Completing the Square: From the Roots of Algebra

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1 Introduction

Most high school students today learn to solve *quadratic equations*, equations of the form

$$ax^2 + bx + c = 0, \tag{$*$}$$

where a, b and c are numbers and a is not 0.

A fair amount of attention in treatments of beginning algebra is given at first to resolving such equations through *factoring*, the sometimes difficult process of reversing the multiplication of linear expressions in the unknown. This procedure aims to replace $(*)$ with an equivalent equation of the form

$$a(x - r_1)(x - r_2) = 0, \tag{$**$}$$

from which the solutions $x = r_1, r_2$ easily follow.

The step that carries us from $(*)$ to $(**)$ is admittedly not straightforward. Even in the simplest situations in which a, b, c and r_1 and r_2 are all integers, holding together all the necessary arithmetic together associated with the factorization procedure in one's head is not easy and typically requires lots of practice to accomplish successfully.

It is not difficult at all to find examples of quadratic equations with small integer coefficients (a, b, c) whose solutions are *irrational* numbers, which makes the factorization process fiendishly hard; it is at this stage that problem solvers tend to "cry uncle" and plead for an easier out. Salvation arrives in the form of the *quadratic formula*,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

a formula whose complexity is counterbalanced by assurances that (1) it is universally applicable for solving all quadratic equations, and (2) it bypasses the entire factoring process! But the price one pays to gain these valuable features is manifested by the inscrutability of the formula. How is it that the formula produces such reliable results? And where does it come from? Things can get even worse: in some cases, the quadratic formula will give *complex numbers* as solutions, numbers which don't correspond to familiar values on the (real) number line (or on some calculators)!

A way out of this mess and one that allows us to see better what is going on comes in the form of yet another method for solving $(*)$, called *completing the square*. Here, the solver must first 'simplify'

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(∗) by dividing through by the number a to reduce the leading coefficient to 1. Then comes the key step, in which the solver performs a very specific (but as-yet unmotivated) calculation: take half of the coefficient of x, square it, and add this to both sides of the equation. This "incantation" serves to recast the equation into a form that is easy to factor and thus straightforward to solve. By carrying out this procedure in its full generality, the quadratic formula itself emerges. In other words, it is by understanding the method of completing the square in its full generality that the quadratic formula is explained.

This Primary Source Project has been designed to help lift the veil on these mysterious procedures. By studying the writings of a man who lived over a thousand years ago, the reader can discover how and why the procedures described above work.

2 The Roots of Algebra

Muḥammad ibn Mūsā al-Khwārizmī[1] (ca. 780–850 CE) is one of the great legendary figures in the history of mathematics. Around the year 820, the caliph al-Ma'mūn[2] summoned al-Khwārizmī to Baghdad to join his team of scholars there, where he made substantial contributions to the study of mathematical astronomy and its applications to preparing calendars and almanacs, which were used to predict the times of celestial events, the dates of Muslim feast days and the *qibla* (direction) for daily prayer.[3] He authored works in cartography and geography to map the known world, and also made key advances in pure mathematics.[4]

The most important of al-Khwārizmī's works, and the source of the texts we will encounter in this project, is the work now considered to be the first written work in algebra. Compiled in about the year 825, it has the title *al-Kitāb al-mukhtaṣar fī hisāb al-jabr wal-muqābala*[5] (*The Compendious Book on Calculation by Restoration and Reduction*)—hereafter referred to by its more common and much shorter name, *Algebra*—and it enjoyed immediate and lasting success as a work in quantitative problem solving. It represented an important advance in the art of arithmetical calculation, in that it offered not just methods of reckoning to work out sums, differences, products, ratios and extraction of roots of numbers for the solutions of many standard practical problems, but it also classified many problem types and laid out systematic methods for solving such problems by means of this classification. For the first time, methods for computational problem solving were presented not problem by problem on an *ad hoc* basis, but for entire families of problems at once.

In what follows, we present selections from al-Khwārizmī's *Algebra*, using two English translations, one by the nineteenth century orientalist Frederic Rosen [al Khwārizmī, 1831], and the other a more recent scholarly edition by Roshdi Rashed[6] [Rāshid, 2009].

