a long tradition of military service. Just days after his birth, Condorcet's cavalry captain father was killed during an Austrian attack on the town of Neuf-Brisach.[61] Yet Condorcet himself rejected a military life, and instead transferred from the Jesuit Collège in Reims to the prestigious Collège de Navarre in Paris in 1758. He then studied at the equally prestigious College de Mazarin, also in Paris. At the age of 16, he successfully defended a dissertation on mathematical analysis (written in Latin) to a committee whose members included d'Alembert. Following this, Condorcet established himself as an independent scholar, living in Paris on a small allowance provided by his mother. As an independent scholar, Condorcet continued his study of mathematics and wrote several works, including the *Essai sur le calcul intégral* [Condorcet 1765] that helped to secure his 1769 election to the Academy of Sciences.

Condorcet also fully immersed himself in Parisian Enlightenment society, attending the most important *salons* of the period and establishing friendships with d'Alembert and other important *philosophes*. Through d'Alembert, Condorcet was recruited to write mathematical articles for supplements to the *Encyclopédie*, an Enlightenment project aimed at spreading light across French society by way of educating its citizens. Condorcet's friendship with d'Alembert was so close that, upon his death in 1783, d'Alembert left his property to Condorcet, thereby ending the financial difficulties that Condorcet faced up to that time. Soon thereafter, in 1786, Condorcet married the influential *salonnière* Sophie de Grouchy, whose salon at the Hôtel des Monnaies was attended by foreign dignitaries such as Thomas Jefferson, who served as the American Minister to France from 1785 to 1789. The intellectual partnership between Condorcet and de Grouchy played a critical role in the social reform efforts in which Condorcet engaged during the last decade of his life, which we discuss in a later section of this Appendix. (We note here in passing that, in contrast to Condorcet, Borda neither married nor participated in salon culture.)

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time that his *protégé* Condorcet was elected as "an immortal" in 1782. D'Alembert was also a highly influential member of the Academy of Science, to which he was elected in 1743. In addition to his contributions to mathematics, d'Alembert is remembered today for his legacy in music theory, philosophy and literature.

[61] **Neuf-Brisach** is a fortified town in Alsace, near the Rhine River and the German border. It was designed by Sébastien Le Prestre de Vauban (1633–1707), a French military engineer whose principles for applying rational and scientific methods in fortification design were widely used for nearly a century. Neuf-Brisach is considered to be Vauban's masterpiece, and it is now a **UNESCO World Heritage site.**

Another of his close *philosophe* friends influenced Condorcet to become increasingly involved in public life: the reform economist Jacques Turgot (1727–1781). Turgot served as the national Minister of Finance under Louis XVI from 1774 to 1776, during which time he promoted policies aimed at mitigating the economic condition of the French people. After appointing Condorcet as the General Inspector of the French Mint in 1775, Turgot later persuaded him to remain in that post even after Turgot's own dismissal from all governmental duties by Louis XVI. Condorcet remained head of the French Mint until 1790, serving for a time under Turgot's political nemesis, Jacques Necker (1732–1804).

At least on the surface, then, our two authors shared a number of similarities: noble family backgrounds (both with long military traditions), initial training in the Jesuit tradition (but with no religious commitment of their own), early recognition of their mathematical talents (for which each earned the praise of d'Alembert), and prolonged service to their nation (Borda as a military man and Condorcet as an administrator). There were, however, marked differences between the two.

With regard to academic orientation, Borda's interests were quite pragmatic, as one might expect of an engineer and military man. In addition to his study of applied topics such as projectile motion and fluid mechanics, he developed a number of utilitarian instruments, most notably the repeating circle, a celestial navigational device that employed a rotating telescope and a system of repeated observations to greatly reduce measurement error. Borda also published, in 1778, a pre-chronometer method for computing the longitude using lunar distances that is regarded as the best of several similar mathematical procedures that were available at that time; it was used, for example, by the 1803–1806 Lewis and Clark Expedition. His other contributions in the area of marine navigation and cartography included development of a series of trigonometric tables, conduct of chronometer tests in the Caribbean, and creation of maritime charts of the Azores and Canary Islands.

