Braess' Paradox in City Planning: An Application of Multivariable Optimization

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On December 5, 1990, The New York Times published an article titled *What if They Closed 42d Street and Nobody Noticed?* [Kolata, 1990] Two of the early paragraphs in this article summarize what happened.

> On Earth Day this year, New York City's Transportation Commissioner decided to close 42d Street, which as every New Yorker knows is always congested. "Many predicted it would be doomsday," said the Commissioner, Lucius J. Riccio. "You didn't need to be a rocket scientist or have a sophisticated computer queuing model to see that this could have been a major problem."
>
> But to everyone's surprise, Earth Day generated no historic traffic jam. Traffic flow actually improved when 42d Street was closed.

This counterintuitive phenomenon, in which the removal of an edge in a congested network actually results in improved flow, is known as Braess' Paradox. This paradox had actually been studied decades earlier not by rocket scientists, but by mathematicians. In the 1968 paper "Über ein Paradoxon aus der Verkehrsplanung" [Braess, 1968] or in English, "On a paradox of traffic planning" [Braess, 2005], Dietrich Braess[1] described a framework for detecting this paradox in a network.

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[1]Dietrich Braess is a German mathematician, born in 1938 in Hamburg. He earned his Ph.D. in theoretical physics in 1964 at Universität Hamburg. He is currently professor emeritus at Ruhr-Universität Bochum. You can find out more about Braess through the Mathematics Genealogy Project website (www.genealogy.math.ndsu.nodak.edu/id.php?id=23190), or from Braess' university webpage (https://homepage.ruhr-uni-bochum.de/Dietrich.Braess/#eng).

When Braess' paper appeared in 1968, the application of mathematics to traffic planning was still a relatively new idea.[2] Today, the paradox named in his honor continues to be studied by mathematical researchers and used by transportation specialists in the design of traffic networks. Other applications of Braess' paradox have also been found in the modeling of telecommunication networks and the Internet, as well as the design of electrical power systems and electronic circuits, mechanical and fluid systems, metabolic networks and ecosystems, and even sports analytics [Nagurney and Nagurney, forthcoming]. In this project, we see how the examples he provided can be analyzed using standard optimization techniques from a multivariable calculus course.

The Basic Idea

How could it be possible that the removal of a road could improve traffic flow? Braess explained the general idea in the abstract to his paper [Braess, 2005].

> If every driver takes the path that looks most favorable to him, the resultant running times need not be minimal. Furthermore, it is indicated by an example that an extension of the road network may cause a redistribution of the traffic that results in longer individual running times.

Task 1 What did Braess assume about the behavior of drivers? Was that assumption realistic?

Braess' Example Network

We now look at the example Braess provided of this phenomenon. Describing this in a precise manner took a good bit of notation, which for convenience we organize here:

- Braess used the word *nodes* to represent destinations to or from which drivers travel.
- He used arrows to represent the roads connecting these nodes. Note that the direction of an arrow indicated a one-way connection.
- He used $t_i(\phi)$ to represent the time it takes to traverse road i with ϕ drivers on it.

[2] The first efforts to formulate traffic flow problems mathematically began only in the 1950s, and two publications from that decade (Wardrop 1952 and Beckmann, McGuire and Winsten 1956) represented the state of the art when Braess himself entered the field. Surprisingly, Braess was completely unaware of those important prior works! His interest in the mathematical modeling of traffic flow instead was inspired by a 1967 seminar talk in which the German mathematician W. Knödel presented a certain algorithm that roused Braess' curiosity. At the time, Braess was 29 years old and had only recently turned to the study of mathematics after completing his doctorate in theoretical physics just three years earlier. Nearly 40 years later, in 2006, Braess delivered his first North American lecture on the paradox named in his honor at the Virtual Center for Supernetworks at University of Massachusets Amherst. In that lecture, he remarked that both his lack of knowledge of the then-current state of transportation science and his background in physics, which had trained him to look for a counterintuitive symmetry-breaking argument, were important factors in shaping the work that led to his discovery of the paradox that now bears his name.

