Fourier's Heat Equation and the Birth of Modern Climate Science

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February 7, 2022

It is often said that Joseph Fourier gave birth to modern climate science. His 1827 paper "Mémoire sur les Températures du Globe Terrestre et des Espaces Planétaires" (translated to English as "On the Temperatures of the Terrestrial Sphere and Interplanetary Space" in [Pierrehumbert, 2004]), contained the following very influential passage:

> The Earth is heated by solar radiation . . . . Our solar system is located in a region of the universe of which all points have a common and constant temperature, determined by the light rays and the heat sent by all the surrounding stars. This cold temperature of the interplanetary sky is slightly below that of the Earth's polar regions. The Earth would have none other than this same temperature of the Sky, were it not for . . . causes which act . . . to further heat it.

The above passage is widely considered to be the first demonstration of the existence of the greenhouse effect. Fourier's claim was far from speculative; rather, it was based on his groundbreaking study of heat published five years earlier, *Théorie analytique de la chaleur (The Analytical Theory of Heat)* [Fourier, 1822]. The goal of this project is to give the reader an insight into the techniques Fourier employed therein, as they have become the basis of modern thermodynamics as well as enormously consequential in mathematics itself. In particular, this project tells the following story:

- **Section 1.** We see what Fourier's starting assumptions were for his heat investigation.
- **Section 2.** We retrace one of Fourier's primary examples: determining the temperature of a square prism of infinite length. Part of the way through, we find that Fourier snapped his fingers and solved a differential equation in just one step.
- **Section 3.** The magical incantation that Fourier used to solve his differential equation was some old magic due to Leonhard Euler (1707–1783). In this section, we read this technique in Euler's own words.
- **Section 4.** We return to Fourier's infinite square prism problem to solve it, using Euler's work.
- **Section 5.** We present Fourier's more general heat equation. Note that we do not present the full derivation of this equation (which is in *The Analytical Theory of Heat*, Chapter II, Section V, for interested readers) but rather offer just an intuitive explanation.

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- **Section 6.** In this section, we consider Fourier's solution of the heat problem for an infinite rectangular solid—a geometric object that seems only slightly more complicated than a square prism of infinite length, but for which the century-old algorithm handed to him by Euler couldn't simply be applied. Instead Fourier's work took a very surprising turn. While studying the heat propagation in an infinite rectangular solid, he invented a whole new theory of infinite series, now called *Fourier series*.

- **Section 7.** In this brief section, students are prompted to explore connections between Fourier's work and modern climate science.

All primary source excerpts that follow are from the 1878 translation of *The Analytical Theory of Heat* by Alexander Freeman,[1] with the exception of the Euler excerpts in Section 3. Some minor corrections have been made to his translation, though for the most part we follow it verbatim.

1 Introducing Fourier and Fourier's Introduction

Jean-Baptiste Joseph Fourier (1768–1830) was born into a working-class family in Auxerre, France, but was orphaned in childhood. Luckily, he obtained admission to a local military school, where he received an education from the Benedictine monks of Saint-Maur. In 1790, they gave him a mathematics teaching appointment at their school in Auxerre, where he also taught rhetoric, history, and philosophy. He later became a founding faculty member at the Ècole Polytechnique[2] in Paris, where Napoleon sometimes attended lectures. This led to Napoleon's request for Fourier's help in the administration of Egypt after its occupation by France in 1798. Upon his return to France, Fourier served as the chief administrator of the region of Isère, where he led extensive infrastructure projects to quell chronic infections that were emanating from marshes in the area. In 1817, he was elected to the Académie des Sciences, and five years later he became their perpetual secretary.[3]

Thus, one can hardly imagine someone with a broader background than Fourier, more uniquely situated to simultaneously tackle problems of pure thought as well as in the physical world around him, perhaps in the same stroke of the pen. In the introduction of *The Analytical Theory of Heat*, he made no secret about the fact that he intended to do just that, with mathematics as his language and tool. Of mathematics, he said the following, found in [Fourier, 1822, pp. 7–8].

> Its chief attribute is clearness; it has no marks to express confused notions. It brings together phenomena most diverse, and discovers the hidden analogies which unite them.

[1] Alexander Freeman (1838–1897) was born in Blackheath, Surrey, England. He was an astronomer and mathematics teacher. In addition to providing the English translation of Fourier's work we use here, he was a frequent correspondent of James Clerk Maxwell (1831–1879), the creator of the modern theories of electricity and magnetism.

[2] Founded during the French Revolution in 1794 (the same year as Fourier's arrest for having defended a member of a particular political faction) in part by mathematician Gaspard Monge (1746–1818), Ècole Polytechnique remains one of the most well-respected institutions of mathematics in the world today.

[3] This title implies being chairperson and chief representative for life, with the option to step down, after which one becomes known as the honorary perpetual secretary. For more on Fourier's life, see [O'Connor and Robertson, 1998b].

Task 1 Do you agree that mathematics has "no marks to express confused notions" as Fourier claimed? Why or why not? What about his second claim? In your mathematical studies, where have you witnessed a case in which mathematics was able to express "hidden analogies which unite" seemingly different phenomena? (Or, if you don't think you have witnessed any such case, what do you think Fourier might have meant by this?)

Fourier also made clear the necessity of looking at heat through a mathematical lens [Fourier, 1822, p. 1].

> Heat, like gravity, penetrates every substance of the universe, its rays occupy all parts of space. The object of our work is to set forth the mathematical laws which this element obeys. The theory of heat will hereafter form one of the most important branches of general physics.

He then stated specifically what problem he was trying to solve [Fourier, 1822, p. 14].

> The problem of the propagation of heat consists in determining what is the temperature at each point of a body at a given instant, supposing that the initial temperatures are known.

