Bhāskara's Approximation to and Mādhava's Series for Sine

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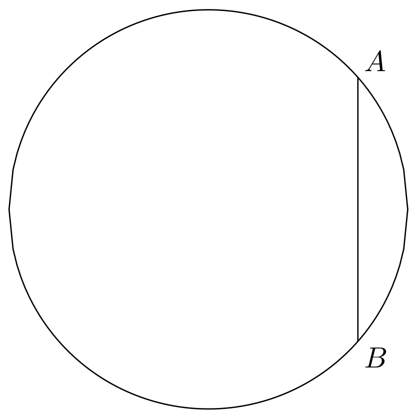
A *chord* is a very natural construction in geometry; it is the line segment obtained by connecting two points on a circle. Chords were studied extensively in ancient Greek geometry. For example, Euclid's *Elements* [Euclid, c. 300 BCE] contains plenty of theorems relating the circle's arc AB to the line segment AB (shown below).[1]



Indian mathematicians, motivated by astronomy, were the first to specifically calculate values of half-chords instead [Gupta, 1967, p. 121]. This work led very directly to the function which we call *sine* today.

Task 1 **Half-chords and sine.** Suppose the circle above has radius 1. Mark the midpoint of the chord AB, and call it M. Mark the center of the circle with a point C, and draw the segments AC and MC. Explain why the half-chord AM is equal to the sine of the radian measure of $\angle ACM$. Explain furthermore how the radian measure of that angle relates to the length of a corresponding arc on the circle.

In this project, we first visit the incredibly accurate seventh-century approximation for sine given

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[1] One particularly important theorem in the *Elements* with regards to the discussion of today's trigonometry is Proposition 29 of Book III [Euclid, c. 300 BCE, Vol. II, p. 60]. It states that "In equal circles equal circumferences are subtended by equal straight lines." One could interpret this as a theorem about sine: if you compute the length of a chord (or half of that length) corresponding to some radian measure of an arc on the unit circle, it doesn't matter where on the circle that arc is chosen. We use this fact all the time in modern-day trigonometry courses when we choose a first-quadrant reference angle for a larger or negative angle in a sine calculation.

by Bhāskara I.[2] Second, we will see the fourteenth-century infinite series expansion for sine given by Mādhava, which is the standard power series formula still taught in calculus courses today.

1 Bhāskara's Approximation for Sine

Little is known about Bhāskara I (c. 600–c. 680) other than that he was most likely a Marathi[3] astronomer[4] who followed the religion today known as Hinduism (his first work opens with a verse in praise of the Hindu god Śiva) and that he wrote several important treatises and commentaries on astronomy and mathematics.[5] His academic writing was in Sanskrit, which served as the standard language for academic writing on the Indian subcontinent (much as Latin was the standard language for scholarship in Europe for many centuries, regardless of what someone's native tongue was).

1.1 Translating the Approximation.

In Bhāskara I's first work, now called *Mahābhāskarīya*, he gives a stunningly accurate approximation for the sine function. The original Sanskrit description of his sine approximation is shown below, taken from a publication by the historian of mathematics Radha Charan Gupta [Gupta, 1967, p. 122].

मख्यादि रहितं कर्म वक्ष्यते तत्समासतः ।
चक्रार्धांशक समूहाद्विशोध्या ये भुजांशका ।। १७ ।।
तत्छेष गुणिता द्विष्ठाः शोध्याः खाभ्रेषुखाब्धितः ।
चतुर्थांशेन शेषस्य द्विष्ठमन्त्य फलं हतम् ।। १८ ।।
बाहु कोट्योः फलं कृत्सनं क्रमोत्क्रम गुणस्य वा ।
लभ्यते चन्द्रतीक्ष्णांश्वोस्ताराणां वापि तत्त्वतः ।। १९ ।।

(***Mahābhāskarīya***, **VII, 17–19)**

[2] He is often referred to as Bhāskara I to disambiguate from another prominent Indian mathematician of the same name, now often called Bhāskara II (1114–1185) or Bhāskarācārya.

[3] The Marathi people is an ethnolinguistic group of India, primarily living in the state of Maharashtra. Today, the various dialects of Marathi are spoken by roughly 83 million people, mostly in the western part of India, making it the tenth most spoken language in the world [Office of the Registrar General, 2011, p. 7].