[1]Pronounced *mu-HAM-ad ib'n MOO-suh ahl-qua-RIZ-mee.*

[2]Pronounced *ahl-mah'-MOON*

[3]Muslims are required to pray five times each day—at dawn, at noon, at mid-afternoon, at sunset, and at night. These prayers are to be performed by facing in the direction of Mecca, a direction called the *qibla*. Of course, this direction differs depending on where on the earth the supplicant stands or prostrates, and it is determined for each place by using methods of mathematical astronomy.

[4]For biographical information on al-Khwārizmī, the reader is directed to [Katz, 1998], [Lindberg, 1992] or [Saliba, 2007]. For a more general history of Islamic science in the middle ages than what appears in the following paragraphs, see, in addition to the above, [Berggren, 2003] and [Katz et al., 2007].

[5]Pronounced *ahl ki-TAHB ahl-muk-TAH-sahr fih hih-SAHB ahl-JAB'r wal-moo-KAH-bah-lah.*

[6]Rashed (b. 1936), also identified as Rushdi Rashid, is an eminent Egyptian historian of mathematics.

After giving praise to God, to the Prophet Muhammad, and to Caliph al-Ma'mūn (in that order, of course!), al-Khwārizmī opened his *Algebra* as follows:

> When I considered what people generally want in calculating, I found that it always is a number. ... I observed that the numbers which are required in calculating by Restoration and Reduction are of three kinds, namely, roots, squares, and simple numbers relative to neither root nor square.
>
> A root is any quantity which is to be multiplied by itself, consisting of units, or numbers ascending, or fractions descending.
>
> A square is the whole amount of the root multiplied by itself.
>
> A simple number is any number which may be pronounced without reference to root or square.
>
> A number belonging to one of these three classes may be equal to a number of another class; you may say, for instance, "squares are equal to roots," or "squares are equal to numbers," or "roots are equal to numbers."

To help us make sense of what al-Khwārizmī meant by the "three kinds" of "numbers which are required in calculating by Restoration and Reduction," it will help to reflect on the words he used to name each class. In the same way that a plant grows up from its roots, or that someone's personal history is referred to as her roots, he used the word "root" here in a similar but mathematical sense, to refer to a quantity from which another is generated.[7] Likewise, to name the second type of number, he used the word "square" in a very suggestive sense, one that explicitly described the process for numerically computing the area of a geometric square that has "the root" for its side.

Finally, al-Khwārizmī used the term "simple" in order to qualify numbers of a third and somewhat less complicated sort, namely those "which may be pronounced without reference to root or square." To better understand what he meant by this, it is helpful to know that, in his later discussion of problem solving, the "root" was always identified with the unknown quantity, which al-Khwārizmī also called the "thing." For example, in a legacy problem that al-Khwārizmī solved later in the *Algebra*, he spoke of "three *dirhams*[8] plus one-third of a *dirham* plus one-third of a thing." In the phrase "one-third of a thing," the number "one-third" is not "simple" since it specifies the number of "roots." On the other hand in the phrase "three *dirhams* plus one-third of a *dirham*," the numbers "three" and "one-third" are "simple" because they can "be pronounced without reference to root or square;" in other words, they are not associated with a number of "roots" or a number of "squares." Note that in al-Khwārizmī's highly practical science of problem solving, a "simple number" was almost always identified as a number of *dirhams*, not as a unitless quantity.

[7] This use of the word *root* long predates al-Khwārizmī, and can be found in the works of ancient Greek mathematicians and geometers.

[8] A *dirham* was the standard coin of the day in al-Khwārizmī's time in Baghdad, likely the largest city on the planet at the time, and was typically struck from silver. The Arabic word *dirham* was derived from the name of the ancient Greek coin, the *drachma*, which would also have been the standard currency of the Middle East in pre-Islamic times.

This example points to another feature of the context of problem solving that was part of the perspective of al-Khwārizmī and his contemporaries: *numbers were (almost always) assumed to be positive*. They were used to count with, or to measure out quantities of stuff, especially when they entered into problems to be solved. And although they might not have come directly from real-life problems, they were at least realistic in form. In particular, a number of *dirhams* would only be considered negative in the special context in which they were simultaneously being considered as part of a debt, but the root of a square could never be negative, since it represented a length.