Lastly, Fourier laid out some starting assumptions regarding how heat transfer can be modeled (and commented that his assumptions were verified by experiment and observation). His primary assumption is often referred to today as *Newton's Law of Cooling* or the first law of thermodynamics, which loosely says that the rate of transfer of heat between two objects will be proportional to the difference in temperature between the two objects. Fourier stated this in Section III of his first chapter, calling it the "Principle of the communication of heat." In his own words, he said the following:

> The action of two molecules, or the quantity of heat which the hottest communicates to the other, is the difference of the two quantities which they give up to each other.

Task 2 Have you seen Newton's Law of Cooling in your previous studies of mathematics or physics? If so, what kind of functions (i.e., polynomial, rational, etc.) are used when modeling temperature using Newton's Law of Cooling? If you haven't seen it previously, feel free to look it up in the index of a precalculus textbook or on the internet.

2 Heating a Square Prism of Infinite Length

Fourier first studied heat transfer in a figure which has a square base (called A) that then extends in **just one direction** to infinity, creating a prism of infinite length. Although the shape itself is three-dimensional, he here threw away all unknowns except one, in order to look for a temperature function $v(x)$ where x represents the distance from the square base. As you read through Fourier's treatment of this situation, pause to work each of the tasks that appear between the various excerpts from his work.

§73 ... A metal bar, whose form is that of a rectangular parallelepiped infinite in length, is exposed to the action of a source of heat which produces a constant temperature at all points of its extremity A. It is required to determine the fixed temperatures at the different sections of the bar.

The section perpendicular to the axis is supposed to be a square whose side $2l$ is so small that we may without sensible error consider the temperatures to be equal at different points of the same section. The air in which the bar is placed is maintained at a constant temperature 0, and carried away by a current with uniform velocity.

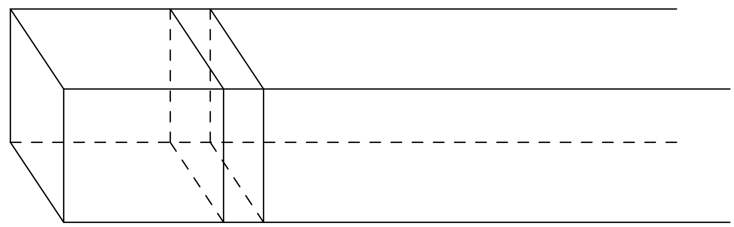
Within the interior of the solid, heat will pass successively all the parts situated to the right of the source, and not exposed directly to its action; they will be heated more and more, but the temperature of each point will not increase beyond a certain limit. This maximum temperature is not the same for every section; it in general decreases as the distance of the section from the origin increases: we shall denote by v the fixed temperature of a section perpendicular to the axis, and situated at a distance x from the origin A.

Task 3 There are really four unknowns here: time t and three spatial dimensions x, y, and z. How did Fourier manage to represent the temperature function v only in terms of x?

Before every point of the solid has attained its highest degree of heat, the system of temperatures varies continually, and approaches more and more to a fixed state, which is that which we consider. This final state is kept up of itself when it has once been formed. In order that the system of temperatures may be permanent, it is necessary that the quantity of heat which, during a unit of time, crosses a section made at a distance x from the origin, should balance exactly all the heat which, during the same time, escapes through that part of the external surface of the prism which is situated to the right of the same section. The lamina whose thickness is dx, and whose external surface is $8ldx$, allows the escape into the air, during a unit of time, of a quantity of heat expressed by $8hlv.dx$, h being the measure of the external conducibility of the prism.[4]

[4] Note that in this time and place, it was fairly common to use a lower dot (period) as a multiplication symbol in mathematics writing instead of a centered dot as we more often do today.

$\boxed{\textbf{Task 4}}$ Consider the diagram below, of Fourier's "rectangular parallelepiped infinite in length," showing the "source of heat" as the square base on the left along with the "section made at a distance x from the origin, ... whose thickness is dx." Label the diagram with the measurements Fourier used in the passages above, and use that labeled diagram to verify his claim that the "external surface is $8ldx$."



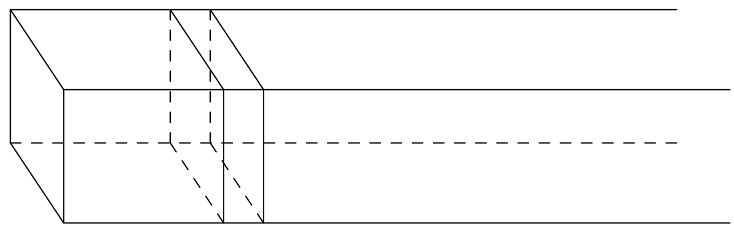
Hence taking the integral $\int 8hlv.dx$ from $x = 0$ to $x = \infty$, we shall find the quantity of heat which escapes from the whole surface of the bar during a unit of time; and if we take the same integral[5] from $x = 0$ to $x = x$, we shall have the quantity of heat lost through the part of the surface included between the source of heat and the section made at the distance x. Denoting the first integral by C, whose value is constant, and the variable value of the second by $\int 8hlv.dx$; the difference $C - \int 8hlv.dx$ will express the whole quantity of heat which escapes into the air across the part of the surface situated to the right of the section.

$\boxed{\textbf{Task 5}}$ Why could Fourier consider the first integral to be a constant value C?

On the other hand, the lamina of the solid, enclosed between two sections infinitely near at distances x and $x + dx$, must resemble an infinite solid, bounded by two parallel planes, subject to fixed temperatures v and $v + dv$, since, by hypothesis, the temperature does not vary throughout the whole extent of the name section. The thickness of the solid is dx, and the area of the section is $4l^2$: hence the quantity of the heat which flows uniformly, during one unit of time, across a section of this solid, is, according to the preceding principles, $-4l^2K\frac{dv}{dx}$, K being the specific internal conducibility:

[5] What Fourier did here is considered invalid (or at least a very poor choice of notation) today, using the same letter x both as the independent variable of the function being integrated as well as in the bounds. Today one would replace the independent variable of the function with some other letter, like τ, and then take the integral from $\tau = 0$ to $\tau = x$. It may be helpful to think of it this way in the calculations that follow.