[4] An enormous amount of work in Indian mathematics was dedicated to sine calculations, and this work was primarily motivated by astronomy. In particular, sine was used to calculate a planet's true position relative to its mean position in circular orbital models. In most early Indian texts, the method given was to list a table of values for sine at 3.75 degree increments (for twenty-four values between 0 and 90 degrees), and then use linear interpolation for values not in the table. (An example of such a table, encoded in a stanza of Sanskrit verse, is explored in Daniel E. Otero's Primary Source Project "Varāhamihira and the Poetry of Sines" [Otero, 2021].) What is so interesting about Bhāskara's method is that it is a quadratic method rather than linear. For more detail on how and why these values were used in astronomy, see [Plofker, 2009, pp. 94–102].

[5] For example, see [Keller, 2006] for an English translation of one of Bhāskara's works.

Just below the original Sanskrit, Gupta also included an English translation of the passage, which he credited to another historian of Hindu mathematics and astronomy, Kripa Shankar Shukla (1918–2007). We include the relevant parts of the translated passage below. Note that one can take the word "bhuja" to mean "angle."

> Now I briefly state the rule ... Subtract the degrees of the bhuja from the degrees of half of a circle ... Then multiply the remainder by the degrees of the bhuja and put down the result at two places. At one place subtract the result from 40500. By one-fourth of the remainder (thus obtained) divide the result at the other place ...

Task 2 **Translating into Symbolic Algebra.**

While it was common to write out calculations in rhetorical algebra during Bhāskara's era, today we tend to write such formulas instead in symbolic algebra. Let x be the "bhuja", measured in degrees, as stated in Bhāskara's rule above. Translate his calculation into a formula involving x. (**Hint!** Do not read too much into the word "place." In this passage, it does not carry a particularly specific or technical meaning, like it would in a phrase such as "place value." Instead, think of it as simply a "place" on the page where you write something.)

Perhaps one surprising part of the formula above was the number 40,500. It appears a bit out of the blue! This particular value, coming up in the context of studying circles, looks mysterious in part because today we typically write our angles using radian measure rather than degrees. Let us do one last bit of translation.

Task 3 **Bhāskara's Formula for Sine in Modern Notation.**

Let x now represent the radian measure of an angle. Show that the approximation

$$\sin(x) \approx \frac{16x(\pi - x)}{5\pi^2 - 4x(\pi - x)}$$

is equivalent to the one you came up with in the previous task (with that ridiculous 40,500 in it) if one converts the constants in that formula from degrees to radians. In particular, take the formula you came up with in the above task and use the conversion $180^\circ = \pi$ radians. (**Hint!** The units on the number 40,500 should be (the rather unusual!) *degrees squared* in order to match the units of the term being subtracted from it.)

The version of Bhāskara's formula stated in the above task is the version we will work with for the rest of this project. For convenience, we give it the name

$$B(x) = \frac{16x(\pi - x)}{5\pi^2 - 4x(\pi - x)}.$$

Task 4 **Comparing the Approximate to the Exact.**

Plot both functions using a graphing utility: $\sin(x)$ and the approximation $B(x)$. How do the graphs compare? What is the worst the error ever is on the interval $[0, \pi]$? (For our purposes in this project, just a visual estimate of the error will suffice.) Note that it may be easier to see the error by plotting the difference of the two functions in question instead of each curve individually.

1.2 How Might Bhāskara Have Found That Approximation?

Whether stated in the original degree version given by Bhāskara (with that ridiculous 40,500 in it), or the radian version given by $B(x)$ (with the much smaller but no less mysterious constant $5\pi^2$ in it), the natural question to ask about this incredibly accurate approximation is: how would one ever come up with such a thing? Historians are unsure of how Bhāskara came upon this formula, as he only published the result and not his derivation. Here we present a plausible line of reasoning that a mathematician might follow today to come up with such a formula.

Suppose we want to approximate sine. Realistically, we don't gain anything by approximating it outside of the interval $[0, \pi]$ since any other value of sine could be computed via a reference angle.[6] Let us list two key properties that sine has on that interval:

1. The only zeroes of the function are 0 and π.
2. The function is symmetric across the vertical line $x = \pi/2$. That is to say, just as $\sin(x) = \sin(\pi - x)$, our approximation should also be the same upon substituting $\pi - x$ for x.

