Leonhard Euler and Johann Bernoulli Solving Homogenous Higher Order Linear Differential Equations With Constant Coefficients

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Imagine, for a moment, what mathematics communication was like in the early 1700s. There was (obviously) no email, texting, or internet. There were (obviously) no phones, radio, or faxes. There were (obviously) no planes, trains, or automobiles. Journals were very rare and expensive. Conferences didn't exist as we know them. The best case scenario was that you were a mathematician in a major academic hub, such as Berlin, St. Petersburg, London, or Paris, that housed an active scientific academy, such as the Académie Royale des Sciences et Belles-Lettres de Prusse, the Imperial Russian Academy of Science, the Royal Society, or the Académie Royale des Sciences de Paris. Long distance communication required letters. Lots of letters.[1] Often in Latin.

Instead of writing directly to a colleague who belonged to an academy, an eighteenth-century mathematician might write to the secretary of that academy, who would file and copy the letter to distribute to interested members. This made an inefficient process slightly better, and could settle arguments over the priority of discoveries. However, people also wrote plenty of letters directly to friends and colleagues. In this project, we'll be looking at three of the 38 known letters between Leonhard Euler[2] (1707–1783) and his teacher Johann Bernoulli[3] (1667–1748), starting with a letter written September 15, 1739. These letters, and 14 more, were published by Gustaf Eneström (1852–

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[1]According to the Euler Archive (https://scholarlycommons.pacific.edu/euler/), there are 2829 known letters to and from Euler. There were certainly more since sometimes "lost" letters are referenced in the correspondences.

[2]I think all that needs to be said about the influence of the Swiss mathematician Leonhard Euler compared to all thee mathematicians in history is expressed by the following ordering:

10. You can't
9. Rank them
8. Because the
7. Importance of
6. Their contributions
5. Is
4. Relative to
3. Their respective
2. Fields
1. Leonhard Euler

[3]Johann Bernoulli was a very talented Swiss mathematician. His unpleasant personality and desire for fame eventually ruined his relationships with his brother Jacob (1655–1705) and his son Daniel (1700–1782), both of whom were also mathematicians.

1923) in a series of three articles in the early twentieth century that are referred to as E863.[4]

Not surprisingly, since mail delivery was slow and difficult, letters were long and contained lots of information. (Imagine how much you would include if you could only send one text a month!) Eneström noted the topics that Euler discussed in this 1739 letter [Eneström, 1905, p. 34]:

> Plans for the second section of his *Dissertatio hydraulica*; completion of new parts of the *Commentarii* for the Petersburg Academy; Euler's method for summing the series
>
> $$\frac{1}{1\pm n}+\frac{1}{4\pm n}+\frac{1}{9\pm n}+\frac{1}{16\pm n}+\ldots;$$
>
> integration of incomplete nth order linear differential equations with constant coefficients.

Most mathematicians would be thrilled to make a contribution to one or two of these areas, much less all of them. This project will deal with the last item of solving what we now call homogenous higher order linear differential equations with constant coefficients.

Here is what Euler wrote about this problem to "the most celebrated esteemed Sir JOHANN BERNOULLI."[5]

∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞

> I have recently found a remarkable[6] way of integrating differential equations of higher degrees in one step, as soon as a finite [algebraic] equation has been obtained. Moreover this method extends to all equations which ... are contained in this general form:[7]
>
> $$y+\frac{A\,dy}{dx}+\frac{B\,ddy}{dx^2}+\frac{C\,d^3y}{dx^3}+\frac{D\,d^4y}{dx^4}+\frac{E\,d^5y}{dx^5}+\text{etc.}=0.$$

∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞

Their conversation about solving this particular type of differential equation played out over two more letters: a December 9, 1739, Bernoulli response letter[8] [Eneström, 1905, pp. 38–43] and a

[4] Gustaf Eneström was a Swedish mathematician and historian best known for surveying and numbering 866 distinct works by Euler. His numbering system, known as the Eneström index, is still used to reference works by Euler. We have used it here.

[5] All translations of the excerpts from these letters in this project were prepared by Danny Otero (Xavier University), 2020. The translations found in [Fauvel and Gray, 1987, pp. 447–449] have also been consulted.

[6] If Euler says it's remarkable, it must be remarkable.

[7] In his 1739 letter, Euler used lower case letters $a, b, c, \ldots$ to represent the constants in this equation, but he later replaced these by upper case letters $A, B, C \ldots$ in the published paper that we will read in the rest of this project. In the interest of consistency, we have changed the notation used in his letter to match that in his published paper.

[8] According to Eneström, other topics discussed by Bernoulli in this letter included [Eneström, 1905, p. 38]:

> On the delay in completing the second section of Johann Bernoulli's *Dissertatio hydraulica*, on the sum of the series
>
> $$\frac{1}{1\pm n}+\frac{1}{4\pm n}+\frac{1}{9\pm n}+\frac{1}{16\pm n}+\ldots;$$
>
> especially when n is a perfect square; on the integration of incompletely linear differential equations with constant coefficients; on the vibrations of floating bodies; on two problems in hydrodynamics that pertain here; a meteorological observation.

January 19, 1740, return letter from Euler[9] [Eneström, 1905, pp. 43–52]. Over the course of this conversation, Euler and Bernoulli hashed out some details and examples (which we will see in Section 3 of this project). Unfortunately, the analysis in these letters was incomplete and the notation was still evolving (as we will see in Task 2 below).

Instead of reading the details of the correspondence between Euler and Bernoulli, we therefore follow Euler's published presentation of the solution in his 1743 paper (which is E62), "De integratione aequationum differentialium altiorum graduum" ("On the integration of differential equations of higher orders") [Euler, 1743].[10] A careful reader may note that we don't follow the order of this publication either. Euler concisely stated the problem in §28 of this paper,[11] after which he gave a summary of his method. Today, such summaries are typically followed by proofs. But in Euler's paper, the proofs actually appeared *earlier*, roughly in Sections §14–§23, intertwined with his development of the various steps of the solution method. We proceed in today's more typical order of problem statement, method development with proof, and summary; we then go back to the Euler-Bernoulli letters to look at some concluding examples.

The purpose of this project is to solve the following Problem, as stated in [Euler, 1743].

PROBLEM I

§28

If a differential equation of order n were proposed of the form

$$0 = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + \cdots + N\frac{d^ny}{dx^n}$$

in which ... the letters $A, B, C, D, \ldots, N$ denote arbitrary constant coefficients, to find the integral[12] of this equation in finite real terms.

[9] Here is Eneström's description of the contents of this letter [Eneström, 1905, p. 43].

> Euler regrets that it is very difficult for him to make a copy of the second section of the *Dissertatio hydraulica*; the sum of the series
>
> $$\frac{1}{1 \pm n} + \frac{1}{4 \pm n} + \frac{1}{9 \pm n} + \frac{1}{16 \pm n} + \ldots;$$
>
> especially when n is a perfect square; integration of incompletely linear differential equations with constant coefficients and of another differential equation of a similar kind; on the vibrations of floating bodies; solution of the two problems posed by Johann Bernoulli in his last letter; the meteorological observation mentioned by Johann Bernoulli in the same letter.