Task 6 Label the diagram of the prism and section, shown again below. Use it to verify Fourier's claim that "the area of the section is $4l^2$," as well as the claim that "the quantity of the heat which flows uniformly, during one unit of time, across a section of this solid, is ... $-4l^2K\frac{dv}{dx}$." (Note the original translation is inconsistent with regards to the use of capital versus lowercase K; here we have changed them all to capital letters for readability.)



... we must therefore have the equation

$$-4l^2K\frac{dv}{dx} = C - \int 8hlv.dx,$$

whence

$$Kl\frac{d^2v}{dx^2} = 2hv.$$

Task 7 (a) Where did Fourier's first equation above come from? He came up with two formulas that both represent what quantity?

(b) What did Fourier do to get the second equation from the first? (**Hint!** You will need to apply a very famous theorem from your first-semester Calculus course in order to justify that little "whence" that Fourier brushed past!)

§76. The integral of the preceding equation is

$$v = Ae^{-x\sqrt{\frac{2h}{lK}}} + Be^{+x\sqrt{\frac{2h}{lK}}},$$

A and B being two arbitrary constants;[6]. . . .

[6] Be aware that this A is just a generic real number and has nothing to do with the A that represents the plate serving as origin of the heat. Sometimes, $A \neq A$.

After he obtained the equation

$$kl\frac{d^2v}{dx^2} = 2hv, \tag{1}$$

Fourier solved it without even pausing to take a breath. He showed no work as to how he obtained the general solution. This was certainly not due to incomplete exposition, however. Rather, it was because the technique for solving such a differential equation was already quite well-known in Fourier's time, having been worked out almost eighty years earlier by none other than the great Leonhard Euler (1707–1783).[7] We take a look at Euler's solution to this equation in the next section.

3 Euler's Solution to Linear Homogeneous Constant-Coefficient Differential Equations

Though much of the groundwork was done in his correspondence with Johann Bernoulli (1667–1748), Euler's method for solving such differential equations was fully written up in the paper "De integratione aequationum differentialium altiorum graduum" ("On the integration of differential equations of higher orders") [Euler, 1743]. Here we present very (very!) selected excerpts which provide just the tiny slice of his method that Fourier actually used above.[8] The first excerpt we show is Euler's statement of the problem.

§28

If a differential equation of order n of this kind was propounded

$$0 = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + \cdots + N\frac{d^ny}{dx^n}$$

in which the element dx is put constant, and the letters $A, B, C, D, \ldots, N$ denote arbitrary constant coefficients, to find the integral of this equation in finite real terms.

Task 8 Take Fourier's Equation (1) and write it in the form Euler gives above. What is n? What are "the letters $A, B, C, D, \ldots, N$" in this instance?

[7] Leonhard Euler was born in Basel, Switzerland to Marguerite (née Brucker) and Paul Euler, a Protestant minister who had attended Johann Bernoulli's lectures at University of Basel. Paul wished for his son to follow him into the ministry, but Johann persuaded Paul to allow Leonhard to study mathematics instead after witnessing his incredible potential for the subject.

[8] For a comprehensive treatment, see Adam Parker's "Leonhard Euler and Johann Bernoulli Solving Homogenous Higher Order Linear Differential Equations With Constant Coefficients" [Parker, 2020].

§12 ...[9] the following algebraic equation will result

$$0 = A + Bp + Cp^2 + Dp^3 + Ep^4 + \cdots + Np^n$$

... if from which any value of p is found, one will at the same time have a particular integral $y = e^{px}$ satisfying the propounded differential equation.

In modern differential equations courses, the equation that Euler showed above is typically called either the *characteristic equation* or the *auxiliary equation*. Because of connections to linear algebra and eigenvalues, typically the symbol λ is used in place of p. Notice what Euler's writing tells us: the function $y = e^{px}$ will solve the differential equation in his §28 whenever p is a numerical solution to the equation shown in §12.

Task 9

(a) Find the characteristic equation corresponding to Fourier's Equation (1).

(b) What are the values of p that solve this characteristic equation?

(c) Euler claimed that "one will at the same time have a particular integral[10] $y = e^{px}$ satisfying the propounded differential equation." For each of your values of p, substitute it into the formula $y = e^{px}$, and check that it is a valid solution to Fourier's Equation (1) as claimed.

Euler then described how to stitch together separate solutions to build a general solution, which he called "the complete value for y."

§15 Therefore, if all roots of this algebraic equation of order n were real, then the complete value for y will ... be the aggregate of n exponential formulas of this kind αe^{px}, ...[11]

Task 10 Notice that in the passage above, Euler allowed us to attach a constant α to the front of our exponential function e^{px} (and for exponential functions corresponding to different values of p, you could have a different constant in front). He also said we should take the "aggregate of n exponential formulas" to obtain "the complete value for y." One can take the word "aggregate" in this context to simply mean "sum." Verify that performing these steps results in Fourier's general solution to Equation (1).

[9]The reader may wonder why Euler's §12 showed the method that answered the question asked in his §28. The answer is simply style! Euler often wrote and published work in an order that showed his messy discovery process of the results, as opposed to today's more standard style in which one first states a result and then follows it with the cleanest possible proof of such a result—rarely how it was happened upon in the first place!

[10]Note that Euler uses the word "integral" in this context where today we would use the word "solution." If a differential equation is of the form $y' = f(x)$, then the solution of the differential equation and the indefinite integral of $f(x)$ are one and the same, hence the somewhat interchangeable words.