Now refer back to the description of the various classes of quantities that al-Khwārizmī enumerated at the beginning of his *Algebra*. In the last sentence of that description, he identified the three kinds of equations which can be formed by setting some two of these three classes of quantity equal to each other: "squares are equal to roots," "squares are equal to numbers," or "roots are equal to numbers."

Task 1

(a) Al-Khwārizmī provided the following three examples of such polynomial equations. Which of the three types listed just above corresponds to each example?

(i) the half of the square is equal to eighteen

(ii) five squares are equal to ten roots

(iii) four roots are equal to twenty

(b) Translate each of the three examples above into symbolic form and solve the equations, both using al-Khwārizmī's sense of the word (numbers must be positive) and in the broader modern sense. Be careful: the equations may have different numbers of solutions depending on this sense!

(c) The most general representation of the types of monomial described here will have the symbolic forms ax^2 (squares), bx (roots), and c (simple numbers). Use these to express symbolically the general form of the three types of al-Khwārizmī's equations. Solve each one for x, making sure to identify what conditions will have to be satisfied in order that the steps you employ are valid. (For instance, you may not divide by 0 or take the square root of a negative number if your answer is to be a real number.)

(d) The case "roots are equal to numbers" produces an equation in which the highest power of x that appears is the first power. Such equations are called *linear* equations in modern mathematics. Why is this?

Task 2

Consider the object of al-Khwārizmī's concern, which he announced in the first sentence of the excerpt above. He called it "what people generally want in calculating." In the context of the problems examined in Task 1(a), what was it that al-Khwārizmī wanted to calculate? Reflecting on this, what do you think al-Khwārizmī meant more generally by the phrase "what people generally want in calculating?"

3 Solving a Quadratic Equation

As Task 1 shows, the first three equation forms identified by al-Khwārizmī involved only two classes of numbers at a time, and are thus rather straightforward to solve. His next task was to consider more complicated problems that can be formed involving all three classes of numbers.

> I found that these three kinds, namely roots, squares, and numbers, may be combined together, and thus three compound species arise; that is, "squares and roots equal to numbers"; "squares and numbers equal to roots"; "roots and numbers equal to squares".

Task 3

(a) Write out symbolic forms for the equations given by these "three compound species." Notice that the word "and" in each phrase is represented symbolically in the same way.

(b) In modern mathematics, all three "compound species" would be subsumed under a single form, namely,

$$ax^2 + bx + c = 0,$$

where none of a, b or c can be zero. Why did al-Khwārizmī require three different "species" to describe this single problem?

Al-Khwārizmī gave examples of his "three compound species" and in each case provided a recipe for solving the corresponding problem, which for him meant identifying a value both for the "thing" (which is the "root"), and for its "square." Here is the first of the cases that he treated. (Circled numbers have been inserted into the text below only to make it easier to reference the steps of the solution; *they are not part of the original source.* The task that follows this excerpt will have us work through each of these steps in order to make sense of al-Khwārizmī's method of solution.)

> Roots and Squares are equal to Numbers: ① for instance, "one square, and ten roots of the same, amount to thirty-nine *dirhams*;" that is to say, what must be the square which, when increased by ten of its own roots, amounts to thirty-nine? ② The solution is this: you halve the number of the roots, which in the present instance yields five. This you multiply by itself; the product is twenty-five. ③ Add this to thirty-nine; the sum is sixty-four. ④ Now take the root of this, which is eight, ⑤ and subtract from it half the number of the roots, which is five; ⑥ the remainder is three. This is the root of the square which you sought for; ⑦ the square itself is nine.

Task 4 Using modern algebraic symbolism, let us follow al-Khwārizmī's steps to set up the problem and produce our own solution from the source text above. We will also be analyzing, by means of this same symbolism, how al-Khwārizmī "missed" an alternate solution to the problem.

