Investigations Into d'Alembert's Definition of Limit*

(Calculus 2 Version)

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1 Introduction

The modern definition of a limit evolved over many decades. One of the earliest attempts at a precise definition is credited to Jean-Baptiste le Rond d'Alembert (1717–1783), a French mathematician, philosopher and physicist.[1] Among his many accomplishments, d'Alembert was a co-editor of the *Encyclopédie*, an important general encyclopedia published in France between 1751 and 1772. This work is regarded as a significant achievement of the Enlightenment movement in Europe.

D'Alembert argued in two 1754 articles of the *Encyclopédie* that the theory of limits should be put on a firm foundation.[2] As a philosopher, he was disturbed by critics who pointed out logical problems with limits and the foundations of calculus. D'Alembert recognized the significant challenges posed of these criticisms; in [d'Alembert, 1754b], for example, he wrote that[3]

> This metaphysics [of calculus], about which so much has been written, is even more important, and perhaps more difficult to develop than the rules of calculus themselves.

In this project we will investigate d'Alembert's limit definition and study the similarities and differences with our modern definition.

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[1]Early chapters of d'Alembert's biography read like something out of *Masterpiece Theater*. He was born out of wedlock and left as an infant at the church Saint Jean le Rond in Paris. His mother, Claudine Guérin de Tencin, was a runaway nun who established a well-known Paris *salon*, a carefully orchestrated social gathering that brought together important writers, philosophers, scientists, artists and aristocrats for the purpose of intellectual and political discussions. Tencin never acknowledged d'Alembert as her son, and his father, Louis-Camus Destouches, found another woman to raise young Jean. Destouches died in 1726, but left funds for Jean's education. D'Alembert did well in school and became active as an adult in the philosophy, literature, science and mathematics of his day, standing "at the very heart of the Enlightenment with interests and activities that touched on every one of its aspects" [Hankins, 1990].

[2]The first of these articles was entitled "Limite (Mathématiques)," and the second "Calcul différentiel."

[3]All translations of d'Alembert excerpts in this project were prepared by Janet Heine Barnett, Colorado State University Pueblo, 2022.

2 D'Alembert's Limit Definition

By 1754 mathematical techniques using calculus were quite advanced. D'Alembert won a 1747 prize for his work in partial differential equations, but became embroiled in arguments with Leonhard Euler (1707–1783) and others over methodology and foundational issues. These squabbles contributed to his interest in clearing up the foundations of limits and convergence.

Here is d'Alembert's limit definition from the *Encyclopédie* [d'Alembert, 1754a]:

> Limit (Mathematics). We say that a magnitude is the *limit* of another magnitude, when the second can approach the first more closely than any given magnitude, as small as we wish to assume, yet without the approaching magnitude ever being allowed to surpass the magnitude to which it approaches; so that the difference of such a quantity and its *limit* is absolutely unassignable.
>
> For example, suppose we have two polygons, one inscribed and the other circumscribed about a circle, it is clear that we can increase the [number of] sides as much as we wish; and in that case, each polygon will always come closer and closer to the circumference of the circle, the contour of the inscribed polygon increasing, and that of the circumscribed [polygon] decreasing; but the perimeter or the contour of the first will never surpass the length of the circumference, and that of the second will never be smaller than that same circumference; the circumference of the circle is thus the *limit* of the increase of the first polygon, and of the decrease of the second.

Let's examine some examples.

Task 1 Draw a diagram for a circle of radius 1 and an inscribed regular polygon with $n = 8$ sides. Use some basic trigonometry to find the exact length of the polygon's perimeter. How close is it to the circle's circumference?

Task 2 Consider d'Alembert's "inscribed polygon → circle" limit example and his definition.

Assume for simplicity that the inscribed polygons are regular with n sides centered at the circle's center. These polygons have perimeter formula

$$perimeter = 2n \cdot radius \cdot \sin(\pi/n)$$

(a) For "given magnitude" 0.1 and a circle of radius 1, how many sides are needed for the regular inscribed polygon to guarantee that "the second can approach the first more closely than" the given magnitude 0.1? Technology will be helpful!