Condorcet's commitment to Enlightenment ideals, on the other hand, significantly influenced the direction of his scholarly work towards questions related to social issues. His interest in probability in the early 1780s, for example, was motivated in part by his vision of the role that probability could play in understanding and ameliorating social and economic conditions. Increasingly, however, Condorcet's energies and interests were diverted away from considerations of how mathematics could promote social change, and towards active involvement in social reform efforts themselves. The extent to which this had occurred even before the start of the Revolution is highlighted by the author of Condorcet's entry in the *Dictionary of Scientific Biography*, who remarked that "after 1787 Condorcet's life is scarcely of interest to the historian of science" [Granger 1970–1990].

Borda and Condorcet also differed with regard to their general views on the nature of science and mathematics. Borda considered himself a disciple of Georges-Louis Leclerc, count de Buffon (1707–1788), a fellow Academician whose views about mathematics and science often came into conflict with those of d'Alembert.[62] As d'Alembert's *protégé*, Condorcet held views about mathematics, science and its correct conduct that thus frequently ran opposite to those of Borda and Buffon. Within the political realm, the differences in their ideological beliefs were even more pronounced. A staunch monarchist, Borda supported the traditionalist views of Necker, Turgot's political rival. Condorcet's alignment with Turgot's politics, on the other hand, placed him among the "modernists" who believed there was a need for constitutional restructuring in France.

[62] See, for example, the discussion of the views held by Buffon and d'Alembert regarding rigor in mathematical proof in [Richards 2009].

Taken together, these differences were sufficiently divisive that Borda and Condorcet may have been seen as (and possibly were) rivals within the Academy. While Condorcet (in his letters to Turgot and elsewhere) was openly critical of many of Borda's ideas, we will see later in this Appendix that the two did work together on at least one occasion. And, as the economist Duncan Black (1908–1991) has noted, "during Borda's prolonged absences from Paris [the two] corresponded on scientific matters" [Black 1958, p. 169].[63]

Whatever the personal relationship between the two men, Borda's military and scientific travels did indeed often take him away from Paris and the day-to-day workings of the Academy. In contrast, Condorcet became intimately involved in the administrative work of the Academy relatively early, serving first as its Assistant Secretary (1773–1776) and then as its Permanent Secretary[64] (1776–1793). As Permanent Secretary, he was responsible for assembling the Academy committees that prepared responses to government requests for scientific studies related to proposed public works (e.g., prisons, canals) and composed reports on evaluations of new scientific discoveries. Another of Condorcet's responsibilities as Permanent Secretary was the preparation of official eulogies for Academicians and other important mathematicians and scientists. In all, he composed over fifty such eulogies, including those of his friend d'Alembert and his enemy Buffon, often inserting comments about mathematics, science and their role in society that were reflective of his own views, rather than those held by the subject of the eulogy.

Condorcet's administrative role in the Academy of Sciences also placed him in the position of helping to organize its various elections, including a vote to allow the publication his own *Essai* on voting theory [Condorcet 1785] by the *Imprimerie Royale*. In the next section of this Appendix, we briefly comment on how the circumstances surrounding its publication relate to the publication of Borda's own "Memoire" on voting theory.

Who was first to study voting: Borda or Condorcet?

At first glance, Borda's work had priority, by more than a decade, over that of Condorcet. Borda's "Memoire sur les elections par scrutin" ("Memoir on elections by ballot") appeared in the *Annual Proceedings* of the French Academy of Sciences for the year 1781, together with an "Analysis" of

[63] Black opened this sentence by claiming that "throughout their lives, the two were close friends" [Black 1958, p.179]. However, this claim is not repeated (or even mentioned) by any other historian that the project author has consulted, and it seems unlikely given the vast differences in their personal values and beliefs. Black provided little evidence for this assertion beyond providing a reference to a letter that is described and quoted in part by Borda's biographer, French astronomer and mathematician Jean Mascart (1872–1935). According to Mascart [1919, p. 96], Borda wrote this letter to Condorcet as a response to the latter's criticisms of the former's work on fluid dynamics, and concluded it by writing [again, as quoted by Mascart]:

> Here is a response to your profession of faith: I am angry that you are not of my belief, but we will not both go to Paradise, except to argue about fluids. I embrace you with all my heart.

Whether the closing salutation [in French: *Je vous embrasse de tout coeur*] was merely a standard courtesy of the day or a genuine expression of fondness, the existence of the correspondence itself does suggest that relations between the two men were of a sufficiently cordial professional nature, even if not as personally close as Black suggested.