Euler's integration of the "differential equation of a similar kind" in this letter was the first appearance of the Cauchy-Euler equation $0 = y + \frac{axdy}{dx} + \frac{bxxddy}{dx^2} + \frac{cx^3d^3y}{dx^3} + \text{etc.}$ [Parker, 2016, pp. 196–197].

[10] All translations of the excerpts from this paper in this project were prepared by Danny Otero (Xavier University), 2021. The translation completed by Alexander Aycock for the Euler Circle-Mainz project has also been consulted.

[11] When we write §X in this project, we are referring to the corresponding section in [Euler, 1743].

[12] A reviewer of this project noted that, "Solving a differential equation" was often referred to as "integration," since this is the operation used to solve a first order differential equation.

1 The Method of Solution

Here is our first time travel, where we move to §12 of [Euler, 1743] in which Euler described the general method.

§12 Now let all the letters A, B, C, D etc. denote constant quantities, in order that this differential equation

$$0 = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + \cdots + N\frac{d^ny}{dx^n} \tag{1}$$

of order n be integrated. As y with its differentials each represent their own dimension everywhere, if we put $y = e^{\int pdx}$, this differential equation is reduced by one order, according to my method as presented in *Commentariorum Academiae Pertrpolitanae*,[13] Vol. III . . . .

§13 But first, it is clear here that if one takes p to be constant, so that its differentials dp, ddp, d^3p, &c. vanish, then because A, B, C, D, etc. are constants, the variable x will absolutely disappear from the equation; and by this hypothesis, what results is the algebraic equation:

$$0 = A + Bp + Cp^2 + Dp^3 + Ep^4 + \cdots + Np^n :$$

if from it any value of p is found, then there will be obtained at the same time an equation for the particular integral $y = e^{px}$ satisfying the proposed differential equation; whence, as we have seen, it $[y]$ also satisfies this equation $y = \alpha e^{px}$ whenever p is a constant quantity that is also a root of the algebraic equation[14]

$$0 = A + Bz + Cz^2 + Dz^3 + Ez^4 + \cdots + Nz^n. \tag{2}$$

We will refer to Euler's equation (2) as the *auxiliary* equation, though sometimes it is called the *characteristic* equation. Let's work this equation out for ourselves.

Task 1

(a) Consider the function $y = e^{\int pdx}$ (with p a constant).[15] Repeatedly differentiate this function and substitute into Equation (1).

(b) Explain what Euler meant by "the variable x will absolutely disappear from the equation." In other words construct the auxiliary equation (2).

(c) Finally, explain why, if p is a root of Equation (2), then $y = \alpha e^{px}$ is a solution to the differential equation.

[13] While not relevant to this Primary Source Project, readers who are interested in a more complete citation of the work that Euler mentioned here are referred to [Euler, 1732].

[14] In this section of his paper, Euler used 'p' for both the variable in equation (2) and also a root of that equation. To avoid confusion, we've changed the variable to 'z' in order to match the notation that Euler used after §13.

[15] Notice this is just $y = e^{px}$.

Task 2 Here is a good place to note that even though both Euler and Bernoulli started with the same Differential Equation (1) in their letters, they developed different algebraic equations from it; in fact, the algebraic equation that Euler gave in his letters was different from the one he gave in his 1743 paper. The two algebraic equations that appeared in their correspondence were analogous to the above Equation (2) in that factoring them also served as an auxiliary step towards solving the original differential equation. Of course factoring one of these different auxiliary equations will give different values for p, and hence the solutions to Equation (1) won't be the simple $y = e^{px}$. Accordingly, we take just a quick look at these alternative equations.

Again, both Euler and Bernoulli started with a differential equation of the form

$$y + \frac{A\,dy}{dx} + \frac{B\,ddy}{dx^2} + \frac{C\,d^3y}{dx^3} + \frac{D\,d^4y}{dx^4} + \frac{E\,d^5y}{dx^5} + \text{etc.} = 0.$$

Euler stated in the September 15 letter [Eneström, 1905, pp. 33–38]:

> "To find the integral of this equation I consider this equation or algebraic expression:
>
> $$1 - Ap + Bp^2 - Cp^3 + Dp^4 - Ep^5 + \text{etc.} = 0."$$

Bernoulli's December response [Eneström, 1905, pp. 38-43] contained an auxiliary equation of the form

$$1 + \frac{A}{p} + \frac{B}{p^2} + \frac{C}{p^3} + \frac{D}{p^4} + \text{etc.} = 0,$$

for which he said one should "multiply by the highest dimension of this p to obtain an algebraic equation whose roots p ... we seek."

How do these two alternate auxiliary equations differ from Equation (2) and from each other? How are they similar?

It might appear that §12 and §13 have given us everything we need to solve the original differential question: simply factor the auxiliary equation to find the roots $p_1, p_2, \ldots, p_n$ then the solutions of the differential equation will be $y = e^{p_i x}$. However in §15, Euler noted that there are at least two complications that needed to be examined more closely.

§15 Thus, if all the roots of this algebraic equation of dimension n are real, then the complete value of y will be expressible in real terms, and it will be the aggregate of n exponential formulas of the form $\alpha e^{qx:p}$,[16] and in this case the complete integral may be

[16] Here again, Euler changed notation. Previously 'p' referred to a root of the auxiliary equation (2). However, in setting up the variables for §15, Euler stated that, "For, if $pz - q$ was a divisor of that equation, from which $z = \frac{q}{p}$ results, it will be $y = \alpha e^{\frac{qx}{p}}$; this particular value contains one arbitrary constant α." In other words, now the root in question is $\frac{q}{p}$.

expressed only by means of a logarithm, or by the quadrature of the hyperbola.[17] But if some of the roots of this algebraic equation are imaginary, then imaginary exponential formulas will enter into the complete integral; I will show below how to construct these by means of the quadrature of the circle.[18] The chief difficulty in this matter occurs whenever two or more roots of the equation are equal; for then, because of the several equal exponential formulas, the number of arbitrary constants is reduced and for that reason the integral found is no longer complete.[19]

Task 3 Factor each of the following three equations. Based on the roots you found, what factoring issues may arise that create difficulties of the kind described by Euler?

(a) $x^2 - 4x + 3 = 0$

(b) $x^2 - 4x + 4 = 0$

(c) $x^2 - 4x + 5 = 0$

1.1 Distinct Roots of the Auxiliary Equation

Before Euler addressed the difficulties that he described in §21, he discussed the case which didn't cause him concern.