[11]The author made a slight change to the translation here, writing αe^{px} where Euler had $\alpha e^{qx:p}$ to make the notation of Euler's §12 and §15 consistent with each other.

4 Back to the Future[12]

Having seen where Fourier's solution of his differential equation (1) came from, we return to his treatment of heat transfer in a square prism of infinite length, picking up where we left off at the end of Section 2 of this project.

§76. The integral of the preceding equation is

$$v = Ae^{-x\sqrt{\frac{2h}{lK}}} + Be^{+x\sqrt{\frac{2h}{lK}}},$$

A and B being two arbitrary constants; now, if we suppose the distance x infinite, the value of the temperature v must be infinitely small; hence the term $Be^{+x\sqrt{\frac{2h}{lK}}}$ does not exist in the integral: thus the equation $v = Ae^{-x\sqrt{\frac{2h}{lK}}}$ represents the permanent state of the solid; the temperature at the origin is denoted by the constant A, since that is the value of v when x is zero.[13]

Task 11 Let us analyze the above passage in modern terminology and notation. Thinking of v as a function of x, what did Fourier claim about the value of $\lim_{x\to\infty} v(x)$? What then does that claim imply about the value of B, and why?

A few sections later in his book, Fourier made the following claims.

§80. It is easy to ascertain how much heat flows during a unit of time through a section of the bar arrived at its fixed state: this quantity is expressed by $-4Kl^2\frac{dv}{dx}$, or $4A\sqrt{2Khl^3}.e^{-x\sqrt{\frac{2h}{KL}}}$, and if we take its value at the origin, we shall have $4A\sqrt{2Khl^3}$ as the measure of the quantity of heat which passes from the source into the solid during a unit of time; thus the expenditure of the source of heat is, all other things being equal, proportional to the square root of the cube of the thickness.

We should obtain the same result on taking the integral $\int 8hlv.dx$ from x nothing to x infinite.

[12]Though this future is 200 years in the past.

[13]In this same section of his *The Analytic Theory of Heat*, Fourier mentioned that this solution was in fact verified empirically! He wrote the following:

> This law according to which the temperatures decrease is the same as that given by experiment; several physicists have observed the fixed temperatures at different points of a metal bar exposed at its extremity to the constant action of a source of heat, and they have ascertained that the distances from the origin represent logarithms, and the temperatures the corresponding numbers.

Task 12 We proceed to verify Fourier's claims from the section above.

(a) Take the formula for v established in §76 and substitute it into $-4Kl^2\frac{dv}{dx}$ in order to verify Fourier's claim about "how much heat flows during a unit of time through a section of the bar" as well as "the measure of the quantity of heat which passes from the source into the solid during a unit of time."

(b) Take that same formula for v established in §76, and substitute it into the integral

$$\int_{x=0}^{x=\infty} 8hlv.dx$$

and verify that we "obtain the same result" as Fourier claimed.

5 Fourier's Heat Equation

In Section 2 of this project, the differential equation that modeled the given heat problem ended up being an *ordinary differential equation* (ODE). It is called "ordinary" because the derivatives are "ordinary." That is, it is a differential equation in which the solution is a function of just one independent variable, and thus all derivatives (be it a first derivative or a higher-order derivative) are calculated with respect to that one variable. This was possible because the temperature of our object was really only dependent on how far you were from the heat source, represented by x. Fourier realized, however, that this would be insufficient for more complicated scenarios, and he proceeded to introduce *partial differential equations*, equations in which the solution will be a function of several independent variables and the derivatives may be taken with respect to any number of those independent variables.

In Chapter II, "Equations of the Movement of Heat," Fourier presented his much-celebrated heat equation:

$$\frac{dv}{dt} = \frac{K}{CD}\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right).$$

The derivation is based on a simple idea: any change in the temperature of a location with respect to time (the left-hand side of his equation) must somehow correspond to heat moving in or out of that location through any one of three directions, x, y, or z (the right-hand side of his equation). The quantities K, C, and D represent properties of the material being studied.

6 Heating an Infinite Rectangular Solid

In Chapter III, "Propagation of Heat in an Infinite Rectangular Solid," Fourier's work took a surprising turn. Solving the heat problem in this "infinite rectangular solid," which in terms of a geometric object seems only slightly more complicated, did not amount to simply applying a century-old algorithm handed to him by Euler. Rather, he invented a whole new theory, Fourier series, in order to study this shape. Loosely speaking, it is the idea of constructing infinite series built out of trigonometric functions.[14] We shall see in the following passage how and why Fourier constructed these!

[14] It should be noted that Fourier was not the first to construct an infinite series out of trigonometric functions. The first such example actually appeared in 1744 in a letter from Euler to the German mathematician and lawyer Christian Goldbach (1690–1764). In English translation (taken from [Lemmermeyer and Mattmüller, 2015, p. 834]), Euler wrote

Note that in the passage below, one may imagine an x, y, and z axis in three-dimensional space. The line of intersection of planes A and B is parallel to the z-axis (and similarly the line of intersection of planes A and C). Thus, the figure is completely uniform in the z direction, hence Fourier's remark that "abstraction is made of the co-ordinate z," essentially meaning that you can draw just one two-dimensional slice of the region (that slice being perpendicular to the z-axis) and have all the information you need.[15]

§164 Suppose a homogeneous solid mass to be contained between two planes B and C vertical, parallel, and infinite, and to be divided into two parts by a plane A perpendicular to the other two . . . ; we proceed to consider the temperatures of the mass BAC bounded by the three infinite planes A, B, C. The other part $B'AC'$ of the infinite solid is supposed to be a constant source of heat, that is to say, all its points are maintained at the temperature 1, which cannot alter. The two lateral solids bounded, one by the plane C and the plane A produced, the other by the plane B and the plane A produced, have at all points constant temperature 0, some external cause maintaining them always at that temperature; lastly, the molecules of the solid bounded by A, B and C have the initial temperature 0. Heat will pass continually from the source A into the solid BAC, and will be propagated there in the longitudinal direction, which is infinite, and at the same time will turn towards the cool masses B and C, which will absorb a large part of it. The temperatures of the solid BAC will be raised gradually: but will not be able to surpass nor even to attain a maximum of temperature, which is different for different points of the mass. It is required to determine the final and constant state to which the variable state continually approaches.