[64] Although his personal friendship with Turgot led to preferments (e.g., Turgot's earmarking of a significant portion of the government's funding for the Academy specifically to support Condorcet's research) complicating Condorcet's early relations with other Academicians, his 1776 election as the Academy's Permanent Secretary was unanimous.

that memoir that was likely written by Condorcet (in an editorial capacity derived from his position as Permanent Secretary). In a footnote to the title of his paper, Borda further asserted that he had already presented the ideas it contained to the Academy some 14 years earlier, on June 16, 1770—a claim repeated by the author of the "Analysis" of Borda's "Memoire" (again, likely Condorcet himself), but for which no archival evidence has yet emerged.[65] In fact, based on the available archival documentation, Brian [2008, p. 2] has shown that:

> The date of 1781 is a fake. The story really begins the 14th of July of 1784, exactly five years before a date of some importance in the history of democracy.

It was on this "Bastille Day" date in 1784 that Condorcet, then Permanent Secretary of the Academy, announced his intention to publish a book on the application of probability to the analysis of electoral decisions. To do so under the auspices of the *Imprimerie Royale*, both a positive report by two referees and a vote of approval by the members of the Academy was required. The two referees were chosen by Condorcet, and the required report was read to a meeting of the Academy just three days later—such an astonishingly short turnaround time given the length and complexity of Condorcet's tome that historians generally believe the report was written by Condorcet himself.[66] Regardless of who actually wrote that report, it sufficed to secure the vote of approval necessary for Condorcet to go forward with publication. A possible delay then arose when Borda read his own paper on the arithmetic of voting to the Academy on July 21, 1784, apparently as a means to establish priority. In response to Borda's paper, Condorcet took quick action to avoid a potential stumbling block to his own publication plans: using his authority as Permanent Secretary, he arranged for Borda's paper to be included in the Academy's proceedings that were then in preparation, as those would appear in print prior to the intended publication date for his own book. To quote Brian [2008, p. 3] once more:

> This is why a paper, supposedly read in 1770, actually presented in 1784, was published in an official academic volume for the year 1781—the one under press in July, 1784.

Because Condorcet's *Essai*—actually written, supposedly reviewed by referees and already approved for publication by July 1784—was not officially published until 1785, Borda thus technically published on voting theory a year before Condorcet.

The Revolutionary Period

As noted above, a landmark event that is often hailed as the official start of the French Revolution took place exactly five years after Condorcet announced his intention to publish on voting theory to the Academy, when the Bastille prison was destroyed by a group of Parisians on July 14, 1789. The events of the next decade significantly shaped the lives and works of all French citizens, including Borda and Condorcet. In this section, we complete our biographical sketch of their professional lives during that decade, by the end of which both had died. We again begin with some general historical context.

[65] Brian does note that the topic of elections was under discussion in the Academy in 1770, and that it is likely that Borda had shared the idea behind his voting method at that time, but not in a published paper [Brian 2008, p. 3].

[66] Even over a century after its publication, Condorcet's text has been described as "one of the most frequently cited, least-read and poorly-understood works in voting theory" [Urken 2008, p. 1].

While the causes and events behind the French Revolution are numerous and complex, these certainly included the citizenry's discontent with both the French monarchy in general and the fiscal crisis in which the monarchy found itself in particular. As a result of that crisis, the government of Louis XIV proposed the enactment of new tax measures in May 1789, as a means to stave off bankruptcy. Circumstances related to that proposal in turn led to the formation of a National Assembly (later renamed the National Constitutional Assembly) by a group of individuals who demanded governmental restructuring, and, further, that the new structure be guaranteed by a written constitution. The formation of the National Assembly was instigated by delegates of the Third Estate who were summoned to Versailles by Louis XVI for an emergency meeting of the General Estates, a parliamentary entity that was convened (for the first time since 1614) with the express purpose of securing the necessary approval to enact his government's tax proposals. Traditionally, each of the three Estates voted as a single group, despite the fact that the Third Estate represented a far greater proportion of the population than either the First or Second Estates, which represented the clergy and the aristocracy respectively. Although members of the Third Estate are generally referred to as "commoners," its representatives at the Estates General predominantly belonged to the bourgeoisie, particularly the legal professions. After their demand that each deputy instead be given a vote was overruled by a declaration from the First Estate, the representatives of the Third Estate constituted itself as a National Assembly on June 17, 1789; they were later joined in their commitment to create a constitutional monarchy by members of the other two estates.