§16 We may deal with both of these inconveniences if we more carefully consider the connection between the proposed differential equation,

$$0 = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + \cdots + N\frac{d^ny}{dx^n},$$

and the thus-formed algebraic equation

$$0 = A + Bz + Cz^2 + Dz^3 + \cdots + Nz^n.$$

For as the latter arises from the former, if one puts z^0 for y, z instead of $\frac{dy}{dx}$, and in general, replaces z^k by $\frac{d^ky}{dx^k}$, so in the same way a differential equation is formed from each of the factors of the algebraic equation, each of which is necessarily connected to the proposed

[17] "Quadrature" is an historical word that means "area." Hence, the phrase "the quadrature of the hyperbola" refers to calculating the area under (i.e., integrating) the hyperbola $y = \frac{1}{x}$. This is why Euler mentioned the logarithm. Later, when he stated "by means of the quadrature of the circle," he was saying that trigonometric functions will be needed.

[18] It was these imaginary roots that gave Johann Bernoulli concern. As historian of mathematics Victor Katz explained: "In essence, [Bernoulli] did not understand how complex roots of the characteristic polynomial could lead to solutions involving the 'real quadrature of the circle.' Euler finally showed him in 1740 that in fact $2\cos x$ and $e^{ix} + e^{-ix}$ were equal." [Katz, 1987, p. 322]. Euler provided Bernoulli with this explanation in his January 19, 1740, letter [Eneström, 1905, p. 76].

[19] Euler used the phrase "complete integral equation" as we use the phrase "an arbitrary linear combination of a fundamental set of solutions."

differential equation, and from which particular values for y are found. Thus, if either $pz - q$ or $q - pz$ is a divisor of the algebraic equation, then from it there arises, by the rule of [this] connection, the differential equation

$$qy - \frac{pdy}{dx} = 0,$$

whose integral yields

$$y = \alpha e^{\frac{qx}{p}},$$

which is the same as what we had determined as coming from the factor $pz - q$.

Task 4 Consider the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$.

(a) What is the corresponding auxiliary equation?

(b) What are the roots of the auxiliary equation?

(c) What are two "complete" or "fundamental" or "linearly independent" solutions to the differential equation?

1.2 Repeated Roots of the Auxiliary Equation

The first difficulty occurs when the auxiliary equation has a repeated root $r_1 = r_2$. In this case, the algorithm would not provide us with two solutions as expected in a "complete" solution, but rather just the single solution

$$y = e^{r_1 x} = e^{r_2 x}.$$

Euler determined what the required second solution will look like.

§17 Hence, it is understood that if one has any divisor of that algebraic equation, say $p + qz + rzz$, then the equation

$$py + \frac{qdy}{dx} + \frac{rddy}{dx^2} = 0$$

which arises from this divisor gives a value for y that also satisfies the proposed differential equation. From this we therefore can remove that difficulty which occurs when the algebraic equation has two or more equal factors. Let therefore $(p - qz)^2$ be a divisor of the algebraic equation from which in expanded form will result this differential equation [of second order]

$$ppy - \frac{2pqdy}{dx} + \frac{qqddy}{dx^2} = 0.$$

Let us put[20]

$$y = e^{\frac{px}{q}} u$$

[20] Assuming a second or particular solution is a multiplicative factor of a known solution was frequently done at the

and on having made the substitution, we will obtain $ddu = 0$, whence $u = \alpha + \beta x$. thus, from the quadratic factor $(p - qz)^2$ there arises the following value,

$$y = e^{\frac{px}{q}}(\alpha + \beta x),$$

which includes two arbitrary constants.

Task 5 (a) Verify Euler's statement that a repeated factor $(p-qz)^2$ in the auxiliary equation corresponds to the differential equation

$$ppy - \frac{2pqdy}{dx} + \frac{qqddy}{dx^2} = 0.$$

(b) If $(p-qz)^2$ is a repeated factor, then $r = \frac{p}{q}$ is a (repeated) root. By our algorithm, we know that one solution will be of the form $y = e^{\frac{px}{q}}$. Verify this by calculating y' and y'' and substituting into the above differential equation.

(c) As per Euler's suggestion, consider $y = e^{\frac{px}{q}}u$, where u is an unknown function. Take derivatives (remembering product rules) and substitute into the differential equation above. Recover Euler's claim that $u'' = 0$ and hence $u = \alpha + \beta x$.

Thus we have shown that if $\frac{p}{q}$ is a repeated root of the auxiliary equation, then it corresponds to the solutions

$$y_1 = \alpha e^{\frac{px}{q}} \qquad \text{and} \qquad y_2 = \beta x e^{\frac{px}{q}}.$$

Euler calculated what happens for higher order repeated roots in §18.

§18 If the algebraic equation has the cubic divisor $(p - qz)^3$, then it will be connected with this proposed differential equation

$$p^3y - \frac{3ppqdy}{dx} + \frac{3pqqddy}{dx^2} - \frac{q^3d^3y}{dx^3} = 0,$$

which on putting

$$y = e^{\frac{px}{q}}u,$$

is transformed into this one: $d^3u = 0$; consequently it yields $u = \alpha + \beta x + \gamma xx$, from which the particular value

$$y = e^{\frac{px}{q}}(\alpha + \beta x + \gamma xx).$$

will satisfy this proposed equation. In a similar way, if the algebraic equation

$$0 = A + Bz + Cz^2 + Dz^3 + \cdots + Nz^n$$

time. See, for example, the Primary Source Projects "Solving Linear First Order Differential Equations: Gottfried Leibniz' 'Intuition and Check' Method" (by the author of this project) and "Fourier's Heat Equation" (by Kenneth M Monks), both available at https://digitalcommons.ursinus.edu/triumphsdiffer/.

has fourth power divisor $(p - qz)^4$, then from this comes the particular satisfying equation

$$y = e^{\frac{px}{q}} \left(\alpha + \beta x + \gamma xx + \delta x^3\right).$$

And in general, if the divisor is $(p - qz)^k$, the resulting value will therefore be[21]

$$y = e^{\frac{px}{q}} \left(\alpha + \beta x + \gamma x^2 + \delta x^3 + \cdots + \xi x^{k-1}\right),$$

so that it involves k imaginary constants.[22]

Task 6 Consider the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$.

(a) What is the corresponding auxiliary equation?

(b) What are the roots of the auxiliary equation?

(c) What are two "complete" or "fundamental" or "linearly independent" solutions to the differential equation?

1.3 Complex Roots of the Auxiliary Equation

Euler began his discussion of the case where the auxiliary equation has complex roots by rewriting an irreducible quadratic in a different form. As suggested by his reference in §15 to "the quadrature of a circle," this form and the resulting solution of the differential equation both involve the use of trigonometric functions.[23]

§20 But having found the values of y that come from any simple divisors of the equation[24]

$$0 = A + Bz + Czz + Dz^3 + \cdots + Nz^n,$$

which are equal to each other, another difficulty remains for us to resolve, [namely] whether this equation has imaginary roots. However, it is well known that if a certain equation has imaginary roots, their number will always be even; also, elsewhere I have shown that these imaginary roots can always be viewed this way as pairs, conjugated two at a time, in such a ways that their sum and their product is a real quantity. Hence, instead of imaginary divisors there are produced composite divisors of two dimensions whose form

$$p - qz + rzz,$$

[21] In the original Latin publication, the following equation was mistakenly printed with kx^{k-1} as the last term of the polynomial. Euler later used ξ for this coefficient, and so we use it here as well.

