

# Euler's Rediscovery of $e$

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## 1 Introduction

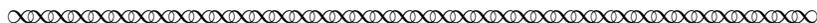
The famous constant  $e$  is used in countless applications across many fields of mathematics, and resurfaces periodically in the evolution of mathematics. In 1683, Jacob Bernoulli essentially found  $e$  while studying compound interest and evaluating the sequence  $(1 + 1/j)^j$  as  $j \rightarrow \infty$ . By 1697, his brother Johann Bernoulli was working with the calculus of exponentials [Bernoulli, 1697]. However, a full understanding was missing. The connection between logarithms and exponential functions was still not well understood, and mathematicians couldn't agree on how to define logarithms of negative numbers. Leonhard Euler would later clear up the confusion on logarithms of negative numbers, and clarify the idea of a logarithmic *function* [Euler, 1749]. In 1748, Euler published one of his most influential works, *Introductio in Analysin Infinitorum* [Euler, 1748]. This was translated into English by John Blanton as *Introduction to Analysis of the Infinite* [Blanton, 1988] and we shall quote his translation with a few minor changes. In Chapter VI, Euler discussed logarithms for various bases and their properties. Logarithms were well known in Euler's day, and tables of common logarithms (base 10) had been compiled, as no scientific calculators were available in 1748. Euler examined exponential and logarithmic functions in Chapter VII, especially as infinite series. We are particularly interested in how  $e$  appears naturally in his development of these functions.

## 2 Euler's Definition of $e$

Part of Euler's challenge in working with logarithmic functions was to find a logarithmic base  $a$  for which infinite series expansions are convenient. It is here that Euler derived  $e$ , both as the limiting value of  $(1 + 1/j)^j$  and as the infinite series  $1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$ . As was common in his day, Euler worked with *infinitely small and large numbers*, a practice that has largely been abandoned with the modern definition of limit. Nevertheless, Euler used his infinitely small and large numbers with great skill, as we shall see.

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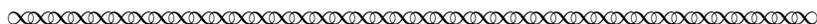


Section 114. Since  $a^0 = 1$ , when the exponent on  $a$  increases, the power itself increases, provided  $a$  is greater than 1. It follows that if the exponent is infinitely small and positive, then the power also exceeds 1 by an infinitely small number. Let  $\omega$  be an infinitely small number, ...,  $a^\omega = 1 + \psi$  where  $\psi$  is also an infinitely small number. ... we let  $\psi = k\omega$ . Then we have  $a^\omega = 1 + k\omega$ , and with  $a$  as the base for logarithms, we have  $\omega = \log(1 + k\omega)$ .

#### EXAMPLE

In order that it may be clearer how the number  $k$  depends on  $a$ , let  $a = 10$ . From the table of common logarithms,<sup>1</sup> we look for the logarithm of a number which exceeds 1 by the smallest possible amount, for instance,  $1 + \frac{1}{1000000}$ , so that  $k\omega = \frac{1}{1000000}$ .

Then  $\log\left(1 + \frac{1}{1000000}\right) = \log\frac{1000001}{1000000} = 0.00000043429 = \omega$ . Since  $k\omega = 0.00000100000$ , it follows that  $\frac{1}{k} = \frac{43429}{100000}$  and  $k = \frac{100000}{43429} = 2.30258$ . We see that  $k$  is a finite number which depends on the value of the base  $a$ . If a different base had been chosen, then the logarithm of the same number  $1 + k\omega$  will differ from the logarithm already given. It follows that a different value of  $k$  will result.



#### Task 1

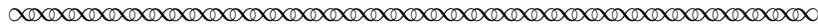
To gain some visual insight into what Euler was doing, plot  $y = a^\omega$  and  $y = 1 + k\omega$  for  $a = 10$  and  $k = \frac{100000}{43429} \approx 2.30258$ . Euler claimed these quantities  $a^\omega$  and  $1 + k\omega$  should be identical for “infinitely small”  $\omega$ . Would changing the  $k$  value to something else, say  $-3$ , change anything about your plot and this claim?

#### Task 2

Use Euler’s ideas and a scientific calculator to estimate  $k$  for  $a = 2$ . Get a visual check by plotting  $y = 2^\omega$  and  $y = 1 + k\omega$  together.

Euler was interested in finding an  $a$  value for which exponential and logarithmic expansions are nice and easy to work with. He derived a series expansion in his Section 115.

<sup>1</sup>These are base 10 logarithms. As recently as the 1970s, most students used tables rather than calculators to find logarithms. Such a table would have an entry that exceeds 1 by the “smallest possible amount” for that table.