If this final state were known, and were then formed, it would subsist of itself, and this is the property which distinguishes it from all other states. Thus the actual problem consists in determining the permanent temperatures of an infinite rectangular solid, bounded by two masses of ice B and C, and a mass of boiling water A; the consideration of such simple and primary problems is one of the surest modes of discovering the laws of natural phenomena, and we see, by the history of the sciences, that every theory has been formed in this manner.

§165 . . . It is supposed that there is no loss of heat at the surface of the plate, or, which is the same thing, we consider a solid formed by superposing an infinite number of plates similar to the preceding: the straight line Ax which divides the plate into two equal parts is taken as the axis of x, and the co-ordinates of any point m are x and y; lastly, the width A of the plate is represented by . . . π, ...

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§I. Given an arbitrary arc a of a circle, let its sine be $= \alpha$, the sine of the double arc be $= \beta$, the sine of the triple arc $= \gamma$, the sine of the quadruple $= \delta$, of the quintuple $= \epsilon$, and so on: I say that the sum of the infinite series

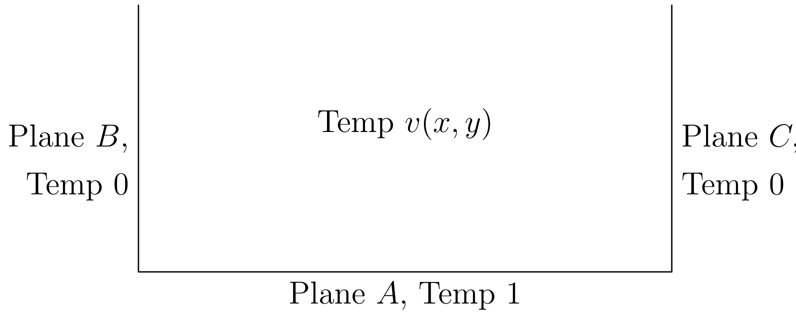
$$\frac{1}{2}a + \alpha + \frac{1}{2}\beta + \frac{1}{3}\gamma + \frac{1}{4}\delta + \frac{1}{5}\epsilon + \cdots$$

always expresses the length of an arc of 90° in the same circle.

Why then, one may ask, are these called Fourier series and not Euler series? Well, on one hand Euler did an unbelievable amount of work with infinite series, so it would be hard to name any one particular type after him. But perhaps the better reason is that Euler's perspective was quite different: he presented this trigonometric series as subdivisions of an arc of a circle into chords. Fourier, on the other hand, was thinking of it as it is more often thought of today: one has a function in mind (in Fourier's case, the solution to a partial differential equation) which is then expressed as a summation of trigonometric functions, much as one finds a power series for a function by expressing it as a summation of powers of the independent variable.

[15] A diagram of this appears below the passage, as a replacement for the figure that Fourier himself provided.

We now pause the reading of the primary source to draw a diagram that illustrates the region Fourier described above. Recall that Fourier's region extended uniformly along the z-axis, so what we draw below is a representation of a cross section perpendicular to the z-axis. That is, one may think of this page as the xy-plane, with the z-axis invisible to us, extending vertically from the paper.



Plane B can be represented by the equation $y = -\frac{\pi}{2}$ and plane C by $y = \frac{\pi}{2}$. Plane A is given by $x = 0$. All three extend indefinitely in the z direction (which again, is perpendicular to the page, which is why we draw A, B, and C as line segments even though they are planes).

Task 13 Describe, in words, how the shape of the region and the heat sources (described in the excerpt we've just read) for this example differ from the one Fourier considered in §73 (found at the beginning of Section 2 of this project). What traits do you think will make this new problem more or less challenging than the previous one?

Let's now go back to reading Fourier. In the passage that follows, he started with his general heat equation and found that, in this particular example, two of the terms conveniently are zero.

Imagine a point ... of the solid plate BAC, whose co-ordinates are x and y, to have the actual temperature v, and that the quantities v, which correspond to different points, are such that no change can happen in the temperatures, provided that the temperature of every point of the base A is always 1, and that the sides B and C retain at all their points the temperature 0.

§166 To apply the general equation

$$\frac{dv}{dt} = \frac{K}{CD}\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right),$$

we must consider that, in the case in question, abstraction is made of the co-ordinate z, so that the term $\frac{d^2v}{dz^2}$ must be omitted; with respect to the first member $\frac{dv}{dt}$, it vanishes, since we wish to determine the stationary temperatures; thus the equation ... is the following:

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0 \ldots\ldots\ldots\ldots (a).$$

Task 14

(a) Explain in words why Fourier claimed "the term $\frac{d^2v}{dz^2}$ must be omitted."

(b) Explain in words why Fourier claimed that the term $\frac{dv}{dt}$ "vanishes."

(c) Setting those two terms equal to zero, and conveniently choosing the constants $K = C = D = 1$, verify that Fourier's general heat equation becomes his simpler equation (a).