(a) Formulate the problem symbolically in an equation (step ①). Note that this is not what al-Khwārizmī did! He made no reference to expressing his solution in terms of symbols of any sort. What you are doing here is a *modern treatment* modeled on what al-Khwārizmī reported in his solution. (We will comment on this again below.)

(b) At step ②, al-Khwārizmī performed some arithmetic associated with the equation you have formulated in part (a). You should tie this operation to your equation by modifying one side of the equation according to the instruction at step ③, making sure to modify the other side of your equation accordingly. Note that the number on one side of the resulting equation is a perfect square. Crucial for our solution method, however, is that *the opposite side of the equation is also a perfect square*! Factor this polynomial to verify that it is indeed a perfect square.

(c) At step ④, al-Khwārizmī performed another arithmetical operation; you should mimic this by performing the same operation on both sides of your last equation.

(d) Note that this last step has reduced the solution of the quadratic equation to that of *two* linear equations. Complete your solution process by executing steps ⑤ and ⑥ as indicated by al-Khwārizmī on one of the linear equations you obtained from part (c). Do the same for the other equation as well.

(e) Al-Khwārizmī only recognized one of the numbers that you found in part (d) as a solution to his problem. Why did he not consider the other one to also be a solution?

(f) For al-Khwārizmī, the original equation included both the "root" (which is the "thing") and its "square"; step ⑦ is needed to report the value of the latter of these. What are the two values of the square of the unknown?

4 The Demonstration: Completing the Square

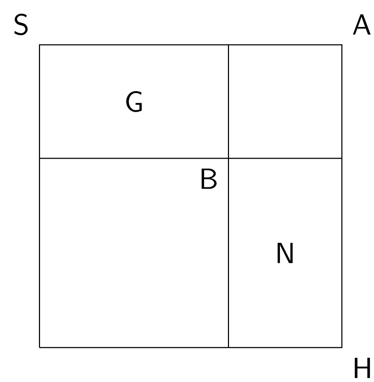
Despite al-Khwārizmī's careful analysis in the selection above, there is still something a bit unsatisfying about his solution method for quadratic equations. In the solutions he presented for "roots and squares are equal to numbers," he began by halving the number of the roots, multiplying this quantity by itself and then adding this quantity to the "simple number" from the equation. This corresponds to steps ② and ③ of the solution he presented above (p. 5). While the computation of this number is a critical step in solving the problem, it's an odd choice of arithmetic to perform at this stage. How did al-Khwārizmī know that adding the square of half the number of roots was precisely what was needed to effect the solution of the problem?

We get a peek into the reason for this by looking at al-Khwārizmī's justification of his solution. As a well-trained mathematician, he knew that a deductive proof, in the manner of the Greek geometers of antiquity, like Euclid (c. 325-265 BCE) and Archimedes (287-212 BCE), would convince his readers that the methods he proposed were sound and unassailable.

Demonstration of the Case "a Square and ten Roots are equal to thirty-nine *Dirhams*": The figure to explain this is a quadrate,[9] the sides of which are unknown. It represents the square, the which, or the root of which, you wish to know...

We proceed from the quadrate AB, which represents the square. It is our next business to add to it the ten roots of the same. We halve for this purpose the ten, so that it becomes five, and construct two rectangles on two sides of the quadrate AB, namely G and N, the length of each of them being five, as the moiety[10] of the ten roots, whilst the breadth of each is equal to a side of the quadrate AB. Then a quadrate remains opposite the corner of the quadrate AB. This is equal to five multiplied by five, this five being half of the number of the roots which we have added to each of the two sides of the first quadrate.

Thus we know that the first quadrate, which is the square, and the two rectangles on its sides, which are the ten roots, make together thirty-nine. In order to complete the great quadrate, there wants only a square of five multiplied by five, or twenty-five. This we add to thirty-nine in order to complete the great square SH. The sum is sixty-four. We extract its root, eight, which is one of the sides of the great quadrate. By subtracting from this the same quantity which we have before added, namely five, we obtain three as the remainder. This is the side of the quadrate AB, which represents the square; it is the root of this square, and the square itself is nine. This is the figure:



In the following project tasks, we will take a closer look at al-Khwārizmī's geometric demonstration and its relation to his numerical algorithm.