Grudgingly acknowledged by Louis XVI in June 1789, the National Assembly began to function as a governing body on July 9, 1789. Meanwhile, popular insurgencies began to arise throughout France, including the Paris uprising that culminated in the storming of the Bastille. On August 26, 1789, the Assembly approved the Declaration of the Rights of Man and of the Citizen,[67] premised on the belief that "the ignorance, neglect, or contempt of the rights of man are the sole cause of public calamities and of the corruption of governments." The "natural, unalienable and sacred rights of man" set forth in the Declaration's seventeen articles served as a set of guiding principles for the aspirations of the revolutionaries. The Declaration later became the preamble of the constitution that was ratified by the Assembly on September 3, 1791, thereby establishing France as a constitutional monarchy with sovereignty residing effectively in the Assembly.

Simultaneously with the development of a new constitution for France, the National Constitutional Assembly began to institute reforms aimed at addressing a range of national concerns. Among these

[67] Like the Declaration of Independence of the United States of America, mainly written by Thomas Jefferson some 13 years earlier, the Declaration of the Rights of Man and of the Citizen clearly embodies Enlightenment ideals. The full text of the latter is available at https://www.refworld.org/docid/3ae6b52410.html. It begins with a preamble that describes the Assembly's goal in setting forth a set of "simple and incontestable principles" as an effort to combat the "ignorance, forgetfulness or contempt of the rights of man [that are] the only causes of public misfortunes and the corruption of Governments." By serving as a unceasing reminder to "all members of the body politic . . . of their rights and their duties," its authors asserted the desire that "the acts of the legislative power and those of the executive power . . . may . . . be the more respected" and that "the demands of the citizens . . . may always be directed toward the maintenance of the Constitution and the happiness of all." Jefferson, who was then living in Paris as the US ambassador to France, served as a consultant to those authors, Emmanuel-Joseph Sieyès (1748–1836) and the Marquis de La Fayette (1737–1834), the latter of whom had served as a military consultant to the US during its war of independence from Britain. Unlike the US document, which served in a sense as a rationale for that country's declaration of war aimed at ousting an oppressor, the French document was initially intended by its authors to serve as a rationale for a transition from an absolute to a constitutional monarchy, rather than an attempt to overthrow the monarchy *per se.*

was a problem to which both Borda and Condorcet contributed their technical expertise: the development of a national system of weights and measures. At the time of the revolution, in France and elsewhere, there existed a profusion of different systems of weights and measures, with units varying both from town to town and within the different trade guilds. In addition to hindering trade and tax collection, this disorganized state of affairs drew frequent complaints from French citizens concerned about the potential it created for unfair manipulation of weights and measures (e.g., landowners collecting more than their share of the harvest from the peasants farming their lands). These complaints were quite literally registered in the *Cahiers de Doléances* (*Complaints Books*) that were filed with King Louis XVI (at his invitation) by individuals, trade guilds and legislative units across France in preparation for the 1789 meeting of the General Estates. For example, of the 50 topics found among the complaints submitted by the Third Estate, weights and measures ranked fourteenth [Shapiro et al .1998, p. 381]. Other frequently mentioned topics of complaint included personal liberties, taxes and the judiciary system.

In May 1790, the Assembly charged the Academy of Sciences with the task of developing a single system that would resolve the problem of non-uniform weights and measures in France, and provided it with funding for that effort.[68] The Commission on Weights and Measures appointed to study the issue was chaired by Borda, with Cordorcet and three other mathematicians serving as its other members. Although the commission gave serious consideration to the advantages of a base-12 duodecimal system (e.g., greater ease in the partitioning of goods into fractional parts), they ultimately proposed a decimal system with the unit of length to be based on a fractional arc of a quadrant of the Earth's meridian (eventually defined to be one ten-millionth of the distance from North Pole to Equator measured along the meridian through Paris). The Assembly approved the proposed plan in March 1791, and charged the Academy with its implementation. Based on a proposal by Borda, five separate groups were established to complete different aspects of the necessary scientific work. This included in particular a survey of the meridian arc that defined the meter, an undertaking that took over six years and considerable expense[69] to complete; the construction of the necessary equipment (including three copies of Borda's repeating circle) alone took one full year. The proposed definitions (and names) of the new units of the metric system were adopted into French law in 1793, although the official standard for the meter stick itself was not recognized until 1800 and the widespread use of the metric system across France took even longer.[70]

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[68] Condorcet was influential in convincing the National Assembly to approve funding for the Academy in connection with metric reform, although the proposal itself was put before the Assembly by Bishop Charles Maurice de Talleyrand, a former member of the First Estate who aligned himself with the revolutionary cause early on.