Fourier's equation (a) is what is today called a *partial differential equation*, in which an unknown function of several variables is sought, given some relationship between the function and its partial derivatives. In this case, we wish to find $v(x, y)$, an unknown function of two variables, such that the sum of the second partial derivatives with respect to x and y is zero. In modern notation, one typically makes the d "curly" in order to indicate a partial derivative. So, Fourier's equation (a) would today be written as

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0,$$

where one interprets $\frac{\partial^2 v}{\partial x^2}$ as the second derivative of v with respect to x, treating y as a constant, and similarly for $\frac{\partial^2 v}{\partial y^2}$, mutatis mutandis.[16]

Task 15 Similar to an ordinary differential equation (ODE), a function is a solution to a PDE if and only if substituting that function into the equation makes the left-hand side equal to the right-hand side. To build up a little intuition for Fourier's heat equation written above, consider each of the following functions $v(x, y)$ and decide whether or not it is a solution to that PDE.

(a) $v(x, y) = 1$

(b) $v(x, y) = 2x - y$

(c) $v(x, y) = x^2 + xy - y^2$

(d) $v(x, y) = e^y \cos(x) + e^x \sin(y)$

Task 16 In ODEs, we often try to come up with a general solution to a given differential equation. In light of your work in the previous task, how feasible do you think it would be to come up with a general solution to Fourier's heat equation?

Seeing just how all-over-the-place the general solution to a PDE can be, it seems quite reasonable that Fourier's next step was to set a more modest goal. Rather than finding all solutions to his PDE, he found just the solutions that can be factored into a function of x times a function of y.

[16]This special case of Fourier's heat equation is sometimes called the two-dimensional Laplace equation, named after the great French mathematician and scientist Pierre-Simon, marquis de Laplace (1749–1827).

§167 … Functions of two variables often reduce to less complex expressions, … which in this particular case, take the form of the product of a function of x by a function of y. … We shall then write $v = F(x)f(y)$; substituting in equation (a) and denoting $\frac{d^2F(x)}{dx^2}$ by $F''(x)$ and $\frac{d^2f(y)}{dy^2}$ by $f''(y)$, we shall have

$$\frac{F''(x)}{F(x)} + \frac{f''(y)}{f(y)} = 0;$$

we then suppose[17] $\frac{F''(x)}{F(x)} = m^2$ and $\frac{f''(y)}{f(y)} = -m^2$, with m being any constant quantity, and as it is proposed only to find a particular value of v, we deduce from the preceding equations $F(x) = e^{-mx}$, $f(y) = \cos my$.

Task 17 Notice that Fourier was not claiming above to have found general solutions to the equations[18]

$$\frac{F''(x)}{F(x)} = m^2$$

and

$$\frac{f''(y)}{f(y)} = -m^2,$$

as he wanted "only to find a particular value of v." Verify that each of the solutions he showed for $F(x)$ and $f(y)$ satisfies the corresponding equation.

At this point, Fourier substituted his particular solutions for $F(x)$ and $f(y)$ back into his equation $v(x, y) = F(x)f(y)$, and thus built the form of a particular solution to his heat equation:

$$v(x, y) = e^{-mx} \cos my.$$

Next, he wished to determine what this constant m could be.

Recall that after finding a general solution to an ordinary first-order differential equation, one typically has an unknown constant C, which can be solved for by plugging in an initial condition. For PDEs, one does something very similar, plugging in what are called *boundary conditions* to find out more information about the solution. Let us read Fourier's boundary conditions for this particular PDE, after which we will use them to determine the possible values of m.

[17]Note that in the original, Fourier simply wrote m and not m^2 in the equations that follow. We made this minor change to the translation to improve its readability, since otherwise it requires following Fourier's slightly awkward step of essentially redefining m as its own square root in his particular solutions.

[18]To do so would require Euler's technique from Section 3, as well as more techniques from that same paper that we did not present here.

The function[19] of x and y, $v(x, y)$, which represents the permanent state of the solid BAC, must, 1st, satisfy the equation (a); 2nd become nothing when we substitute $-\frac{1}{2}\pi$ or $+\frac{1}{2}\pi$ for y, whatever the value of x may be; 3rd, must be equal to unity when we suppose $x = 0$ and y to have any value included between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.

Further, this function $v(x, y)$ ought to become extremely small when we give to x a very large value, since all the heat proceeds from the source A.

Task 18 Fourier listed four conditions that our temperature function $v(x, y)$ must satisfy, labeled by "1st", "2nd", "3rd", and "Further." The "1st" referred to the PDE itself, but the other three specified boundary conditions. Write out each of the three boundary conditions in three ways:

(a) in words, exactly as Fourier wrote it,

(b) in symbols, using formulas and notation from calculus wherever possible, and

(c) in terms of the corresponding assumptions being made about the temperature of the infinite rectangular solid. That is, what does the boundary condition tell us about the real-world heat transfer problem that is being modeled?

Fourier then applied the boundary conditions, one at a time, to extract information about m.

We could not suppose m to be a negative number, ...

Task 19 Which of Fourier's boundary conditions justified the above claim, and why?

The exponent m which enters into the function $e^{-mx} \cos my$ is unknown, and we may choose for this exponent any positive number: but ... m must be taken to be one of the terms of the series, $1, 3, 5, 7, \&c.$;

Task 20 Which of Fourier's boundary conditions justified the above claim, and why?

[19] Note that Fourier used v to indicate the unknown function and $\phi(x, y)$ to indicate the solution for this unknown, using the two somewhat interchangeably. For readability, we use only v, or $v(x, y)$, eliminating $\phi(x, y)$ entirely.

A more general value of v is easily formed by adding together several terms similar to the preceding, and we have

$$v = ae^{-x}\cos(y) + be^{-3x}\cos(3y) + ce^{-5x}\cos(5y) + de^{-7x}\cos(7y) + \&c. \dots\dots\dots (b).$$

It is evident that the function $v(x, y)$ solves the equation $\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0$, and the condition $v\left(x, \pm\frac{\pi}{2}\right) = 0$.

Task 21 Verify the two claims Fourier made about the function v. At this point, which is the only boundary condition that Fourier had not yet used?

A third condition remains to be fulfilled, which is expressed thus, $v(0, y) = 1$, and it is essential to remark that this result must exist when we give to y any value whatever included between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.