[22] There are k constants because we started with αx^0.

[23] In this section, Euler's notation for these trigonometric functions has been modified slightly, as described at the start of Section 2.

[24] What follows assumes the coefficients in the auxiliary equation are real.

is real, and which have simple[25] imaginary divisors. Therefore, in such a composite divisor $qq < 4pr$; hence

$$\frac{q}{2\sqrt{pr}} < 1.$$

Therefore ... I assume that the cosine of some real angle, which shall be $= \varphi$, is $\frac{q}{2\sqrt{pr}}$; and so $q = 2\sqrt{pr}\cos(\varphi)$, from which the general form of the compound imaginary divisors which are therein contained will be thus: $p - 2z\sqrt{pr}\cos\varphi + rzz$.

Let's unpack what Euler was saying in this excerpt in the next two tasks.

Task 7 Euler's comment

> elsewhere I have shown that these imaginary roots can always be viewed this way as pairs, conjugated two at a time, in such a ways that their sum and their product is a real quantity

is tied to a fact we typically take for granted. Namely, if we factor the real quadratic

$$x^2 + bx + c = 0$$

into complex $(x - r_1)$ and $(x - r_2)$, then r_1 and r_2 are complex conjugates. We will show this in two ways.

(a) First calculate r_1 and r_2 by applying the quadratic formula to $x^2 + bx + c = 0$ to show that the roots are indeed of the form $\alpha \pm \beta i$.

(b) Secondly, show that if $x^2 + bx + c = (x - r_1)(x - r_2)$, then $b = -(r_1 + r_2)$ and $c = r_1 r_2$. In other words we know that the roots "can always be viewed this way as pairs, conjugated two at a time, in such a ways that their sum and their product is a real quantity."

(c) Finally if $r_1 = \alpha + i\beta$ and $r_2 = \gamma + i\delta$ and their sum and products are real, show that $\alpha = \gamma$ and $\beta = -\delta$, i.e. that r_1 and r_2 are complex conjugates.

Task 8 Euler began with the quadratic equation $rz^2 - qz + p$. If this quadratic has complex roots, what must be true about the discriminant $q^2 - 4pr$? Use this to derive

$$q < 2\sqrt{pr}.$$

Explain how this relates to Euler's "assumption" that $\cos\varphi = \frac{q}{2\sqrt{pr}}$ for some real angle φ. Then solve this last equation for q to obtain Euler's final form for the quadratic: $rz^2 - 2z\sqrt{pr}\cos\varphi + p$.

[25] "Simple" here means multiplicity one. Also, Euler is assuming that p, q, r are all positive real numbers.

In §21 and §22, Euler used the ideas of §20 to derive two different forms of the solutions. Euler himself preferred the form given in §21.[26] However, his version from §22 is much closer to the modern version, and so we follow that derivation.

§22 The same or an equivalent expression for y [as that found in §21] is derived from the simple but imaginary factors of the equation

$$0 = p - 2z\sqrt{pr}\cos\varphi + rzz,$$

which, on putting $f = \sqrt{\frac{p}{r}}$, transforms into

$$0 = ff - 2fz\cos\varphi + zz,$$

whose roots are

$$z = f\cos\varphi \pm f\sqrt{-1}\sin\varphi. \tag{3}$$

Hence, for y they yield the values

$$e^{fx\cos\varphi + fx\sqrt{-1}\sin\varphi} \quad \text{and} \quad e^{fx\cos\varphi - fx\sqrt{-1}\sin\varphi},$$

which when combined becomes

$$y = e^{fx\cos\varphi}\left(\eta e^{fx\sqrt{-1}\sin\varphi} + \theta e^{fx\sqrt{-1}\sin\varphi}\right).$$

Task 9 Euler quickly made three claims we should verify. We assume we've rewritten our irreducible quadratic in the form of §20:

$$0 = p - 2z\sqrt{pr}\cos\varphi + rzz.$$

(a) Make this equation monic[27] and make the change $f = \sqrt{\frac{p}{r}}$ to derive the equivalent form

$$0 = ff - 2fz\cos\varphi + zz.$$

(b) Apply the quadratic equation to the previous quadratic to derive Euler's roots (3).

(c) Euler then stated two solutions to the differential equation. He used the fact that an arbitrary linear combination of solutions is again a solution. In §15 presented above, he referred to this as the "aggregate" of the solutions. Show that

$$\eta e^{fx\cos\varphi + fx\sqrt{-1}\sin\varphi} + \theta e^{fx\cos\varphi - fx\sqrt{-1}\sin\varphi}$$
$$= e^{fx\cos\varphi}\left(\eta e^{fx\sqrt{-1}\sin\varphi} + \theta e^{-fx\sqrt{-1}\sin\varphi}\right).$$

[26] Euler wrote, "But that transformation seems to be most convenient in which the values of y are reduced to the form found in §21" [Euler, 1743, p. 210].

[27] This means that the coefficient on the highest power of z is 1. In other words divide by r.

Still at issue is that these are *complex* solutions to a *real* differential equation and we would like to have *real* solutions (as mentioned in Euler's statement of Problem I). Euler gave more guidance here. In what follows, it is a bit confusing to unravel his use of η, θ, α and β. What is important is that we have derived two solutions to the differential equation and any combination of those solutions is again a solution. By carefully choosing the combination, we can assure that the new solution is real. In fact, we can do it twice and introduce two arbitrary constants as expected.

And converting these exponentials into series, there results

$$y = e^{fx\cos\varphi}\begin{cases}(\eta+\theta)\left(1-\frac{ffxx\sin^2\varphi}{1\cdot 2}+\frac{f^4x^4\sin^4\varphi}{1\cdot 2\cdot 3\cdot 4}+\text{etc.}\right)\\(\eta-\theta)\sqrt{-1}\left(fx\sin\varphi-\frac{f^3x^3\sin^3\varphi}{1\cdot 2\cdot 3}+\text{etc.}\right)\end{cases}$$

Therefore, putting

$$\eta+\theta=\alpha \qquad \text{and} \qquad (\eta-\theta)\sqrt{-1}=\beta$$

and summing these infinite series yields

$$y = e^{fx\cos\varphi}\left(\alpha\cos fx\sin\varphi+\beta\sin fx\sin\varphi\right).$$

which expression easily reduces to the first.

Let's convert "these exponentials into series."