[9] "Quadrate" is another old word, used by Frederic Rosen in his nineteenth century translation [al Khwārizmī, 1831] of the *Algebra*, which means "square or rectangular figure." Its use here is handy, for if we were to use the more common word "square" instead, we would likely be confused between whether the word signified the shape—which is what is meant in this instance—or a type of number which is the product of another number by itself—which is what the shape is meant to represent, as al-Khwārizmī goes on to mention. In passing, I note that while the translation here is from Rosen, the diagram shown in this excerpt is taken from a different translation of the *Algebra* [Rāshid, 2009].

[10] This is a lovely old word that simply means "middle," or "half," related to the French word *moitié*, also meaning "half."

Task 5 (a) Draw a copy of the diagram of the square that appears with al-Khwārizmī's Demonstration. Denote by C and D the other two corners of the "quadrate AB," C being the one on segment AS and D being the one on segment AH. Denote by E the unlabeled fourth corner of the "great quadrate SH." According to the explanation given in the text that accompany the problem "a square and ten roots are equal to thirty-nine *Dirhams*," identify next to your diagram the numerical values of the lengths of these line segments:

(i) AC
(ii) AD
(iii) CS
(iv) DH
(v) AS

(b) Also identify the numerical values of the areas of these regions in your diagram:

(i) quadrate AB
(ii) rectangle G
(iii) rectangle N
(iv) quadrate SH
(v) quadrate BE

(c) Which of the elements of the diagram above correspond respectively to the "square," the "roots" and the "simple number" of "*dirhams*" in the original problem? Which of them corresponds to the "thing," which we would represent as x?

Task 6 Consider another problem that al-Khwārizmī solved in the *Algebra*: "a square and five roots of the same are equal to twenty-four *dirhams*." Draw a diagram similar to the one above for this problem. Which segment or region in your diagram corresponds to the "thing" of the solution? Which is the "square"? the "five roots"? the "twenty-four *dirhams*"?

Task 7 (a) What did al-Khwārizmī mean when he said in the opening section of the text above that the "quadrate [AB], the sides of which are unknown, ... represents the square"?

(b) Why is it geometrically necessary to "halve for this purpose" the number of "roots" of the equation?

(c) Why is this method of solving quadratic equations known today as *completing the square*?

(d) Today we call the single equation form

$$ax^2 + bx + c = 0$$

(and any specific equation obtained from this by giving particular values to the three coefficients[11]) a *quadratic equation*. Look up the etymology of the word *quadratic*; why is this an appropriate term to use to describe such equations?

[11] This includes the possibility that either b or c equals 0, which al-Khwārizmī had relegated to the cases "squares equal to roots" and "squares equal to numbers."

Task 8 Draw a diagram similar to the one presented by al-Khwārizmī above to accompany the quadratic equation $x^2 + 14x = 32$. Instead of labeling the diagram with letters for the points or regions, label the elements of the diagram that correspond to the quantities $x, x^2, 14$ and 32 that appear in the equation. Show how this helps to illustrate the process of completing the square which solves the equation.

5 Conclusion

The general form of the species "roots and squares are equal to numbers" which was analyzed above by al-Khwārizmī is representable using modern notation as having the form

$$ax^2 + bx = c.$$

In this representation, the coefficients a, b, c, which normally stand for particular numbers—in al-Khwārizmī's earlier prototypical example, $a = 1, b = 10, c = 39$—are also called *parameters*. The term "parameter" is used to emphasize that a higher level of generality is being applied, at which the quantities a, b, c are considered fixed but unspecified. This is in contrast to the way in which the *unknown* quantity x is being treated. For while x is also unspecified in the problem, it cannot be freely chosen by the solver, as it is subject to the conditions of the equation. Indeed, these conditions are partly established by the choice of the parameters that appear in the equation. The unknown, x, is restricted by and must be derived from these conditions.