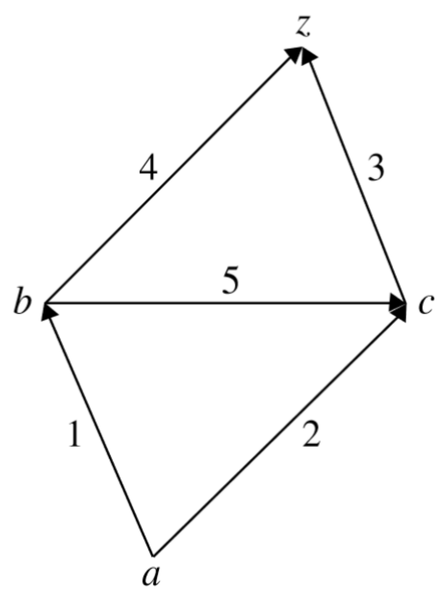
- He used the notation $|\Phi|$ to represent the total number of drivers. This total number of drivers was then split into potentially different quantities of drivers taking different paths. This was written $\Phi_{a_1 a_2 \ldots a_n}$, to represent the number of drivers on the path from a_1 to a_2 to $\ldots$ to a_n.
- The expression $T(\Phi)$ denoted the maximum travel time for a given distribution of drivers. That is to say, $T(\Phi)$ represented the worst-case travel time among all the drivers.

We now turn to a model example with four nodes. For convenience, the link travel times $t_\alpha(\phi)$ are linear functions...

$$t_1(\phi) = t_3(\phi) = 10\phi,$$

$$t_2(\phi) = t_4(\phi) = 50 + \phi,$$

$$t_5(\phi) = 10 + \phi.$$



(a) If a total flow of $|\Phi| = 2$ is to be guided from a to z, the optimal solution is

$$\Phi_{abcz} = 2, \quad \Phi_{abz} = \Phi_{acz} = 0, \quad T(\Phi) = 52.$$

(b) If a total flow of $|\Phi| = 6$ is to be guided from a to z, the optimal solution is

$$\Phi_{abcz} = 0, \quad \Phi_{abz} = \Phi_{acz} = 3, \quad T(\Phi) = 83.$$

(c) If a total flow of $|\Phi| = 20$ is to be guided from a to z, the optimal solution is

$$\Phi_{abcz} = 0, \quad \Phi_{abz} = \Phi_{acz} = 10, \quad T(\Phi) = 160.$$

Obviously, all the solutions are unique.

Task 2 Verify the claims Braess made above. Specifically, in each of the three cases, calculate the travel times and verify the value given for $T(\Phi)$. In each case, can you make an argument regarding why any redistribution of the drivers will cause the maximum travel time to increase?

Task 3 (a) Consider the following list of connections that appear in road networks:

- 1 km of three-lane interstate highway,
- 1 km of three-lane interstate highway that shrinks from three lanes to two lanes halfway through,
- the toll booth at the entrance to a tunnel, and
- the stretch of the tunnel following the tollbooth.

Suppose the travel time for each of those connections was modeled by a linear function $t(\phi) = m\phi + b$, where ϕ represents the number of drivers on that road. In each case, describe what you think the sign and relative magnitude[3] of each of m and b might realistically be.

(b) Can you come up with a road map, labelled with some key geographic features (rivers, cities, mountains, etc.), that could be represented by Braess' example network?

We consider once more the model example from last section. If the volume of traffic is as in cases (a) or (c), then it is most profitable to move in accordance with the optimal flow. This is different in case (b). The optimal flow moves along paths (abz) and (acz). There exists, however, a path for which travel time is lower. Specifically, $T_{abcz} = 70 < 83 = |T(\Phi)|$. Suppose that the vehicles are distributed as the optimal flow. Those drivers to which the link travel times are known would move to path $(abcz)$ and destroy the optimality.

Task 4 Explain why in case (b) we have the travel time on path $abcz$ as $T_{abcz} = 70$ (as a hint, notice that the function t_5 outputs a travel time of 10 when there are no cars on the highway). Suppose all drivers realize this and everyone switches to now take path $abcz$. What is the new travel time? How does it compare to the previous travel time of 83?