This will require the Taylor expansions for e^x, $\cos x$ and $\sin x$. On the off chance that you've forgotten them, they are:

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}, \qquad \cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}, \qquad \sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

By substitution,

$$e^{fxi\sin\varphi}=\sum_{n=0}^{\infty}\frac{(fxi\sin\varphi)^n}{n!}.$$

Since the powers of i cycle as $1\to i\to(-1)\to(-i)\to(1)\to\dots$, we have

$$e^{fxi\sin\varphi}=1+i\frac{fx\sin\varphi}{1!}-\frac{ffxx\sin^2\varphi}{2!}-i\frac{f^3x^3\sin^3\varphi}{3!}+\frac{f^4x^4\sin^4\varphi}{4!}+\dots.$$

Task 10 (a) In a similar way, derive

$$e^{-fxi\sin\varphi}=1-i\frac{fx\sin\varphi}{1!}-\frac{ffxx\sin^2\varphi}{2!}+i\frac{f^3x^3\sin^3\varphi}{3!}+\frac{f^4x^4\sin^4\varphi}{4!}+\dots.$$

(b) Show that $\eta = \theta = \frac{1}{2}$ gives the purely real function:

$$\frac{1}{2}e^{fxi\sin\varphi} + \frac{1}{2}e^{-fxi\sin\varphi} = \left(1 - \frac{ffxx\sin^2\varphi}{1\cdot 2} + \frac{f^4x^4\sin^4\varphi}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}\right)$$
$$= \cos(fx\sin\varphi).$$

(c) Show that $\eta = \frac{-i}{2}$ and $\theta = \frac{i}{2}$ give another purely real function:

$$\left(\frac{-i}{2}e^{fxi\sin\varphi} + \frac{i}{2}e^{-fxi\sin\varphi}\right) = \left(fx\sin\varphi - \frac{f^3x^3\sin^3\varphi}{1\cdot 2\cdot 3} + \text{etc.}\right)$$
$$= \sin(fx\sin\varphi).$$

Thus we have shown that two real solutions to the differential equation are

$$e^{fx\cos\varphi}\cos(fx\sin\varphi) \qquad \text{and} \qquad e^{fx\cos\varphi}\sin(fx\sin\varphi).$$

We now show that this answer is the same as the one from our text.

Task 11 The modern method considers a quadratic

$$az^2 + bz + c = 0$$

with complex roots $m \pm in$. It then derives the two solutions

$$e^{mx}\cos nx \qquad \text{and} \qquad e^{mx}\sin nx.$$

(a) Write the roots $m \pm in$ in terms of the coefficients a, b, c.

(b) Then write the solutions in terms of a, b, c.

Task 12 Euler started with the quadratic

$$rz^2 - qz + p = 0$$

and after a few changes of variables (e.g., $f = \sqrt{\frac{p}{r}}$, $\cos\varphi = \frac{q}{2\sqrt{pr}}$), derived the roots (3) which led to the solutions

$$e^{fx\cos\varphi}\cos(fx\sin\varphi) \qquad \text{and} \qquad e^{fx\cos\varphi}\sin(fx\sin\varphi).$$

(a) Write the roots (3) in terms of the coefficients r, q, p.

(b) Now write the solutions in terms of these coefficients of r, q, p.

Task 13 Show that the solutions from Task 11 and Task 12 are the same.

Task 14 Consider the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$.

(a) What is the corresponding auxiliary equation?

(b) What are the roots of the auxiliary equation?

(c) What are two "complete" or "fundamental" or "linearly independent" solutions to the differential equation?

Sections §23–§26 dealt with the case of repeated complex roots for the auxiliary equation. We won't derive this, though the statement of the solution is found at the end of the next section of this project (Section 2) and an example is given that you can work through in Task 20.

2 Putting It All Together

We now return to §28 where Euler followed his statement of Problem I with a concise summary. If you look at his article or letters in either the original Latin or a faithful translation, you will see notation of the form $\cos A.\varphi$ and $\sin A.\varphi$. In these cases, A was not to be understood as a constant. Rather the A stood for the Latin word "Arcus," and the trigonometric functions that this notation represented for Euler and Bernoulli are equivalent to our current $\cos\varphi$ and $\sin\varphi$. In the previous section, the A has been omitted to make the text easier to read. To make things more authentic, we revert to the original notation for just this section.

Solution

One should write 1 in place of y, z in place of $\frac{dy}{dx}$, z^2 in place of $\frac{ddy}{dx^2}$; and in general z^k in place of $\frac{d^ky}{dx^k}$; consequently, the following algebraic equation of order n is formed:

$$0 = A + Bz + Cz^2 + Dz^3 + \cdots + Nz^n.$$

Then, find all the simple real divisors which are involved in this equation; and if it has imaginary divisors, take the real composite divisors for these, in which z has two dimensions, as imaginary factors in [conjugate] pairs always constitute one composite real factor. Then from each factor form particular values of y in the following way. From any simple factor of the form $f - z$ which is not equal to any other there comes the value

$$y = \alpha e^{fx}.$$

But the values of y must be jointly determined from any two or more factors which are identified as equal. For instance, from the factor $(f - z)^2$ comes [the value]

$$y = (\alpha + \beta x)e^{fx};$$

and from the factor $(f - z)^3$ comes [the value]

$$y = (\alpha + \beta x + \gamma xx)e^{fx};$$

and, in general, from the factor $(f-z)^k$ one deduces [the value]

$$y = e^{fx}(\alpha + \beta x + \gamma xx + \cdots + \xi x^{k-1}).$$

Should any composite factors be found, then if the algebraic equation has a factor

$$ff - 2fz \cos A.\varphi + zz,$$

which has no other factor equal to it, then the value arising from it will be

$$y = e^{fx \cos A.\varphi} \alpha \sin A.(fx \sin A.\varphi + \mathfrak{A}).$$

If the algebraic equation has two such factors which are equal, then it will be divisible by

$$(ff - 2fz \cos A.\varphi + zz)^2,$$

so from this quadratic divisor the following value:

$$y = \alpha e^{fx \cos A.\varphi} \sin A.(fx \sin A.\varphi + \mathfrak{A}) + \beta x e^{fx \cos A.\varphi} \sin A.(fx \sin A.\varphi + \mathfrak{B}).$$

Moreover, if any power

$$(ff - 2fz \cos A.\varphi + zz)^k,$$

of such a factor is a divisor of the algebraic equation, then from this arises the following value:

$$\begin{aligned} y &= \alpha e^{fx \cos A.\varphi} \sin A.(fx \sin A.\varphi + \mathfrak{A}) + \beta x e^{fx \cos A.\varphi} \sin A.(fx \sin A.\varphi + \mathfrak{B}) \\ &+ \gamma x^2 e^{fx \cos A.\varphi} \sin A.(fx \sin A.\varphi + \mathfrak{C}) + \delta x^3 e^{fx \cos A.\varphi} \sin A.(fx \sin A.\varphi + \mathfrak{D}) \\ &+ \cdots + \xi x^{k-1} e^{fx \cos A.\varphi} \sin A.(fx \sin A.\varphi + \mathfrak{E}). \end{aligned}$$

And having found in this way the respective values of y from each divisor of the algebraic equation, nothing remains but that all these values be collected into a single sum, whereby the complete value of y is produced; moreover, it is the very one which would have been produced had the propounded differential equation of order n been integrated n times.