It may come as a surprise to the reader that the use of symbols, like letters $a, b, c, \ldots, x, y, z$ for quantities and $+, -, =$, etc., for arithmetical operations and relationships, is nowhere to be found in the writings of "the Father of Algebra." Indeed, symbolism did not enter into algebraic procedures for another 700 years after al-Khwārizmī's publication of his *Algebra*! To modern sensibilities, the use of symbolism has become an integral part of the practice of algebraic problem solving. Instead, al-Khwārizmī was able to communicate precisely the same operations and relationships entirely in words; we call this *rhetorical algebra* to distinguish it from our now more familiar *symbolic algebra*.

In addition to the absence of symbolic notation, al-Khwārizmī's work was influenced by the prevailing understanding in his day of what numbers meant. As we have seen already, he was required to classify all quadratic equations into five types:

- squares equal to roots;
- squares equal to numbers;
- squares and roots equal to numbers;
- squares and numbers equal to roots;
- squares equal to roots and numbers.

Contemporary algebraic practice now makes do with only one type, namely

$$ax^2 + bx + c = 0.$$

This remarkable efficiency is made possible because today's concept of number is such that the equation's coefficients need not be positive numbers only, but may take the value 0 (save that a may not be 0, else the equation reduces to a linear equation) or, indeed, even some negative value.

Despite the changes to mathematics that have taken place since his time, al-Khwārizmī's method of completing the square itself is still the key to *deriving* the famous *quadratic formula*,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{†}$$

as is indicated in Appendix A below. Yet while this formula can be applied to every possible quadratic equation, the context of a problem that gives rise to the equation sometimes restricts the type of number that is acceptable as a solution even today. It is not surprising that al-Khwārizmī, the "Father of Algebra," realized that completing the square would not, in a practical sense, produce a solution for every quadratic equation one might think to formulate. Appendix B of this project explores how al-Khwārizmī's treatment of this issue in the *Algebra* over 1000 years ago relates to a similar restriction on our use of the quadratic formula today.

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Appendix A: The Quadratic Formula

In the task outlined in this Appendix, you will apply a method similar to al-Khwārizmī's formulaic solution of the form "squares plus roots are equal to numbers" to develop the standard quadratic formula for the solution of the general symbolic form of the quadratic equation,

$$ax^2 + bx + c = 0,$$

leaving the parameters unspecified.

Task A Let a, b, and c be the coefficients of the general symbolic quadratic equation

$$ax^2 + bx + c = 0,$$

(a) Begin by bringing the equation to the form of the species "squares and roots are equal to numbers." Reduce this further by dividing the equation through by the coefficient of x^2.

(b) Next, halve the number of roots in the new equation, multiply the result of this by itself, and add this square to both sides of the last equation. This step ensures that there is a trinomial expression on one side of the equation which factors as a perfect square; factor the trinomial as a perfect square. On the other side of the equation is some number.

(c) Now take square roots on both sides of the equation. This should produce two new equations, both linear, depending only on which of the two square roots of the number is selected for the equation. In each equation, solve for x. You should obtain something equivalent to the following:

$$x = \pm\sqrt{\left(\frac{b^2}{4a^2} - \frac{c}{a}\right)} - \frac{b}{2a}. \qquad (\ddagger)$$

(d) Finally, "rationalize" the expression by expressing all fractions with a common denominator. Show that the result produces the famous *quadratic formula*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad (\dagger)$$

6 Appendix B: Complex Roots

Al-Khwārizmī believed that the only valid roots one could find to solve a quadratic equation were positive numbers, because for him, all problems derived from real problems with measurable quantities as the "thing" to solve for. Still, he was perceptive enough to identify the problematic situation in which this might not be possible. Here is a comment he made in the midst of his treatment of the case "squares and numbers are equal to roots":

> And know that, when in a question belonging to this case you halved the number of the roots and multiplied the moiety by itself, if the product be less than the number of *dirhams* connected with the square, then the instance is impossible; but if the product be equal to the *dirhams* by themselves, then the root of the square is equal to the moiety of the roots alone, without either addition or subtraction.

Task B

(i) As we have seen, the quadratic formula (†) claims to produce the solutions to any quadratic equation $ax^2 + bx + c = 0$. Considering how al-Khwārizmī formulated his problem case "squares and numbers are equal to roots," why would it be, as he stated above, that if "the product [of half the number of roots by itself] be less than the number of *dirhams* connected with the square," then the solution of the problem would be "impossible"? You might try coming up with values of a, b and c that satisfy this condition and then use the quadratic formula to find a solution.