[69] Heilbron [1989, p. 991] reports that estimates of the actual amount spent by the Academy to complete the survey range from 300,000 to millions of *livres.* An additional 500,000 *livres* in funding was also granted to the Committee of Public Instruction by the National Assembly in order "to defray all expenses relative to the establishment of the new measures, as well as the advances which are indispensable for the success of such work," as stipulated in Article 21 of a National Assembly decree approved on April 7, 1795.

[70] The French decree of August 1, 1793, that legally adopted the metric system as that nation's official system of weights and measures also mandated its obligatory use beginning July 1, 1794. In a new decree approved on April 7, 1795, the latter provision was rescinded indefinitely, due in part to the reluctance of the public to abandon its use of the existing units of measurements. That same decree also indefinitely suspended the 1793 provision requiring the use of the decimal division of the day and the parts thereof. Historian of science Kenneth Alder has offered a convincing explanation for why there was such resistance to actually using the metric system, despite the fact that it was complaints from the French people themselves that helped bring the new system into existence [Alder 1995; Alder 2002]. His book,

Borda's contributions to the development of the metric system essentially constituted the full extent of his participation in revolutionary activities, as one might expect given his conservative political leanings. In contrast, Condorcet's ever-increasing involvement in the political events of the time led to the (non-mathematical) works and the writings for which he is best remembered by the world at-large today. In light of his Enlightenment values, Condorcet's enthusiastic support of the ideals set forth in the Declaration of the Rights of Man and of the Citizen comes as no surprise. Echoes of his belief in the universality of equal rights can be heard in his assertion (as quoted in [Alder 2002]):

> A good law ought to be good for all men,
> as a good proposition [in geometry] is good for all men.[71]

In fact, Condorcet embraced this ideal beyond a literal interpretation of the phrase "all men." He was ahead of his time, for instance, in supporting the abolition of slavery, as well as equal political rights for Jews, Protestants and women.[72] And, his commitment to social equality was much more than academic. When the National Constitutional Assembly was disbanded in September 1791 (its work in drafting a new constitution having been completed) and the Legislative Assembly took its place, Condorcet was elected as one of the delegates representing Paris on the new Assembly, where he quickly established himself as a torch bearer for the revolution. He served, for instance, as the secretary of the Assembly and led the efforts of the Assembly's Committee on Education by drafting a plan for state-supported education based on the principles of meritocracy and equality, in keeping with his belief that social progress required an enlightened citizenry.

It was also Condorcet who composed the Legislative Assembly's declaration that justified the suspension of the monarchy. Despite Louis XVI's reluctant acceptance of France's new constitutional monarchy, growing fears of counter-revolutionaries and mistrust of the King's intentions led to the full arrest of the Royal family in 1792.[73] In September of that year, **the**

The Measure of all Things: The Seven Year Odyssey and Hidden Error that Transformed the World [Alder 2002], also provides a riveting account of the surveying project behind the definition of the meter.

[71] In the original French: *"Une bonne loi doit être bonne pour tous les hommes, comme une proposition vrai est vrai pour tous."*

[72] Condorcet wrote several essays advocating the abolition of slavery, the first of which, "Reflexions sur l'esclavage des nègres" ("Reflections on the enslavement of negros"), was published well before the Revolution, in 1781. He was also a founding member, along with the military commander La Fayette, of the Société des Amis des Noirs (Society of the Friends of Negros), established in 1788. His 1790 essay "Sur l'admission des femme au droit de Cité*"* ("On the admission of women to the right of Citizenship") further argued the ability to reason, a human attribute shared by men and women of all races, justifies granting equal rights to all:

> The rights of men stem exclusively from the fact that they are sentient beings, capable of acquiring moral ideas and of reasoning upon them. Since women have the same qualities, they necessarily also have the same rights. Either no member of the human race has any true rights, or else they all have the same ones; and anyone who votes against the rights of another, whatever his religion, colour or sex, automatically forfeits his own.