If one were to rewrite the complex part of this solution to mimic the modern solution, it would read as follows:

Concerning composite factors, if that algebraic equation has the irreducible factor

$$azz + bz + c,$$

with roots $m \pm in$, the values which must arise from it will be

$$y = e^{mx} \cos nx \qquad \text{and} \qquad y = e^{mx} \sin nx$$

If the algebraic equation has two equal factors of this kind such that it is divisible by

$$(az^2 + bz + c)^2$$

then from this quadratic divisor the following value results

$$y = \alpha e^{mx} \cos nx + \beta e^{mx} \sin nx + \gamma x e^{mx} \cos nx + \delta x e^{mx} \sin nx$$

But if any arbitrary power of this factor, say

$$(az^2 + bz + c)^k$$

was a divisor of the algebraic equation, then from it the following value results

$$\begin{aligned} y &= \alpha e^{mx} \cos nx + \beta e^{mx} \sin nx + \gamma x e^{mx} \cos nx + \delta x e^{mx} \sin nx \\ &\quad + \epsilon x^2 e^{mx} \cos nx + \zeta x^2 e^{mx} \sin nx + \eta x^3 e^{mx} \cos nx + \theta x^3 e^{mx} \sin nx \\ &\quad + \cdots + \psi x^{2k-1} e^{mx} \cos nx + \omega x^{2k-1} e^{mx} \sin nx \end{aligned}$$

Task 15 Summarize Euler's solution method in your own words. You need to understand all the cases (except repeated complex roots) for the next section.

3 Examples

We now travel back in time once more, and return to the letter exchange between Euler and Bernoulli. In Euler's 1739 letter, he gave the following "suitable" example [Eneström, 1905, p. 38].

Let the following be taken as a suitable example[28]

$$y\,dx^4 = K^4\,d^4y \qquad \text{or} \qquad y - \frac{K^4\,d^4y}{dx^4} = 0;$$

Task 16
(a) What is the corresponding auxiliary equation for this differential equation?
(b) Factor the auxiliary equation completely.
(c) What are the four solutions to the differential equation?

[28]This particular differential equation arises in the study of the vibration of an elastic beam with one end fixed to a wall. Indeed, on May 4, 1735, Daniel Bernoulli (the son of Johann) wrote to Euler, "For the curve [of the vibrating elastic lamina] I find the equation $nd^4y = ydx^4$ ... but this matter is very slippery" (as quoted in [Cannon and Dostrovsky, 1981, p. 70]). At that time, Euler reported that he was only able to solve this equation in series form. Some four years later, the method for finding a closed-form solution that he shared with Johann shows that Euler had resolved the slippery matter.

You can check your answer against Euler's solution in that same letter:

... this gives rise to the algebraic expression $1 - K^4 p^4$, whose real factors are these three $1 - K p, 1 + K p, 1 + K^2 p^2$; and from these spring the integrals of the equation

$$y = Ce^{\frac{-x}{K}} + De^{\frac{x}{K}} + E \sin \frac{x}{K} + F \cos \frac{x}{K}$$

in which expression, because a four-fold integration has been done in one operation, there are four new constants as the nature of the integration demands. If it would please you, Most Excellent Sir, I shall write down the method of proof on another occasion.

Perhaps Euler wished he had used an intermediary when he received Bernoulli's response letter, because Bernoulli claimed priority in this discovery [Eneström, 1905, p. 40].

I recall that, many years ago, I had discovered something similar which I noted in the work of my adversaries, but I do not have time now to search for it.[29] From the brief sketch that is here, which, one may add, lacks a demonstration,[30] I fully conclude that you have had the opportunity to meditate on these things.[31]

Of course, Johann was a brilliant mathematician in his own right. However, in his response, Bernoulli described his slightly different method and admitted that it wouldn't solve Euler's proposed equation. The reason is that he only found one root to the characteristic equation [Eneström, 1905, p. 40]:

The example you give of a fourth order differential equation,

$$ydx^4 = K^4 d^4 y, \qquad \text{or} \qquad y - \frac{K^4 d^4 y}{dx^4} = 0$$

is most easily solved this way. For if the letters a, b, c were removed and you set $d = -K^4$, you would have an equation of the fourth dimension, but not affected,

$$p^4 - K^4 = 0, \qquad \text{or} \qquad p = K.$$

... I confess that at the moment I can exhibit in this way only one example of such a logarithmic curve, whereas you have found the many curves

$$y = Ce^{\frac{-x}{K}} + De^{\frac{x}{K}} + E \sin \frac{x}{K} + F \cos \frac{x}{K}.$$

[29] Bernoulli's way of saying: I've known this for years, but can't prove it.

[30] Bernoulli's way of saying: I'm criticizing your rigor, even though in Euler's previous letter he offered, "At some other time, Most Excellent Sir, I shall write up a demonstration of this method, if you would like."

[31] Bernoulli's way of saying: I do believe you've at least thought about this.

Bernoulli then proposed a question back to Euler [Eneström, 1905, p. 41].

I also acknowledge that if the proposed equation were

$$y + \frac{K^4 d^4 y}{dx^4} = 0,$$

my logarithmic curve would be impossible or imaginary; But the same should also apply to your solution, and even more universally, for with you it should happen when k is impossible.[32]

Task 17 Let's follow Euler's method applied to Bernoulli's example. First, note that the proposed differential equation leads to the auxiliary equation $K^4z^4 + 1 = 0$.

(a) This quartic can be factored into two quadratics

$$K^2z^2 + Mz + 1 \qquad \text{and} \qquad K^2z^2 + Nz + 1.$$

What are M and N?

(b) Using the quadratic formula, what are the roots of each of these irreducible quadratics?

(c) What, then, are the solutions to the fourth order differential equation proposed by Bernoulli?

You can check your answer against Euler's solution in the final 1740 letter [Eneström, 1905, p. 47]:

Being indeed led to this algebraic equation $p^4 + K^4 = 0$ which can be resolved into two equations of two real dimensions

$$p^2 + Kp\sqrt{2} + K^2 = 0 \qquad \text{and} \qquad p^2 - Kp\sqrt{2} + K^2 = 0,$$

whence I obtain the complete integral equation

$$y = Ce^{\frac{x}{K\sqrt{2}}} \sin\frac{x}{K\sqrt{2}} + De^{\frac{x}{K\sqrt{2}}} \cos\frac{x}{K\sqrt{2}} + Ee^{\frac{-x}{K\sqrt{2}}} \sin\frac{x}{K\sqrt{2}} + Fe^{\frac{-x}{K\sqrt{2}}} \cos\frac{x}{K\sqrt{2}},$$

this equation having four constants C, D, E and F it is obvious this equation is the complete integral.

[32]Impossible and imaginary are being used interchangeably in these passages. Remember, we will only be concerned with the case when the differential equation has all real coefficients.

Fast-forwarding a bit in time once more, let's work through some of the examples that Euler included in his published paper [Euler, 1743].

Task 18 Solve the differential equation given in Euler's Example 2.