(ii) What did al-Khwārizmī mean by the statement made at the end of this excerpt, that "if the product be equal to the dirhams by themselves, then the root of the square is equal to the moiety of the roots alone, without either addition or subtraction"?

Notes to Instructors

Primary Source Project Content: Topics and Goals

This mini-Primary Source Project (mini-PSP) is designed to serve students who are learning algebraic methods for solving quadratic equations, a topic found in most high school and college algebra courses in the United States and in many precalculus courses as well. It will also be of value to future high school mathematics teachers who will be responsible for teaching algebra in their own classrooms, as well as instructors of higher algebra courses who are interested in conveying a sense of the early history of the theory of equations. Additionally, it is suitable for use in a general history of mathematics course as an introduction to the role of early Islamic-era mathematics in the development of algebra as a major branch of mathematics.

The main goal of this project is a simple one: to give the student a deep understanding of the method of completing the square, the universal procedure for solving quadratic equations. In a pair of appendices, students may further explore these ideas through (1) a derivation of the quadratic formula, and/or (2) consideration of when a quadratic equation produces complex roots.

Student Prerequisites

Nearly nothing in the way of special prerequisites is required of students for this PSP beyond what a typical high school student knows about solving algebraic equations. Indeed, the project may teach the student no mathematical techniques beyond what they come to the project already possessing. Its purpose is to impart conceptual rather than procedural understanding.

PSP Design, and Task Commentary

1 Introduction: The project opens with a brief section that informs the student about the goals of the PSP.

2 The Roots of Algebra: We introduce Muḥammad ibn Mūsā al-Khwārizmī (ca. 780-850 CE), generally regarded as the "Father of Algebra," the author of an immensely influential work, *al-Kitāb al-mukhtaṣar fī hisāb al-jabr wal-muqābala* (*The Compendious Book on Calculation by Restoration and Reduction*), written around the year 825, probably in Baghdad. This influence extended not just within Arab-Islamic culture, but was very important in preparing European scholars to make advances in algebra and problem solving between the twelfth and sixteenth centuries. It is from this work that the project draws all its source texts, using two English translations, one from the nineteenth century [al Khwārizmī, 1831] by the orientalist Frederic Rosen (including some memorable and charming old words, like "moiety" and "quadrate"), and another produced by the eminent Egyptian historian of science, Roshdi Rashed (or Rushdī Rāshid) [Rāshid, 2009].

We turn to the beginning of the *Algebra*, in which al-Khwārizmī classified the kinds of problems that algebra was then capable of resolving, namely quadratic equations. He did this by defining "three classes" of number ("squares," "roots" and "simple numbers"), what today we would call monomial terms in a quadratic polynomial expression. Students come to see that while the terminology is foreign, al-Khwārizmī's mathematics is familiar. In Task 1, they translate the various classes of equations he proposes into the symbolic form that characterizes present-day algebra.

3 Solving a Quadratic Equation: Task 4 invites students to use the circled numbers embedded in the source text as markers to help them parse al-Khwārizmī's text for meaning. Students work through one of his quadratic equation solutions, handled through the process we now call "competing the square."

4 The Demonstration: Completing the Square: Students read through al-Khwārizmī's geometric demonstration of his method which—literally!—completed a square, illustrating how the corresponding algebraic procedure works.

5 Conclusion: In this final section, the student's attention is called to the (by now rather obvious) fact that al-Khwārizmī's text includes not a single x, nor even an equal sign! A discussion of the power of the systematic and analytical process that al-Khwārizmī, the Father of Algebra, used to solve quadratic equations leads to the formulation of the quadratic formula and ends the project.

Appendix A: The Quadratic Formula: In this optional sequel, al-Khwārizmī's method of completing the square is used to derive the quadratic formula. But no source text is employed, as the procedure is handled using modern symbolism.