The full text of an English translation of this essay is available at https://oll.libertyfund.org/titles/condorcet-on-the-admission-of-women-to-the-rights-of-citizenship.

[73] The Royal family had been essentially under house arrest at the Palace of the Tuileries in Paris since the so-called Women's March on Versailles took place on October 6, 1789, in part as a reaction to the chronic hunger and increasing frustration brought about by the costs and scarcity of bread in the city. Concerns about Louis XVI's intentions became

Legislative Assembly was replaced by the National Convention and the (First) Republic of France was declared. Condorcet was again elected as a delegate to the Convention, and he served as both its secretary and vice-president. He also chaired the committee that was charged with drafting the new republic's constitution. Although Condorcet promptly began drafting a version of that constitution, its completion was delayed by his scholarly perfectionism and his extensive involvement in other political activities. In the interim, Louis XVI was condemned to death by the Convention and executed by guillotine in Paris in January 1793. (Although opposed to the death penalty on principle, Condorcet did not vote against the death penalty for Louis XVI; he instead chose to abstain.) By the time Condorcet presented his careful and thorough draft constitution to the Convention (which included an article that prohibited the death penalty), the relatively moderate political group with whom he was associated (the Girondists) was beginning to lose power. For this and other reasons,[74] Condorcet's proposed constitution was rejected in June 1793, in favor of a hastily-drafted "Constitution of Year I" that was put forward by a far more radical group (the Jacobins). With the passage of the Jacobins' constitution, a 9-member Committee of Public Safety[75] gained unlimited executive power and the infamous period of the revolution known as the Reign of Terror began soon thereafter. Between September 1793 and July 1794, Queen Marie-Antoinette and thousands of other French citizens from all social classes and professions were accused of counterrevolutionary actions and met their fate, as had Louis XVI, at the blade of the guillotine.[76]

Although Condorcet himself escaped the guillotine, he did not outlive the Reign of Terror. After openly and passionately accusing the Committee of Public Safety of using fear tactics to gain passage of their version of the constitution, a warrant for his arrest was issued by the Convention in July 1793. Condorcet became a fugitive, and he hid in the Parisian home of Madame Vernet[77] for

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more pronounced following his family's aborted attempt to flee the country during the night of June 20–21, 1791. Nevertheless, the National Constitutional Assembly was willing to allow Louis to retain the throne under a constitutional monarchy at that time.

[74] While the politics of the French revolution are too complex to recount here, the description of Condorcet's abilities (or lack thereof) as a politician offered by historian John Herivel suggests another possible factor in the failure of Condorcet's constitution [Herivel 1975, p. 247]:

> Condorcet was no politician. His uncompromising directness of manner and inability to suffer illogical windbags in silence made him many enemies and few friends. His weak voice, lack of oratorical powers, and tendency to bore the Convention by the excessive height of his arguments was one of the tragedies of the Revolution.

[75] The Committee of Public Safety was actually created in April 1793 to replace the previous Committee of General Defense, an administrative body that was charged with protecting the new republic from internal and external attacks. In July 1793, Robespierre (1758–1794) became a member of the restructured Committee of Public Safety and served as its *de facto* leader until his own arrest and subsequent death by guillotine a year later essentially brought the Reign of Terror to a close.

[76] Estimates of the number of individuals that were executed in France during the Reign of Terror by the guillotine or other means range upwards of 17,000, excluding those who died at the hands of vigilantes or in prison while awaiting their trial. Lynn, who sets the number of official executions at 40,000, further estimates that the number of individuals who were arrested accounted for about 1 in every 50 French citizens, or approximately 300,000 individuals in all [Lynn n.d.]. Members of the clergy and aristocrats were disproportionately represented both among those arrested and those executed by the state.