(b) How many sides are needed for a circle of radius 1 and 'given quantity' 0.01?

(c) *(Optional)* As a bonus, derive the given perimeter formula.

Note that d'Alembert's definition is lacking in precise, modern mathematical notation. Also observe that the polygon/circle example is for the limit of a *sequence*. Here is a standard first-year calculus book definition of limit for a sequence:

First-Year Calculus Definition. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

Rewriting the example in Task 2 in modern limit notation, we thus have

$$\lim_{n\to\infty} 2n \cdot r \cdot \sin(\pi/n) = 2\pi \cdot r,$$

where r represents the radius of the circle. Today's notation for sequences also uses modern subscript notation. For instance, setting $p_n = 2n \cdot r \cdot \sin(\pi/n)$ gives us the sequence $\{p_n\}$. Since $\lim\limits_{n\to\infty} p_n = 2\pi r$, we can also write $p_n \to 2\pi r$ as $n \to \infty$.

Task 3 Use calculus to verify that $p_n \to 2\pi r$ as $n \to \infty$, where $p_n = 2n \cdot r \cdot \sin(\pi/n)$.

Task 4 Consider the sequence $\{a_n\}$ with $a_n = \dfrac{n}{2n+1}$.

(a) Find the limit of this sequence by any means.

(b) For "given magnitude" 0.01, suppose we want a_n to be "as close to its limit" as .01. What is "sufficiently large" for n to guarantee that a_n and its limit differ by 0.01 or less?

(c) Repeat part (b) for "given magnitude" 0.001.

Later in his *Encyclopédie* article on limits [d'Alembert, 1754a], d'Alembert wrote the following:

Strictly speaking, the *limit* never coincides, or becomes equal to the quantity for which it is the *limit*; but that quantity always approaches the *limit* closer and closer, and may differ from it by as little as we wish. The circle, for example, is the *limit* of the inscribed and circumscribed polygons; for it never strictly coincides with them, although they may become infinitely close.

Task 5 Look closely at d'Alembert's phrase "strictly speaking, the *limit* never coincides, or becomes equal to the quantity for which it is the *limit*" and notice that it does not appear in the First-Year Calculus definition. Find a simple convergent sequence that violates this requirement of d'Alembert's limit definition.

Task 6 Look back at d'Alembert's phrase "without the approaching magnitude ever being allowed to surpass the magnitude to which it approaches" in the first excerpt (above Task 1) and notice that it does not appear in the First-Year Calculus definition. Find a simple convergent sequence that violates this requirement of d'Alembert's limit definition.

3 A More Precise Definition of Limit

As we have seen, d'Alembert's 1754 limit definition doesn't fully apply to some types of sequences studied by today's mathematicians. It is interesting to note that during d'Alembert's era there was some debate regarding whether or not a quantity could ever reach or surpass its limit. Based on your work with d'Alembert's definition of limit, what do you think was d'Alembert's opinion on these questions?

During the 1800s mathematicians reached a consensus that limits could be attained, and a convergent sequence could indeed oscillate about its limit. We see the First-Year Calculus definition allows for these possibilities; however, it is too vague for actually constructing complex proofs. We can remedy this problem by clarifying the logic and converting some verbal descriptions into algebraic inequalities.

Task 7 Use inequalities and the quantifier expressions "for all" and "there exists" to help rewrite d'Alembert's limit definition for sequences in a less verbal form. The First-Year Calculus Definition and a graph of the sequence $\{a_n\}$ should be helpful in getting started. You should introduce a variable ϵ to represent the allowable difference or tolerance between a sequence term a_n and the limit itself, and another variable M to measure n being "sufficiently large." Be sure to include d'Alembert's requirements that sequence terms can neither surpass nor coincide with the limit in your answer.