Example 2

§30 To find the integral of this differential equation of third order

$$0 = y - \frac{3a^2ddy}{dx^2} + \frac{2a^3d^3y}{dx^3}.$$

Task 19 Solve the differential equation given in Euler's Example 3.

Example 3

§31 To find the integral of this differential equation of third order

$$0 = y - \frac{a^3d^3y}{dx^3}.$$

Euler's Examples 4 and 5 were exactly the the differential equations from Tasks 16 and 17, which he had discussed earlier with Bernoulli. His Example 6 (Task 20) results in a repeated irreducible quadratic, a case that we have thus far not explicitly considered. However, the modern solution of this case (based on Euler's description of it at the end of his §28 summary) is described at the end of our Section 2.

Task 20 Solve the differential equation given in Euler's Example 6.
Hint: $z + 1$ and $z^2 + z + 1$ are factors of the auxiliary equation. Long divide those factors out to get a quartic, and use the technique from Task 17 (a) to break the quartic into quadratics. Finally, apply the quadratic formula.

Example 6

§34 To find the integral of this differential equation of seventh order

$$0 = y + \frac{ddy}{dx^2} + \frac{d^3y}{dx^3} + \frac{d^4y}{dx^4} + \frac{d^5y}{dx^5} + \frac{d^7y}{dx^7}$$

Task 21 Solve the differential equation given in Euler's Example 7.
Hint: $z^2 + 1$ is a factor of the auxiliary equation.

Example 7

§35 To find the integral of this differential equation of eighth order

$$0 = \frac{d^3y}{dx^3} - \frac{3d^4y}{dx^4} + \frac{4d^5y}{dx^5} - \frac{4d^6y}{dx^6} + \frac{3d^7y}{dx^7} - \frac{d^8y}{dx^8}$$

Task 22 Solve the differential equation given in Euler's Example 8.

Example 8

§36 To find the integral of this differential equation of indefinite order

$$0 = \frac{d^ny}{dx^n}$$

4 Conclusion

It seems only fitting to allow Euler the last word about the method of solution for the higher order differential equations that we have studied in this project. We thus conclude with Euler's parting comment on the topic in his exchange with Bernoulli:

Thus, it appears that my method is distinguished from others, as it is characterized as not requiring me to take as many integrations as there are orders of differentiation, but only one, as it were, and I actually determine the complete integral of the equation and find a real one, which satisfies this differential equation of indefinite degree

$$0 = y + \frac{Ax\,dy}{dx} + \frac{Bx^2\,ddy}{dx^2} + \frac{Cx^3\,d^3y}{dx^3} + \frac{Dx^4\,d^4y}{dx^4} + \text{etc.}\ldots$$

5 Epilogue

A careful reader might notice that Euler referred to

$$y = Ce^{\frac{-x}{K}} + De^{\frac{x}{K}} + E \sin \frac{x}{K} + F \cos \frac{x}{K}.$$

as an "equation" while Bernoulli referred to the exact same expression as a "curve." This is indicative of an important shift happening in mathematics at the time.

For centuries, calculations such as areas, tangent lines, and arc lengths were strictly geometric constructions. As the American historian of science Carl Boyer (1906–1976) noted in his study of the history of calculus [Boyer, 1959, p. 58],

> There was in Greek geometry no idea of a curve as corresponding to a function, nor was there a satisfactory definition of a tangent in terms of the limit concept. There was therefore in the thought of Archimedes no anticipation of the realization that the geometrical notion of tangency is to be based upon the function concept . . . .

Indeed, this dependence on geometry continued even after François Viète (1540–1603) introduced the use of vowels for variables, and René Descartes (1596–1650) and Pierre de Fermat (1608–1665) independently introduced the use of algebraic equations as a means to represent and study geometric curves via analytic geometry.

It wasn't until Euler that functions became the central idea of calculus. Again quoting Boyer [Boyer, 1959, p. 243]:

> Most of his predecessors had considered the differential calculus as bound up with geometry, but Euler made the subject a formal theory of functions which had no need to revert to diagrams or geometrical conceptions. . . . Euler was the first mathematician to give prominence to the function concept and to make a systematic study and classification of all the elementary functions, along with their differentials and integrals.

Perhaps nowhere is this transition from geometry to analysis more obvious than for trigonometric functions. Around 1727, Euler wrote the unpublished treatise *Calculus Differentialis* (*Differential Calculus*) in which he classified all functions as either algebraic or transcendental [Yushkevich, 1983]. In it, Euler recognized exponential and logarithmic functions as transcendental—but made no mention at all of trigonometric functions [Katz, 1987, p. 316]. Thirty years later, Euler did include the trigonometric functions in his transcendental studies, and also acknowledged their importance,

> In addition to the logarithmic and exponential quantities there occurs in analysis a very important type of transcendental quantity, namely the sine, cosine, and tangent of angles, whose use is certainly most frequent. Therefore this type rightly merits, or rather demands, that a special calculus be given, whose invention in so far as the special signs and rules are comprised, the celebrated author of this dissertation is rightly to claim all for himself . . . .
>
> [Euler, 1760] as quoted in [Katz, 1987, p. 316].

So, what changed between 1727 and 1754? Why did Euler start to consider trigonometric functions in his class of transcendental functions? Here is what the historian of mathematics Victor Katz has written about these questions [Katz, 1987, p. 317].

> A consideration of Euler's papers before 1740 provides an answer. The trigonometric functions entered calculus via the study of differential equations. Not only did this study give the sine and cosine the status of "function" in our sense, and give them an equal status with the exponential and logarithmic functions, but it also provided the necessary uses for these functions. The study of differential equations was not just the cause of the sine and cosine functions entering calculus, however. It was Euler's knowledge of these functions which led him, I believe, to the development of the standard method of solving linear differential equations with constant coefficients. The remainder of this paper will be devoted to convincing the reader of the truth of these assertions.[33]

In other words, it was exactly the problem and passages we have been talking about that made the sine and cosine into the trigonometric functions that have since been learned by centuries of high school and college math students!

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[33] I encourage interested students and instructors to indeed continue to read Katz's 1987 paper to be convinced.

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Notes to Instructors

This set of notes accompanies the Primary Source Project "Leonhard Euler and Johann Bernoulli Solving Homogenous Higher Order Linear Differential Equations With Constant Coefficients" written as part of the TRIUMPHS project. (See the end of these notes for details about TRIUMPHS.)

PSP Content: Topics and Goals

This Primary Source Project (PSP) is appropriate for any undergraduate Ordinary Differential Equations (ODE) course. The first general type of higher order differential equations solved in such a course are homogenous, linear, and with constant coefficients. In other words, equations that can be put in the form

$$0 = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + \cdots + N\frac{d^ny}{dx^n}.$$

It turns out that, historically, this was also the first class to be solved. Doing so involves factoring polynomials and it is difficult to make the quadratic formula exciting. Having students work through what amounts to the textbook or modern derivation can help. Euler and Bernoulli are obviously titans of mathematics and seeing their technique rise from correspondence to publication to what today's students see in class can be exciting, as can doing examples that we know they did.