[77]Located on Paris' left bank on the Rue des Fossoyeurs (now 15 Rue Servandoni), Condorcet's refuge was coincidentally located within half a block of the residence of Olympe de Gouges (1745–1793), a playwright and author of the *Declaration des Droites de Femmes et Citoyennes* (*Declaration of the Rights of Women and Citizenesses*). [The

the next eight months. During this time, Condorcet wrote his most famous philosophical work, *Esquisse d'un tableau historique des progrès de l'esprit humain* (*Sketch of a historical picture of the progress of the human mind*),[78] a declaration of his enlightenment belief in the power of rationality and perfectibility of human society. Concerned about the peril in which he was placing his protector, Condorcet fled Paris on foot on March 25, 1794, but he was denounced to local authorities by an innkeeper in the nearby village of Clamart-de-Vignoble. Placed under arrest, he was transferred to a guarded house located the village of Bourg-la-Reine[79] on March 29, 1794. The following day, Condorcet was found dead in his makeshift cell; whether he took his own life, died of natural causes, or was murdered is unknown to this day.[80]

As for Borda, after retiring to his family estate during the Reign of Terror, he returned to Paris and resumed his work on the metric system. He died[81] after a long illness on February 20, 1799 at age 65, just months before the *mètre des Archives*, a platinum bar selected as the first prototype of the meter based on the results of the meridional survey effort that Borda coordinated, was placed in the National Archive.

The final years of Borda's life, from 1796 to1799, also witnessed the military rise of Napoléon Bonaparte (1769–1821), which in turn signaled the end of the revolutionary period in France. In the next section, we return to the history of the Academy of Sciences, where Napoléon also exerted his influence.

English translation of de Gouges' document is available at https://www.olympedegouges.eu/rights_of_women.php.] Although it is unlikely that Condorcet and de Gouges associated with each other during the time that Condorcet was in hiding, the two were almost certainly acquainted, given de Gouges's attendance at the salon of Condorcet's wife Sophie de Condorcet, née de Grouchy, at the Hôtel des Monnaies. Condorcet only met his protector Madame Vernet, the widow of the sculptor Louis-François Vernet (1744–1784), when mutual friends brought him to her home seeking refuge.

[78] Condorcet's *Esquisse* was published posthumously by his wife, Sophie de Condorcet, née de Grouchy, in 1795. In 1799, she also published his *Éloges des academicians* (*Eulogies for the Academicians*). Although de Grouchy had filed for divorce while Condorcet was in hiding, she did so with his secret consent as a move to prevent the seizure of her property by the revolutionary government due to her marital connection to the fugitive Condorcet. This strategy failed when he died before the divorce was finalized. De Grouchy thereafter earned a living for herself and their daughter Eliza by painting miniature portraits. She also continued to edit Condorcet's collected works, and remained active in France's intellectual culture through her re-established salon. Although de Grouchy wrote many articles that were published anonymously, both before and after the revolution, the only one of her works to appear in her own name was the book *Huit Lettres sur la Sympathie* (*The Letters on Sympathy*), which was published alongside her translation of Adam Smith's *Theory of Moral Sentiments* and *Origins of Language* in 1798.

[79] This house was located at what was then 49 Grande Rue (now 81 Ave Le Clerc). The town itself was known by a more revolutionary name at that time: Bourg-de-l'Égalité. Under its current name, Bourg-la-Reine is familiar to students of the history of mathematics as the birthplace of Évariste Galois (1811–1832), one of the founders of modern abstract algebra and a staunch supporter of the French republicans during the post-Napoléonic era, who died of injuries suffered in a duel at the age of 21.

[80] Although the location of his remains are also unknown to this day, memorials of Condorcet's life and work include plaques at his final "residences" in Paris and Bourg-la-Reine, a statue at 15 Quai de Conti in Paris (between the Musée de Monnaie and the Institut du France), and the symbolic interment in the Pantheon in Paris of ashes taken (in 1989) from the cemetery in Bourg-la-Reine.

[81] Memorials of Borda's life and work include a statue in the Place de la Halle in his birthplace of Dax, and a series of French naval vessels harboured in Brest (and nicknamed *The Borda*) that have successively served as the site of French Naval School since its founding in 1830. The current collection of scientific instruments and other objects in the Borda Museum in his hometown of Dax was founded on Borda's own collection of curiosities. Borda's name (but not Condorcet's) is also listed, together with 71 other scientists, on the first floor of the Eiffel Tower. Both men have Paris streets named after them (Borda in the 3rd Arrondisement and Condorcet in the 9th).