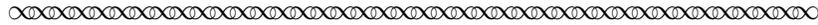


Section 115. Since  $a^\omega = 1 + \psi$ , we have  $a^{j\omega} = (1 + \psi)^j$ , whatever value we assign to  $j$ . It follows that

$$a^{j\omega} = 1 + \frac{j}{1}k\omega + \frac{j(j-1)}{1 \cdot 2}k^2\omega^2 + \frac{j(j-1)(j-2)}{1 \cdot 2 \cdot 3}k^3\omega^3 + \dots \quad (1)$$

If now we let  $j = \frac{z}{\omega}$ , where  $z$  denotes any finite number, since  $\omega$  is infinitely small, then  $j$  is infinitely large. Then we have  $\omega = \frac{z}{j}$ , where  $\omega$  is represented by a fraction with an infinite denominator, so that  $\omega$  is infinitely small, as it should be. When we substitute  $\frac{z}{j}$  for  $\omega$  then

$$a^z = (1 + kz/j)^j = 1 + \frac{1}{1}kz + \frac{1(j-1)}{1 \cdot 2 \cdot j}k^2z^2 + \frac{1(j-1)(j-2)}{1 \cdot 2j \cdot 3j}k^3z^3 + \frac{1(j-1)(j-2)(j-3)}{1 \cdot 2j \cdot 3j \cdot 4j}k^4z^4 + \dots \quad (2)$$



We would like to capture the spirit of Euler's ideas but put his work on modern foundations by avoiding infinitely small and large numbers.

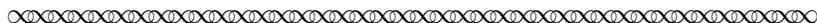
**Task 3** Assume  $a > 1$  and  $\omega$  is a small, positive finite number defined by  $a^\omega = 1 + \psi$  and  $k = \psi/\omega$ .

- (a) What theorem was Euler using to obtain (1)? For what  $\psi$  values is this series known to converge?
- (b) Verify the algebraic details needed to obtain (1) from this theorem.

**Task 4** Assume  $a > 1$  and  $\omega$  is a small, positive finite number defined by  $a^\omega = 1 + \psi$  and  $k = \psi/\omega$  and  $j = z/\omega$ .

- (a) What is the general  $n^{th}$  term in the series (2)?
- (b) Verify the algebraic details needed to obtain (2) from  $j = z/\omega$  and (1).

Euler next used his infinitely large numbers to produce an infinite series expression for his ideal logarithm base  $a$ . At this point in his book, Euler set  $z = 1$  to find his special value for  $a$ .



Section 116. Since  $j$  is infinitely large,  $\frac{j-1}{j} = 1, \dots, \frac{j-2}{j} = 1, \frac{j-3}{j} = 1$ , and so forth.

It follows that  $\frac{j-1}{2j} = \frac{1}{2}, \frac{j-2}{3j} = \frac{1}{3}, \frac{j-3}{4j} = \frac{1}{4}$ , and so forth. When we substitute

these values [into equation (2)], we obtain  $1 + \frac{kz}{1} + \frac{k^2z^2}{1 \cdot 2} + \frac{k^3z^3}{1 \cdot 2 \cdot 3} + \frac{k^4z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$ .

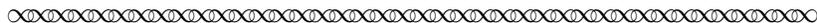
This equation expresses a relationship between the numbers  $a$  and  $k$ , since when we let  $z = 1$ , we have

$$a = 1 + \frac{k}{1} + \frac{k^2}{1 \cdot 2} + \frac{k^3}{1 \cdot 2 \cdot 3} + \dots \tag{3}$$

Section 122. Since we are free to choose the base  $a$  for the system of logarithms, we now choose  $a$  in such a way that  $k = 1$ . Then the series found above in Section 116,

$$1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \dots \tag{4}$$

is equal to  $a$ . If the terms are represented as decimal fractions and summed, we obtain the value  $a = 2.71828182845904523536028\dots$ . When this base is chosen, the logarithms are called natural or hyperbolic. The latter name is used since the quadrature of a hyperbola<sup>2</sup> can be expressed through these logarithms. For the sake of brevity for this number 2.718281828459... we will use the symbol  $e$ , which will denote the base for the natural or hyperbolic logarithms.



**Task 5** Why do you think Euler chose  $a$  “in such a way that  $k = 1$ ” in his series (3)?

We can obtain an expression for Euler’s special  $a$  value as the limit of a *sequence*, and then use modern methods with Euler’s ideas to prove this sequence converges. To justify Euler’s work from a modern point of view, let’s look at the key equation (2)

$$(1 + kz/j)^j = 1 + \frac{1}{1}kz + \frac{1(j-1)}{1 \cdot 2 \cdot j}k^2z^2 + \frac{1(j-1)(j-2)}{1 \cdot 2j \cdot 3j}k^3z^3 + \dots$$

and set  $k = 1, z = 1$  as Euler did, but suppose  $j$  is a *natural number*.