[3]Note we say "relative magnitude" because we could make any nonzero magnitude as large as we like by measuring time in sufficiently small units.

In the quote below, Braess used the symbol u_5 to refer to the road labelled with the number 5 in the figure.

In this framework, we also recognize a paradoxical fact. If the link u_5 is eliminated in the road network, ... the distribution of the traffic flow is improved in this case. This means that for real-life traffic practice: *In unfavorable situations an extension of the road network may lead to increased travel times.*

Task 5 Braess claimed that this paradox occurs only in "unfavorable situations", implying that it does not hold all the time. Verify that this paradox does not hold in cases (a) and (c). That is, check that no removal of an edge in those cases will result in improved flow.

Average Travel Times and Multivariable Optimization

In the same paper, Braess mentioned that there is freedom to change which property of the traffic flow is being optimized. He explored an alternate measure of how good a particular distribution is in the excerpt below. The symbol β in the definition represents a path connecting the origin-destination pair being considered. So, the summation is over all such paths, and the symbol T_β represents the travel time across path β.

The concept and the results do not differ substantially if the mean value of the travel time

$$\frac{1}{|\Phi|}\sum_{\beta}\Phi_\beta T_\beta\left(\Phi\right)$$

and not maximal time, $|T\left(\Phi\right)|$, determines the quality. It cannot be determined by mathematical arguments which specification is more appropriate. This decision must be left to traffic planners.[4]

Task 6 What were the two different measures that Braess used to determine how good a particular traffic flow is? Can you imagine a situation where one of these measures might be more appropriate than the other?

[4]Mathematicians and traffic planners in Germany had quite a bit to study around this time! In the decades that followed the end of World War II in 1945, Germany's cities underwent massive amounts of reconstruction and traffic planning was a huge priority across the nation. Historian Jeffery M. Diefendorf writes "Traffic planning was central to the radical plans for Mainz and Berlin, the conservative plans for Freiburg and Cologne, and the pragmatic plans of cities like Aachen and Kiel. Often traffic planning took precedence over other planning goals. It would be possible to present an almost unending list of citations in which major planners proclaimed that traffic planning formed the heart, the *Kernstück*, of their efforts" [Diefendorf, 1993].

It is slightly harder to compare average travel times using *ad hoc* arguments than it is to compare maximum travel times, even in small networks like the one we discussed above. Thus, this is a great place for the framework of multivariable calculus and optimization to step in!

Task 7 Let us see if we can verify Braess' claim that "the results do not differ substantially" with this new goal. Specifically, let us revisit the same road network from Braess' case (a). We wish to see what the optimal flow looks like when we analyze the travel times using average rather than maximum travel time.

(a) First, we declare our variables:

- Let x represent the number of drivers who use path abz.
- Let y represent the number of drivers who use path acz.
- Let z represent the number of drivers who use path $abcz$.

Briefly explain why those are the only three paths we need to consider.

(b) Explain why the functions

$$\begin{aligned} T_{abz}(x,y,z) &= 10(x+z) + (50+x), \\ T_{acz}(x,y,z) &= (50+y) + 10(y+z), \text{ and} \\ T_{abcz}(x,y,z) &= 10(x+z) + (10+z) + 10(y+z) \end{aligned}$$

describe the travel times on their respective subscripted paths.

(c) Explain why the function

$$A(x,y,z) = \frac{x \cdot T_{abz}(x,y,z) + y \cdot T_{acz}(x,y,z) + z \cdot T_{abcz}(x,y,z)}{2}$$

calculates the average travel time.

(d) Explain why to optimize the function $A(x,y,z)$, we should impose the constraint

$$x + y + z = 2.$$

(e) Use Lagrange multipliers to find the minimum possible average commute time.[5] Note that Braess was only looking at integer values of x, y, and z; Lagrange multipliers will be instead searching over all real numbers x, y, and z. Thus, you may find a better minimum!