After working through this project, we hope students can solve any homogenous higher order linear differential equations with constant coefficients (with reasonable auxiliary equation). This includes cases of distinct real roots, repeated roots of arbitrary multiplicity, and complex roots. While the PSP doesn't go into detail on repeated complex roots, the rule is presented along with an example for those colleagues that do cover that topic.

The author would like to thank the readers, editors, and testers of this PSP who made me aware that not only is this problem and passage interesting as a modern solution to a type of differential equation, but it also lies at an important historical moment where Euler was pushing analysis away from a geometric construct to one that involved functions. Prior to this problem, Euler did not consider trigonometry as a study of functions but afterwards he absolutely did. While decades passed between these points, Katz [1987] argues it is *exactly* this problem that facilitated the change in perspective. Information about this is included in an Epilogue after the Conclusion.

This is a fairly straightforward PSP. The topic isn't difficult and the historical documents mimic the modern derivation well (with the exception of the complex case). As noted by one instructor who typically uses PSPs for *enriching* previously introduced content, this particular PSP can thus be fruitfully used for *introducing* the content in question. Indeed, like most PSPs in the TRIUMPHS collection, it has been designed to be used in that way.

Student Prerequisites

This topic is standard in any ODE course. And it requires little background. Elementary factoring techniques along with very basic ODE notation and definitions are all that is necessary.

PSP Design and Task Commentary

The preamble has no tasks and can be assigned before class. It does serve a purpose and probably shouldn't be skipped entirely. The organization of the PSP is somewhat complicated as the primary sources are spread across three letters and a publication. And, we don't follow the order in publication. The preamble is designed to help clarify that.

Section 1 works through [Euler, 1743]. Task 2 is interesting even if not particularly appropriate for formal homework write-up. It connects the correspondence to the publication (or rather shows why we didn't follow the original letters). It can lead to robust classroom discussion but can also be omitted without interrupting the flow of the project. Three subsections are designated for different ways the auxiliary equation might factor: distinct real roots (Subsection 1.1), repeated roots of arbitrary multiplicity (Subsection 1.2), and complex roots (Subsection 1.3). Euler's presentation mimics our modern derivation for distinct and repeated real roots, but is not the same for the complex case. Rather than simply factoring $rz^2 - qz + p = 0$, he changed variables to rewrite the quadratic as $z^2 - 2z\sqrt{pr}\cos\varphi + ff = 0$ before he found the roots. Because of this, his preferred form of the solutions (from §21) doesn't resemble what we currently teach. The form he derived in §22 is closer, and Tasks 11–13 show this. Instructors should spend time deciding how to cover Subsection 1.3.

While Euler discussed the case of repeated complex roots, I don't cover that case in much detail in this project. Specifically the formula for complex roots is presented in Section 2 and Task 20. However, I skip over the derivation in sections §23–§26 from [Euler, 1743]. Instructors could expand or ignore this depending on their curriculum.

A "modern" example is available in each subsection of Section 1 with seven additional historical examples in Section 3. Most of those additional historical examples include unspecified constants such as a or K. This is sometimes confusing to students. An instructor might consider replacing them with numbers. For example, in Task 16 asking

$$y - 16\frac{d^4y}{dx^4} = 0 \qquad \text{instead of} \qquad y - \frac{K^4\,d^4y}{dx^4} = 0$$

or in Task 18 asking

$$0 = y - \frac{3ddy}{dx^2} + \frac{2d^3y}{dx^3} \qquad \text{instead of} \qquad 0 = y - \frac{3a^2ddy}{dx^2} + \frac{2a^3d^3y}{dx^3}.$$

Suggestions for Classroom Implementation

Please see the "PSP Design and Task Commentary" section above and the "Sample Implementation Schedule" below for suggestions.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Sample Implementation Schedule (based on a 50- or 75-minute class period)

This project is a doable activity in either two 50-minute or one 75-minute class period, provided instructors are cautious about the amount of in-class time spent working on the complex root case in Subsection 1.3.

If using two 50-minute class periods, the preamble along with Section 1 and Tasks 1 and 3 should be assigned as advanced preparation. I begin Day 1 class with a discussion of that material. Students should then complete Subsections 1.1 (possibly skipping Task 2) and 1.2 in groups, and begin Subsection 1.3. I would leave enough time at the end of the first class day for students to read Section 2. In that reading, students should recognize what they derived in Subsections 1.1 and 1.2 and the instructor should highlight the *modern* formula for complex roots. (Alternatively, Section 2 could be assigned as advanced reading to prepare for Day 2, which would start with a brief whole-class discussion of the modern formulation.) Much of Day 2 consists of groups working through Subsection 1.3, where students will derive Euler's version of the complex root case and show it is equivalent to the modern one from Subsection 2. Then, groups can complete as many examples from Section 3 as time allows with the remainder assigned as homework.

The important parts of this PSP can be covered in one 75-minute class period. The preamble along with Section 1 and Tasks 1 and 3 should still be advanced preparation. However, with less time, only the modern presentation of the complex case should be covered, and probably without the repeated complex case. The students could either just accept the modern formula in the PSP, or the instructor could derive it using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$. Again, the class would conclude with groups completing as many examples from Section 3 as time allows, with the remainder assigned as homework. Task 20 might be eliminated as it requires covering repeated complex roots. Alternatively, instructors who choose to have students complete Subsection 1.3 in-class should allow one additional class period under this scenario.

Please do read the above section "PSP Design and Task Commentary," as it contains notes about specific Tasks that can be modified or eliminated or expanded to suit your needs. The actual number of class periods spent on each section naturally depends on the instructor's goals and on how the PSP is actually implemented with students. This project is typically done in groups.

Connections to other Primary Source Projects

The following additional projects based on primary sources are also freely available for use in teaching standard topics in an ODE course. With the exception of the final project in the list (which requires up to 1 full week for implementation), each of these can be completed in 1–2 class days. The first three PSPs listed are designed as a series, but any one of them can be used independently or in conjunction with the other two. Classroom-ready versions of all projects in the list can be downloaded at `https://digitalcommons.ursinus.edu/triumphs_differ/`.

- Solving Linear First-Order Differential Equations: Gottfried Leibniz' Intuition and Check Method, by Adam E. Parker
- Solving Linear First-Order Differential Equations: Johann Bernoulli's (Almost) Variation of Parameters Method, by Adam E. Parker
- Solving Linear First-Order Differential Equations: Leonard Euler's Integrating Factor Method, by Adam E. Parker
- Fourier's Heat Equation, by Kenneth M Monks
- Wronskians and Linear Independence: A Theorem Misunderstood by Many, by Adam E. Parker *(Also suitable for use in Linear Algebra and Introduction to Proof courses.)*
- Runge-Kutta 4 (and Other Numerical Methods for ODEs), by Adam E. Parker

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For more information about TRIUMPHS, visit https://blogs.ursinus.edu/triumphs/.