Task 8 Now use inequalities and the quantifier expressions "for all" and "there exists" to rewrite the First-Year Calculus limit definition for sequences, without the extra requirements that d'Alembert imposed in his definition. Then comment on the differences between this definition and your definition from Task 7.

Task 9 Use your definition from Task 8 to prove that sequence $\left\{\dfrac{n}{2n+1}\right\}$ converges.

Task 10 Suppose that a sequence $\{c_n\}$ converges to limit 1. Use your definition from Task 8 to prove that there exists a natural number M for which $0.9 < c_n < 1.1$ whenever $n \geq M$.

4 Conclusion

Historians have noted that definitions of limit were given verbally by mathematicians of the 1600s and 1700s. However, to make these ideas useful in rigorous proofs, it is important to translate the verbal limit definition into one with clear logic and algebraic language, as you accomplished in Task 8. The mathematician Augustin-Louis Cauchy (1789–1867) is usually credited with being the first to do this, using ϵ and precise inequalities in some of his proofs. Even so, his definition of limit was verbal and similar to d'Alembert's, except that for Cauchy limits could be attained and surpassed, as in the modern definition. The modern limit definition we see today finally matured in the work of Karl Weierstrass (1789–1867) and his students.

How influential was d'Alembert's limit definition? This is hard to say, since d'Alembert only used his definition to carry out one proof. Certainly his advocacy for a precise limit definition may have influenced mathematicians such as Cauchy, and can thus be considered a worthy contribution to the evolution of the rigorous limit definition we use today.

References

Jean le Rond d'Alembert. Limite (Mathématiques) [Limit (Mathematics)]. In *Encyclopédie ou Dictionnaire raisonné des sciences, des arts et des métiers*, volume 9, page 542. Chez Braisson, David, Le Breton and Durand, Paris, 1754a.

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Judith Grabiner. The Calculus as Algebra. In *A Historian Looks Back*, pages 1–124. Mathematical Assocation of America, Washington DC, 2010.

Thomas L. Hankins. *Jean d'Alembert: Science and the Enlightenment*. Gordon and Breach, New York, 1990.

Notes to Instructors

PSP Content: Topics and Goals

This mini-Primary Source Project (mini-PSP) is designed to investigate the definition of limit for sequences, beginning with d'Alembert's definition and a modern Introductory Calculus text definition. Similarities and differences are explored.

Two versions of this project are available, for very different audiences.

- One version is aimed at Calculus 2 students studying sequences for the first time. **This is the version you are currently reading.** D'Alembert's definition is completely verbal, and Section 2 tasks lead students through some examples based on that definition. Other tasks in that section ask students to find examples illustrating the difference between the modern conception of limit and that of d'Alembert. Section 3 examines these differences in a more technical fashion by having students write definitions for each using inequalities and quantifiers; this section is more appropriate for use in honors courses or as extra credit, and could be omitted by instructors who wished to pursue a more informal approach to sequences. Some historical remarks are given in a concluding section.
- A longer version is aimed at Real Analysis students. It includes several tasks based on D'Alembert's verbal definition that are more technical than those that appear in the Calculus 2 version, as well as an additional section that investigates two limit properties stated by d'Alembert (in an excerpt that is not included in the Calculus 2 version). That additional section includes tasks that prompt students to write modern proofs of those properties.

The specific content goals of this version of the project are as follows.

1. Develop familiarity with sequence convergence through examples based on d'Alembert's verbal definition.
2. Analyze subtleties of the limit definition: whether sequence terms can 'surpass' or coincide with the limit.
3. Develop a modern limit definition with quantifiers for sequences based on d'Alembert's definition and an Introductory Calculus text definition.

Student Prerequisites

This version of the project is written for a course in Calculus 2 with the assumption that students have limited familiarity with either sequences or with quantifiers.

PSP Design, and Task Commentary

This mini-PSP is designed to take up to two days of classroom time where students work through tasks in small groups. Some reading and tasks are done before and after class. If time does not permit a full implementation with this methodology, instructors can use more class time for guided discussion and less group work for difficult parts of the project.