Thinking in this manner, a plausible first attempt at approximating sine via something algebraic could be

$$\sin(x) \approx x(\pi - x).$$

Task 5 **A First Approximation.**

(a) Verify this approximation satisfies the two properties listed above. (**Hint!** Notice the given approximation is a degree two polynomial, and thus represents the graph of a parabola. Think back to how one finds the axis of symmetry of a parabola!)

(b) How good of an approximation is this? Plot both and describe what is good about the approximation, as well is what is not so good about it. How large does the error get on that interval?

Task 6 **A Second Approximation.**

(a) As you probably suspect from your work in the previous problem, it seems it would be wise to perform a vertical compression to our approximation to get the heights closer. Scale it by whatever constant is needed to get the y-value correct at $\pi/2$. That is, find a real number a such that the approximation $ax(\pi - x)$ has the correct y-value at $x = \pi/2$.

(b) How does the new approximation $ax(\pi - x)$ compare? How large does the error get using the new approximation?

[6] See footnote 1!

One way we could obtain an even better approximation would be to scale by different amounts at different parts of the interval, rather than just scaling by a constant factor across the whole interval. In interest of preserving symmetry across $x = \pi/2$, we'll scale by something of the form $b + cx(\pi - x)$. Thus, the form of our approximation will be

$$\sin(x) \approx \frac{ax(\pi - x)}{b + cx(\pi - x)}$$

for some real numbers a, b, and c.

Task 7 **Getting Rid of an Unknown.**

(a) Why can we assume $b \neq 0$?

(b) Since b is nonzero, we can divide the top and bottom of the fraction by b, and then rename the unknowns a and c. More specifically, think of the equation

$$\frac{ax(\pi - x)}{b + cx(\pi - x)} = \frac{Ax(\pi - x)}{1 + Cx(\pi - x)},$$

in which we have forced b to equal 1 by dividing the top and bottom by it. What would A and C be in terms of the original a, b, c to make that equation work?

It should be noted that we are certainly using more symbolic algebra than was available in seventh-century India.[7] The spirit of this approach—guessing something reasonable and then adjusting the guess—is a standard mathematical problem-solving strategy widely used today. Observe how similar the method above is to what we do when we find a power series for a function: one sets up a form of an unknown power series and then uses some facts about the function (namely values of the function and its derivatives at the center of the power series) to solve for its coefficients.

Task 8 **Set Up a Guess, then Adjust and/or Solve!**

Can you think of another process in your mathematics coursework that takes that same approach, namely, one sets up the form of a guess and then uses some information to improve the guess and/or solve for unknown coefficients?

At this point, all that remains is to figure out what the values of the unknowns A and C from the above task should be. Wouldn't it be nice if we had two equations that involved these two unknowns? To see where such equations might come from, we visit another primary source to see what other information about the sine function would have been likely used by Bhāskara. The passage below is from the *Āryabhaṭīya*, the only surviving work of the fifth-century Indian mathematician Āryabhaṭa (476–550). Bhāskara would have been well aware of the contents of the *Āryabhaṭīya*, as he wrote a commentary on it.[8]

[7] See [Van Brummelen, 2009, p. 104] for a possible derivation method that Bhāskara himself could have used, based on geometry rather than algebra.

[8] An English translation of Bhāskara's commentary appears in [Keller, 2006].

The chord of a sixth part of the circumference, that is equal to the semi-diameter.

Task 9 **A Friendly Value of Sine.**

Draw a circle of radius 1. On the circle, mark out a "sixth part of the circumference" as well as the corresponding "chord." Use your diagram to explain why Āryabhaṭa's claim is equivalent to the calculation of the exact value of $\sin(\pi/6)$.

We now use the values of $\sin(x)$ at $x = \pi/6$ and $x = \pi/2$ (also certainly a value that Bhāskara would have had at his disposal since at $x = \pi/2$ the half-chord is just the radius of the circle) to solve for our unknown constants A and C.

Task 10 **Solving for the Unknowns.**

(a) Substitute each of $x = \pi/2$ and $x = \pi/6$ into $\frac{Ax(\pi-x)}{1+Cx(\pi-x)}$ and set the resulting expression equal to the corresponding value of sine. This will produce a system of two equations in the unknowns A and C, for which one can then solve. Do that!