(f) Check your work another way: throw out z by substituting $z = 2 - x - y$ into the formula for $A(x,y,z)$. Now that A is a function of only x and y, find the critical points by setting $\partial A/\partial x$ and $\partial A/\partial y$ equal to zero and then applying the second derivative test. Can you confirm the minimum average travel time from the previous part?

[5] Do not be bothered if the results are not whole number values of x, y, and z. Perhaps the 2 which measures the total flow is measured in units of thousands of cars, or tens of thousands even. In these cases, fractional values of x, y, and z might make perfectly good sense.

(g) Is the minimum that you found a stable state? That is, if the drivers were to be distributed as per that optimal arrangement, would it stay that way or would drivers begin to switch their routes? If they re-route, what is the new average travel time, and how does it compare to the previous average?[6]

(h) If you delete the road that connects b to c, what is the new minimum average travel time?

(i) Use the information you gathered throughout the parts of this task to verify that Braess' paradox is still not present in his example (a), using average rather than maximum travel time, thus verifying his claim that the "results do not differ substantially" in this case.

So How Often Does This Actually Happen?

Looking at the strange behavior of Braess' case (b), one might wonder if this is a common enough phenomenon to really think about in traffic planning, as opposed to a pathological example built by Braess just for the fun of exploring interesting mathematical objects. This question was answered in the paper "The Prevalence of Braess' Paradox" [Steinberg and Zangwill, 1983]. In their abstract, they stated their main result.

> The present paper gives, under reasonable assumptions, . . . that Braess' Paradox is about as likely to occur as not occur.

These "reasonable assumptions" include the following:

- There is just a single origin-destination pair on which the flow is being considered.
- The network is congested; in noncongested networks, one can show that Braess' paradox will never occur.

Task 8

(a) Do a bit of research and find a few examples throughout the history of city planning (other than the one described in this project) where Braess' Paradox has been observed.

(b) Pull up a road map of a metropolitan area of interest to you! Perhaps it is the area you grew up in, the area you currently live in, a place you would like to visit, or the landing place of a dart you threw at a map. Look at the map and think about traffic patterns. Do you suspect Braess' Paradox might be occurring in this area? Can you identify a road (and use mathematics to back it up) whose closure might result in improved traffic flow?

[6] One could argue that results like these make a great case for driverless cars with a centralized navigation center which optimizes the minimum average commute time for the entire group rather than individual drivers all acting selfishly!

References

Dietrich Braess. Über ein Paradoxon aus der Verkehrsplanung. *Unternehmensforschung*, 12:258–268, 1968.

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Jeffry M. Diefendorf. *In the Wake of War: The Reconstruction of German Cities after World War II*. Oxford University Press, Inc., 1993. ISBN 0-19-507219-7.

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Richard Steinberg and Willard I. Zangwill. The prevalence of Braess' Paradox. *Transportation Science*, 17(3):301–318, August 1983.

Notes to Instructors

PSP Content: Topics and Goals

This PSP is intended to give students an instance where they can apply the standard techniques of optimization from a multivariable calculus course. The author has found this to be particularly valuable as such courses tend to be full of a plethora of physics and engineering applications (which are lovely) but little beyond that. Not only do students get practice with the standard Lagrange multiplier and critical point/second derivative test techniques, but they get some exposure to key ideas of graph theory and discrete optimization along the way, and get to see that the ideas of multivariable calculus can be applied in areas beyond physics and engineering. The key competencies which come up in this project are as follows:

- Lagrange multipliers
- Partial derivatives
- Critical points and the second derivative test for surfaces

Student Prerequisites

The standard multivariable optimization techniques listed above are the only prerequisites for students attempting this project.

PSP Design, and Task Commentary

This PSP was designed by very carefully threading a reasonable path through Braess' work that a multivariable calculus student can follow, with the absolute minimum amount of extra prerequisite knowledge. Specifically, the author did not wish to derail his multivariable calculus class by turning it into a course on graph theory and combinatorial optimization (or maybe he wished to do so but chose not to for the sake of actually covering the course competencies). Thus, the PSP starts by giving students a bare minimum of required notation for them to be able to understand the graph that Braess used for all of his examples. Then, when analyzing average travel times, the techniques required in Task 7 lie solidly in the standard topics of a first course in multivariable calculus (specifically critical points, second derivative test, and Lagrange multipliers).