Appendix B: Complex Roots: The quadratic formula can be used to produce complex roots for certain quadratic equations. Given al-Khwārizmī's understanding of number, one not wide enough to accept the possibility that complex roots were valid, he would identify such equations as "impossible." But he was clever enough to understand exactly when such equations might arise. This appendix asks students to make sense of al-Khwārizmī's conditions for determining when an equation was "impossible."

Suggestions for Classroom Implementation

Instructors who plan to use this project in the classroom are urged are to carefully work the student tasks within this project before assigning them to students! Many of the tasks are not routine exercises and may pose challenges, even to those adept in algebra.

The project is meant to be completed in two 50-minute classroom periods, plus time in advance for students to do some initial reading and time afterwards for them to write up their solutions to the Tasks. Another period may be needed to include one or both of the Appendices in classroom discussions as well.

The actual number of class periods spent on each section naturally depends on the instructor's goals and on how much of the PSP is implemented with students. Time estimates closer to a full week of classes assume that most PSP work is completed by students working in small groups in the classroom, with less work left for outside the classroom.

Sample Implementation Schedule (based on a 50-minute class period)

Instructors implementing this project should ensure that their students concentrate on working Tasks 1, 4 and 5, where the bulk of the work lies. Students should prepare for class by reading the beginning of the project through Task 1, writing up Task 1 and bringing their work to the classroom so that discussion can start there when class begins. Task 2 may be handled in a brief discussion with the entire classroom before the students partake in a common reading of Section 3, punctuated by brief and quiet individual work on Tasks 3 and 4 when they are encountered, followed by small group work to discuss the results they obtain. Students may need to complete a write-up of Task 4 for homework.

On day 2, the focus is on Task 5 and the linking of the procedure of completing the square with the geometric version that constitutes al-Khwārizmī's demonstration. To prepare for that work, students could be asked to read the source excerpt on page 7 in advance of the class meeting. The first two parts of Task 7 may also be done in class, leaving Tasks 6, 7(c) and (d), and 8 for homework.

Instructors who teach in a 75-minute period may be able to have students work through one of the two Appendices at the end of the second period. Those who teach 50-minute periods would need a third meeting to do this; alternatively, the tasks in this section could be assigned as homework, especially with more advanced students. Note that the second Appendix assumes knowledge of the quadratic formula, whose derivation is the subject of the first Appendix. Still, it is not necessary to complete the first Appendix to take on the content of the second.

Possible Modifications of the Project

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Connections to other Primary Source Projects

This project allows certain connections with other Primary Source Projects, all of which are freely available on the website of the Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS) Project:

- This project is a reduced version of an earlier PSP by this author, *Solving Equations and Completing the Square: From the Roots of Algebra*. The longer version is designed for implementation in the classroom over a full week; it includes more of the source text from al-Khwārizmī's *Algebra* and a deeper presentation of the content above, with further attention to a comparison of al-Khwārizmī's rhetorical algebra with modern symbolic methods.

- The PSP titled *The Pythagorean Theorem and the Exigency of the Parallel Postulate*, by Jerry Lodder, presents a full demonstration of the Pythagorean Theorem as given by Euclid in Book I of his *Elements*, paying special attention to the dependence of the argumentation on the famous Parallel Postulate. Students who want to investigate the nature of geometrical demonstrations, of a similar type to that presented by al-Khwārizmī at the end of the current project, may be interested in investigating this foray into plane geometry.

- Students who are interested in other treatments of algebra that make use of no symbolism may find this PSP an enjoyable study: *Solving Systems of Linear Equations Using Ancient Chinese Methods*, by Mary Flagg. This project is also ideal for students interested in studying other non-Western mathematical works.

Recommendations for Further Reading

The history of the mathematics of the medieval Islamic world is fascinating, especially with regard to how it exploded into activity at roughly the time of al-Khwārizmī through the translation of scientific works from the surrounding civilizations, and how it influenced later developments of mathematics, especially at the time of the European Renaissance. Chapter 9 of [Lindberg, 1992] and Chapter 7 of [Katz and Parshall, 2014] tell this story rather well. The latter book is worth studying in its entirety to learn the full scope of the history of the development of algebra from ancient to modern times.

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For more information about TRIUMPHS, visit https://blogs.ursinus.edu/triumphs/.