<sup>2</sup>The area of a region between the  $x$ -axis and hyperbola  $y = 1/x$

**Task 6** Apply the finite Binomial Theorem for natural number  $j$  to expand  $(1 + 1/j)^j$  as a finite series and show that

$$(1 + 1/j)^j = 1 + \frac{1}{1} + \frac{1(j-1)}{1 \cdot 2 \cdot j} + \frac{1(j-1)(j-2)}{1 \cdot 2j \cdot 3j} + \dots + \frac{1(j-1)(j-2) \cdots (j-(j-1))}{1 \cdot 2j \cdot 3j \cdots (jj)}.$$

Now we form a sequence  $(c_j)_{j=1}^{\infty}$  with  $c_j = (1 + 1/j)^j$ . If we can justify taking the limit of this sequence, we should obtain Euler's number  $e$ .

**Task 7** Show that

$$c_j = 1 + \frac{1}{1} + \frac{(1-1/j)}{1 \cdot 2} + \frac{(1-1/j)(1-2/j)}{1 \cdot 2 \cdot 3} + \dots + \frac{(1-1/j)(1-2/j) \cdots (1-(j-1)/j)}{1 \cdot 2 \cdot 3 \cdots j}.$$

**Task 8** Write out  $c_{j+1}$  and compare it term-by-term with  $c_j$ . What can you conclude about the sequence  $(c_j)_{j=1}^{\infty}$ ?

**Task 9** Compare  $c_j$  to a geometric series to show the sequence  $(c_j)$  is bounded.

Hints: Use the series form for  $c_j$  you found in Task 7, and compare it term-by-term with a geometric series.

**Task 10** Apply the Monotone Convergence Theorem to give a modern proof that  $\lim_{j \rightarrow \infty} (1 + 1/j)^j$  exists.

We now have a modern justification of the constant  $e$  as  $e = \lim_{j \rightarrow \infty} (1 + 1/j)^j$ . You may recall from introductory calculus courses that Euler was correct with the series expansion (4) for  $e$ . A modern justification of this series expansion for  $e$  is beyond the scope of this project.

**Task 11** This task should give you some appreciation for Euler's series (4) when he wanted to find a good decimal approximation for  $e$ . Remember that he had no computers at his disposal!

- Use technology to find both  $c_4 = (1 + 1/4)^4$  and the partial sum  $P_4 = \sum_{n=0}^4 \frac{1}{n!}$ .
- How close are  $c_4$  and  $P_4$  to Euler's Section 122 decimal approximation to  $e$ , and which is more accurate?
- Use technology to find a value of  $j$  for which  $c_j$  is closer to  $e$  than  $P_4$ .

**Task 12** Euler wanted to use his work to express the function  $e^z$  as an infinite series. Use Euler's (2) and his Section 116 infinitesimal methods to find an infinite series expression for  $e^z$ . How does this series compare with the Taylor series for  $e^z$  you learned about in introductory calculus?

Recall (from page 2) that Euler was investigating how the number  $k$  depends on the logarithm base  $a$ , where  $\omega = \log_a(1 + k\omega)$  with “infinitely small”  $\omega$ . We can re-interpret this equation without reference to infinitesimals in terms of the following limit:

$$\lim_{\omega \rightarrow 0} \frac{\omega}{\log_a(1 + k\omega)}. \quad (5)$$

The final task of this project uses this limit to reinforce the connection between  $a$  and  $k$ .

**Task 13** As you may recall, the derivative of  $\log_a x$  is  $\frac{1}{x \ln a}$ . You will need this formula for part (a).

- (a) Use introductory calculus techniques to find the limit (5) in terms of  $a$  and  $k$ .
- (b) Since  $\omega = \log_a(1 + k\omega)$  with “infinitely small”  $\omega$ , explain why this limit (5) should be 1 for any pair  $a, k$  where  $k$  is chosen to depend properly on  $a$ .
- (c) In particular, when  $a = e$ , what value of  $k$  is required for limit (5) to be 1?
- (d) Use limit (5) to find the **exact** values of  $k$  when  $a = 10$  and  $a = 2$ . Use your answers to reflect on the decimal values for  $k$  and the graphs of  $y = a^\omega$ ,  $y = 1 + k\omega$  that you found in Task 1 and Task 2.