Suggestions for Classroom Implementation

It could be helpful to provide students with a bit more guidance in the earlier parts of the project; the later parts are routine applications of multivariable calculus techniques. As an extension, instructors may wish to inform students that they can take whole courses in combinatorial optimization. The particular situation addressed here (optimizing the maximum travel time) was just the tip of the iceberg.

The instructor may encounter the student who wishes to see this a bit more with their own eyes. In this case, we recommend the exploration of Brian Hayes' simulator (which was used to generate the animation for the onscreen article in the *Convergence* publication of this PSP), found at `http://bit-player.org/extras/traffic/` along with corresponding *American Scientist* article at `http://bit-player.org/wp-content/extras/bph-publications/AmSci-2015-07-Hayes-traffic.pdf`. For the student who loves computer programming, an interesting project could involve customization of that simulator (source code available at `https://github.com/bit-player/traffic`). In particular, Hayes' simulator has a two-directional connection that can be added or removed, as opposed to the network in Braess' example which has only one-directional connections. It would be interesting to see how that restriction changes the simulator.

Sample Implementation Schedule (based on a 50-minute class period)

If an in-class activity is desired, the following would be a reasonable breakdown:

- Advance reading: Read the opening sources, up to but not including Task 2.
- First 20 minutes of class time: lead discussion on the opening passages, likely including addressing some questions on what the notation and the figure represent. Together, work out Task 2 for case (a). Then let the students verify cases (b) and (c).
- 30 minutes: Let students carry out the analysis of the network with respect to average travel times, using the standard optimization techniques learned in a multivariable calculus class.

Completed write-up of the PSP can then be due the following week.

The author, however, does not implement this PSP in that manner. Rather, the author uses this as one of a choice of projects. A small group of students (perhaps 2–3) could have time in class one session to prepare solutions and time in class another session to present solutions in a mini-conference. Ideally, the other students would be doing the same but on different projects in parallel. Four further student projects appropriate for the study of optimization in a multivariable calculus course are available upon request from the author (although none of these additional projects are based on primary source material).

The author is happy to provide LaTeX code for this project. It was created using Overleaf which makes it convenient to copy and share projects and can allow instructors to adapt this project in whole or in part as they like for their course.

Connections to other Primary Source Projects

The following additional projects based on primary sources are also freely available for use in teaching standard topics in the calculus sequence. The PSP author name of each is given (together with the general content focus, if this is not explicitly given in the project title). With the exception of the final project in the list (which requires up to 2 full weeks for implementation), each of these is a mini-PSP that can be completed in 1–2 class days. Classroom-ready versions of these projects can be downloaded from `https://digitalcommons.ursinus.edu/triumphs_calculus`.

- The Derivatives of the Sine and Cosine Functions, Dominic Klyve
- Fermat's Method for Finding Maxima and Minima, Kenneth M Monks
- Beyond Riemann Sums: Fermat's Method of Integration, Dominic Klyve
- How to Calculate π: Buffon's Needle (calculus version), Dominic Klyve (integration by parts)
- Gaussian Guesswork: Elliptic Integrals and Integration by Substitution, Janet Barnett
- Gaussian Guesswork: Polar Coordinates, Arc Length and the Lemniscate Curve, Janet Barnett
- Gaussian Guesswork: Infinite Sequences and the Arithmetic-Geometric Mean, Janet Barnett
- Investigations Into d'Alembert's Definition of Limit (xalculus version), Dave Ruch (sequence convergence)
- How to Calculate π: Machin's Inverse Tangents, Dominic Klyve (infinite series)
- Euler's Calculation of the Sum of the Reciprocals of Squares, Kenneth M Monks (infinite series)
- The Radius of Curvature According to Christiaan Huygens, Jerry Lodder

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