The PSP is designed to be used largely in place of a textbook section introducing the definition of limit for sequences. The differences between the d'Alembert and modern definition can help students realize subtleties and the precision of the modern definition.

Task 7 in Section 3 may be difficult for students, even those enrolled in honors courses or those completing it as extra credit. Encouraging students to draw a plot and labels for ϵ and M should help. Leading questions to help them realize that the definition needs to start with "for all $\epsilon > 0$" may also be helpful. Including d'Alembert's requirements that sequence terms can't "surpass" or coincide with the limit is challenging but pedagogically useful.

Suggestions for Classroom Implementation

Advanced reading of the project and some task work before each class is ideal but not necessary. See the sample schedule below for ideas.

LaTeX code of this entire mini-PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The mini-PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Sample Implementation Schedule (based on a 50-minute class period)

This PSP is designed to take 1–2 class days.

Students read through the first d'Alembert excerpt and do preparatory work on Task 1 before the first class. After a class discussion of this task, students work through Tasks 2–6 in groups. (Although Task 3 could instead be assigned as an individual homework task.) As needed, the remainder of these tasks could be assigned as homework for Day 2.

For instructors who choose to complete the optional Section 3, students spend the majority of time during the second class day in group work on Task 7; this task is critical for the remainder of the section, so a class discussion is advisable to make sure everyone understands it before continuing. Tasks 8–10 could be assigned for homework.

Connections to other Primary Source Projects

The following additional projects based on primary sources are also freely available for use in teaching standard topics in the calculus sequence. The PSP author name of each is given (together with the general content focus, if this is not explicitly given in the project title). Each of these projects can be completed in 1–2 class days, with the exception of the four projects followed by an asterisk (*) which require 3, 4, 3, and 6 days respectively for full implementation. Classroom-ready versions of these projects can be downloaded from `https://digitalcommons.ursinus.edu/triumphs_calculus`.

- *L'Hôpital's Rule*, by Daniel E. Otero
- *The Derivatives of the Sine and Cosine Functions*, by Dominic Klyve
- *Three Hundred Years of Helping Others: Maria Gaetana Agnesi on the Product Rule*, by Kenneth M Monks
- *Fermat's Method for Finding Maxima and Minima*, by Kenneth M Monks
- *Gaussian Guesswork: Elliptic Integrals and Integration by Substitution*, by Janet Heine Barnett
- *Gaussian Guesswork: Polar Coordinates, Arc Length and the Lemniscate Curve*, by Janet Heine Barnett

- *Gaussian Guesswork: Infinite Sequences and the Arithmetic-Geometric Mean*, by Janet Heine Barnett
- *Beyond Riemann Sums: Fermat's Method of Integration*, by Dominic Klyve (uses geometric series)
- *How to Calculate π: Machin's Inverse Tangents*, by Dominic Klyve (infinite series)
- *Euler's Calculation of the Sum of the Reciprocals of Squares*, by Kenneth M Monks (infinite series)
- *Fourier's Proof of the Irrationality of e*, by Kenneth M Monks (infinite series)
- *Jakob Bernoulli Finds Exact Sums of Infinite Series (Calculus Version)*,* by Daniel E. Otero and James A. Sellars
- *Bhāskara's Approximation to and Mādhava's Series for Sine*, by Kenneth M Monks (approximation, power series)
- *Braess' Paradox in City Planning: An Application of Multivariable Optimization*, Kenneth M Monks
- *Stained Glass, Windmills and the Edge of the Universe: An Exploration of Green's Theorem*,* by Abe Edwards
- *The Fermat-Torricelli Point and Cauchy's Method of Gradient Descent*,* by Kenneth M Monks (partial derivatives, multivariable optimization, gradients of surfaces)
- *The Radius of Curvature According to Christiaan Huygens*,* by Jerry Lodder

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For more information about TRIUMPHS, visit https://blogs.ursinus.edu/triumphs/.