## References

- J. Bernoulli. Principia calculi exponentialium seu percurrentium. *Acta Eruditorum*, March: 125–133, 1697.
- J. D. Blanton (translator). *Introduction to Analysis of the Infinite (Introductio in Analysin Infinitorum)*. Springer-Verlag, New York, 1988.
- L. Euler. *Introductio in Analysin Infinitorum*. St. Petersburg, 1748.
- L. Euler. De la controverse entre Mrs. Leibniz et Bernoulli sur les logarithmes des nombres negatifs et imaginaires. *Mémoires de l'académie des sciences de Berlin*, 5:139–179, 1749.

## Notes to Instructors

The heart of this project for an introductory real analysis course is giving a modern justification of  $e = \lim_{j \rightarrow \infty} (1 + 1/j)^j$  using Euler's ideas along with some modern theory. The approach using the Monotone Convergence Theorem, as outlined in Tasks 6 – 10, is a common approach in current analysis textbooks. Reading about it in Euler's own words gives context to the exercises and some appreciation of his dexterity with infinitesimals and series, as well as the close connection with  $e$  as a logarithm base to motivate the definition. This series development of  $e^z$  is an interesting alternative to the Taylor series approach students have seen in introductory calculus courses.

One question for instructors and students alike is how formally and thoroughly to treat Euler's manipulations of infinitely large and small numbers. The project author is of the opinion that students already have a personal sense of what these objects are and how they should work, having been through introductory calculus courses. Euler makes a good case for his development in the passages quoted in the project so students can follow his reasoning. Since this project is designed for an introductory real analysis course, a lengthy discussion of infinitesimal calculus is not appropriate. However, instructors for other courses may want to spend more time on these issues.

In Tasks 1 and 2, it is interesting to note that if students try to approximate  $k$  better by using smaller values of  $k\omega$ , they may run into technology problems. For example, a TI-84 calculator evaluates  $\frac{10^{-10}}{\log_{10}(1 + 10^{-10})}$  to be 2.302585093, but the Mathematica 10 computer algebra system does not fare so well, producing 2.30258490259. This is likely the case because the TI calculator uses base 10 floating point arithmetic, while Mathematica uses base 2. Using  $k\omega = 10^{-6}$  accomplishes the main goal while avoiding technology problems. Students revisit these  $k$  values in the last exercise of the project.

### Project Content Goals

1. Rigorously prove that the sequence  $\{(1 + 1/j)^j\}$  converges to  $e$  by modernizing Euler's proof.
2. Develop Euler's idea of  $e$  as an optimal logarithm base.
3. Understand the relationship between the series and sequence expressions for  $e$ , as developed by Euler.

### Student Prerequisites

The project is written for a course in Real Analysis with the assumption that students have studied sequences and are familiar with the Monotone Convergence Theorem. If students are rusty with the Binomial Theorem or the derivative of  $\log_a(x)$ , some quick "Just in Time" review will be needed.

## Suggestions for Classroom Implementation

This project takes around two 75-minute class sessions plus homework using the following methodology, very similar to David Pengelley’s “A, B, C” method described on his website.<sup>3</sup>

1. Students do some advance reading and light preparatory exercises before each class. This should be counted as part of the project grade to ensure students take it seriously. Be careful not to get carried away with the exercises or your grading load will get out of hand! Some instructors have students write questions or summaries based on the reading.
2. Class time is largely dedicated to students working in groups on the project — reading the material and working exercises. As they work through the project, the instructor circulates through the groups asking questions and giving hints or explanations as needed. Occasional student presentations may be appropriate. Occasional full class guided discussions may be appropriate, particularly for the beginning and end of class, and for difficult sections of the project. I have found that a “participation” grade suffices for this component of the student work. Some instructors collect the work. If a student misses class, I have them write up solutions to the exercises they missed. This is usually a good incentive not to miss class!
3. Some exercises are assigned for students to do and write up outside of class. Careful grading of these exercises is very useful, both to students and faculty. The time spent grading can replace time an instructor might otherwise spend preparing for a lecture.

## Sample Implementation Schedule

Day 1. Assign through Task 1 as advance prep work; complete Tasks 2 - 6 in-class.

Day 2. Assign Task 7 as advance prep work; complete Tasks 8, 10, 13 in-class.

Homework. Tasks 9, 11, 12.

If time does not permit a full implementation with this methodology, instructors can use more class time for guided discussion and less group work for difficult parts of the project.

L<sup>A</sup>T<sub>E</sub>Xcode of this entire PSP is available from the author by request to facilitate preparation of ‘in-class task sheets’ based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

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<sup>3</sup><https://www.math.nmsu.edu/~davidp/classroom-dynamics.pdf>

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