Amazingly, just the one condition stated above ended up being enough information to solve for the infinitely many unknowns $a, b, c, d, \dots$!

Equation (b) must therefore be subject to the following condition:

$$1 = a\cos y + b\cos 3y + c\cos 5y + d\cos 7y + \&c.$$

The coefficients $a, b, c, d, \&c$, whose number is infinite, are determined by means of this equation.

Fourier proceeded to solve for all infinitely many unknowns via some very clever, and very messy, infinite series manipulations. These appear in Section II of his Chapter III,[20] which he quite appropriately entitled "First example of the use of trigonometric series in the theory of heat." In fact, the infinite sum of cosines in this equation was the very first use of Fourier series for any purpose! The end results for this particular example are actually surprisingly clean, producing

$$a = \frac{4}{\pi}, b = -\frac{4}{3\pi}, c = \frac{4}{5\pi}, d = -\frac{4}{7\pi}, e = \frac{4}{9\pi}, \dots$$

and so on.

[20] More specifically, you can find Fourier's derivation of these constants in his §171 – §176.

Task 22 Take Fourier's values of $a, b, c, d, \ldots$ written above and substitute them into Fourier's Equation (b) to find the actual formula for v (as an infinite series). Plot a few partial sums for v in a 3D graphing utility (for example, Geogebra 3D). What do these graphs represent in terms of the original temperature problem? On what domain do the graphs make sense?

Of course, there is a somewhat unsatisfying aspect of this: we simply trusted Fourier's work in having found those coefficients rather than verifying the values and finding them ourselves. We fix this in our next task; however, we will not follow Fourier's methods! We will instead follow the suggestion of our translator, Alexander Freeman, who offered a different approach to solving for these constants in a footnote to his translation.

Task 23 In a footnote to his translation of §177 of Fourier's treatise on heat transfer, Freeman noted that the coefficients of equation (b) can instead be determined

> by multiplying both sides of the first equation by $\cos y$, $\cos 3y$, $\cos 5y$, &c., respectively, and integrating from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$.

Apply this method[21] and verify that the same coefficients are produced. That is to say, take Fourier's equation shown above (the one he referred to as "the following condition"), and then multiply both sides by $\cos(y)$ and integrate from $y = -\pi/2$ to $y = \pi/2$ to find a. Then, start back with Fourier's "condition" again, but this time multiply both sides by $\cos(3y)$ and integrate from $y = -\pi/2$ to $y = \pi/2$ to find b. Then, start back with Fourier's equation yet again, but this time multiply both sides by $\cos(5y)$ and integrate from $y = -\pi/2$ to $y = \pi/2$ to find c. You don't need to provide a rigorous argument that the pattern continues forever; just verify that the first few values match, let's say out to coefficient d. **Hint!** The product-to-sum identity for cosine,

$$\cos(A)\cos(B) = \frac{\cos(A+B) + \cos(A-B)}{2},$$

will be very helpful. Be warned that for each coefficient, you will need to integrate this series term-by-term, producing infinitely many integrals. However, do not fret; all but one will be zero!

7 Epilogue

Fourier's 1822 *The Analytical Theory of Heat* was a revolutionary work, serving as the mathematical basis for his creation of modern climate science with 1827's "On the Temperatures of the Terrestrial Sphere and Interplanetary Space."[22] Of course, that was just the beginning! Let us witness just a bit of where it went from there.

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[21]Freeman attributed this method to the Scottish mathematician Duncan Gregory (1813–1844). As it is more algorithmic and requires far less cleverness than what Fourier did, this is the standard method for finding coefficients in a Fourier series taught in mathematics courses today. Here you may assume that an integral of an infinite series can be calculated by integrating each term separately.

[22]It should be noted that Fourier's *The Analytical Theory of Heat* was not the end of the story of mathematicians' use of trigonometric series, but just the beginning. Not only did they turn out to be useful in physical applications, but the incredibly curious examples they produced served as bases for many questions involving rigor and foundations in analysis. Many leading 19th-century mathematicians, including Niels Abel (1802–1829), Augustin-Louis Cauchy (1789–

Task 24 Do a bit of research on each of the names written below. For each, briefly describe their contributions to climate science. Do you see any of Fourier's ideas in their work?

- Svante Arrherius
- Eunice Newton Foote
- Guy Callendar
- One climate scientist of your choice, not among the three listed above!

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1857), Karl Weierstrass (1815–1897), Bernard Bolzano (1781–1848), and Richard Dedekind (1831–1916), proceeded to lay out some of the more rigorous foundations of analysis that have become standard today, in part motivated by Fourier's work.

Raymond Pierrehumbert. Warming the world. *Nature*, page 677, 2004. Includes (as a supplementary document) an English translation of [Fourier, 1827]; that translation is also available at geosci.uchicago.edu/$\sim$rtp1/papers/Fourier1827Trans.pdf.

Notes to Instructors

PSP Content: Topics and Goals

This Primary Source Project (PSP) is intended to give students a guided tour through a portion of Fourier's incredibly influential book, *The Analytical Theory of Heat*. The mathematics that they will engage with is quite broad, and includes the following:

- Higher-order linear homogeneous constant-coefficient differential equations
- Fourier series
- Infinite series
- Partial differential equations
- Improper integrals
- Limits to infinity

Note that this project comes in two versions, a "mini-PSP" and a longer "full-length" PSP, both available at https://digitalcommons.ursinus.edu/triumphs_differ/. Short descriptions follow.

- The mini-PSP shows only two examples of Fourier modeling heat propagation: one in an square prism of infinite length and one in an infinite rectangular solid. However, the substantial amount of mathematics listed above all arises in just these two examples! **This is the version you are currently reading.**
- The full-length PSP includes everything mentioned above, but it then also takes the reader for a joyride with Fourier, following him as he used his trigonometric series to prove a plethora of infinite series identities. This PSP then ends with a substantial Epilogue, showing how this work served as one impetus to future generations of mathematicians as they explored questions regarding rigor in analysis.

Student Prerequisites

Basic techniques of differentiation, partial differentiation, integration, and limit calculation are the only assumed prerequisites for this project, along with some basic idea of what a differential equation is and what a solution to a differential equation is. Hence, this project is quite appropriate near the end of a multivariable calculus course (assuming there was a brief introduction to ODEs somewhere in students' Calculus I or II classes) as well as in a course in differential equations.

PSP Design and Task Commentary

This PSP gives some context and background in the introduction and Section 1. Section 2 describes Fourier's solution to the heat problem in the case of a square prism of infinite length, which results in a second-order linear differential homogeneous equation. Section 3 shows a result from Euler which is applied to Fourier's differential equation in Section 4. Section 5 introduces Fourier's general PDE with respect to time and three-dimensional space, and Section 6 then applies it to an infinite rectangular solid, which motivated Fourier's development of Fourier series. In Section 7, we ask the student to look ahead a bit and investigate how climate science evolved in the decades that followed Fourier's work.

None of the manipulations are terribly messy in this project, except for perhaps the integrals involved in finding Fourier series coefficients in Task 23. The product-to-sum identity for cosine was provided to help with these otherwise-challenging integrals, or if the instructor wishes to streamline implementation further, one could allow evaluation of those integrals via technology. Also, one could omit that task entirely and the PSP will still flow just fine; in particular, if Fourier series is not a topic you are explicitly trying to teach in your course, it might make sense to stop Section 6 at Task 22 and then go to the Epilogue.

One place where the instructor will want to pay close attention to student work is where the student is asked to identify the boundary conditions in the PDE governing the infinite rectangular solid. If the student has even a small transcription error there, they will be setting themselves up for a world of hurt when trying to find m in the tasks that follow.

Suggestions for Classroom Implementation

This project could be implemented any of a number of ways, and would probably take a very different form in a multivariable calculus course versus in differential equations. In a multivariable calculus course, it could actually make a very good project-based assessment, perhaps replacing a cumulative test since it covers such a broad plethora of topics from the calculus sequence, including limits, definite integrals, improper integrals, and partial derivatives (and the only technique from differential equations that is required is self-contained, included in the Euler passage). In a differential equations course, this project might be a little too easy for a test, but it could be used to introduce the content of homogeneous linear constant-coefficient differential equations in a very contextualized applied setting; see the sample implementation schedule provided below.

The author is happy to provide LaTeX code for this project. It was created using Overleaf which makes it very convenient to copy and share projects and can allow instructors to adapt this project in whole or in part as they like for their courses.

Sample Implementation Schedule (based on a 75-minute class period)

The author recommends two full 75-minute class periods for implementation of this PSP in a differential equations class. From a bird's-eye view, the PSP really just consists of two gigantic examples: Fourier's square prism of infinite length (which could be the goal for session 1) and Fourier's infinite rectangular solid (which could be the goal for session 2).

- The readings and tasks of the PSP up to and including Section 1 can be assigned as preparation for class, along with as much of a head start on Section 2 as students can muster.

- Start class with 10 minutes of follow-up discussion on the first two sections. Perhaps get them ready for Section 2 by making sure they've labelled the infinite prism diagram correctly.

- The majority of the class period could then consist of students working in small groups with the assistance of the instructor and/or any learning assistants, completing as much as they can through the end of Section 4.

- During the last 10 minutes of class, perhaps have some discussion around common difficulties and sticking points. Any unfinished work in Sections 1 through 4 can be assigned for homework.

- The class preparation for the second session should be reading Section 5 along with initial reading and attempts of Section 6 tasks.

- The body of the second session can consist of working together on Section 6. If students are getting stuck solving for the coefficients of the Fourier series, the instructor may wish to show the solution for a on the board and then let students do b, c, and d.

- Any unfinished work in Section 6 along with Section 7 could be assigned for homework.

Connections to other Primary Source Projects

There are several PSPs that relate very directly to the content of this PSP (available at the URLs listed in the bibliography). The ways in which they relate are discussed in footnotes throughout this project.

Furthermore, this differential equations PSP is only one of a series of such projects that include student projects on first-order linear DEs, Bernoulli DEs, exact DEs, higher-order linear DEs, Wronskians, and more! In particular, there is a longer version of this same PSP which also showcases some consequences of Fourier series in the study of infinite series as well as Fourier series as a motivating factor in the quest for rigor in analysis. Find all of them at the URL given below:

https://digitalcommons.ursinus.edu/triumphs_differ/

Recommendations for Further Reading

The PSPs mentioned above are perfect further reading for the curious student! For an advanced student looking for a deeper treatment of the subjects discussed here, the author recommends *Fourier Analysis: An Introduction* by Stein and Shakarchi (Princeton University Press, 2003). It takes Fourier's foundational ideas and extends them in incredibly surprising ways (for example, using them to prove Dirichlet's Theorem), and is quite readable for an advanced undergraduate.

Acknowledgments

The development of this project has been partially supported by the Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS) Project with funding from the National Science Foundation's Improving Undergraduate STEM Education Program under Grants No. 1523494, 1523561, 1523747, 1523753, 1523898, 1524065, and 1524098. Any opinions, findings, and conclusions or recommendations expressed in this project are those of the author and do not necessarily reflect the views of the National Science Foundation. The author would especially like to thank Nicholas Scoville, Adam Parker, and V. Frederick Rickey for their very helpful reviews as well as Janet Heine Barnett for guidance throughout the course of writing and editing this project.

For more information about the NSF-funded project TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS), visit

http://blogs.ursinus.edu/triumphs/ .