(b) Simplify your expression with the numerical values for A and C plugged in to obtain Bhāskara's formula!

2 Mādhava's Sine Series

The more familiar power series formula for the sine function taught in second-semester Calculus courses today has been attributed to Mādhava of Saṇgamagrāma[9] (c. 1350–c. 1425) by descendants in his mathematical family tree. Though there are no surviving mathematics writings from Mādhava's own hand, the Kerala school[10] astronomer Keḻallur Nīlakaṇṭha Somayājī (1444–1544) published Mādhava's sine series[11] in 1501 in the *Tantrasaṅgraha* [Ramasubramanian and Sriram, 2011]. Below, we show an English translation from A K Bag [Bag, 1976].

Multiply the arc by the square of itself (multiplication being repeated any number of times) and divide the result by the product of the square of even numbers increased by that number and square of the radius (the multiplication being repeated the same number of times). The arc and the results obtained from above are placed one below the other and are subtracted systematically from one it's above. These together give the chord[12] . . .

[9] Now known as Irinjalakuda, Saṇgamagrāma is a town on the southwest coast of India, near Kochi in the Kerala state.

[10] Mādhava taught in a family compound called an *illam*, where a small group of students memorized his findings in verse and passed along these verses to future generations.

[11] This series enabled Somayājī to produce an incredibly accurate (at least 7 places past the decimal) table of 24 values for sine, as described above [Van Brummelen, 2009, p. 120]. The creation of this table might seem unusual from a modern perspective: once you have that infinite series, why do you need a table? Perhaps because it was already so common to work with a table of values in India, that the readers would be looking for one!

[12] Be aware that the passage contained the word for "chord" but the result gives the half-chord. Perhaps by that time it had become standard enough to seek half-chords rather than chords that it was implied!

Task 11 **Translating to Modern Notation.**

Call the arc by the letter x. One phrase at a time, write out the process that is being described above, using x as the arc. Verify that it results in a formula equivalent to our modern power series for sine (with the radius chosen to equal 1, since we take the convention of doing our sine calculations as half-chords of the unit circle[13] rather than an arbitrary circle). Place your translations in a table similar to the one shown below (with a few entries filled in to get you started).

Primary Source	Modern Formulation
the arc	x
the square of itself	
multiplication being repeated any number of times	$x, x \cdot x^2, x \cdot x^2 \cdot x^2, x \cdot x^2 \cdot x^2 \cdot x^2, \ldots$
square of even numbers increased by that number	$2^2+2, 4^2+4, 6^2+6, \ldots$
multiplication being repeated the same number of times	
divide the result	
results placed one below the other	
The arc and the results obtained from above... subtracted systematically from one it's above	

Just as with Bhāskara's work, this sine series was stated but it was not accompanied by an explanation of how the result was discovered. One interesting thing to note is how it was certainly *not* discovered: via iterated differentiation/Taylor series, as you likely would construct the formula

[13]Note that the choosing radius 1 in trigonometric calculations is a fairly modern convention; historically, sine tables were usually made based on much larger circles. Mādhava, for example, generated his sine table using a circle whose circumference was 21,600, which gives a radius of roughly 3438. (See [Van Brummelen, 2009, p. 120] for more details.)

in a calculus course![14] Even after the development of calculus techniques in 17th-century western Europe, the series given by Mādhava was often the starting point (not the ending point!) for determining other properties of the sine and cosine functions. For example, over three centuries later Euler showed how one could use Mādhava's series[15] to find the derivatives of sine and cosine in his work *Institutiones calculi differentialis* [Euler, 1755, §201]. See Dominic Klyve's Primary Source Project *The Derivatives of the Sine and Cosine Functions* for a detailed presentation of this work [Klyve, 2017].

3 Comparing the Methods

The advantage to Mādhava's series is that you can get arbitrary accuracy by taking more and more terms in the series. Bhāskara's formula does not have a way to adjust and improve the accuracy. What is absolutely astounding, though, is just how many terms you need in the infinite series to match the accuracy Bhāskara achieves!

Task 12 **Comparing the Methods.**

How many terms would you need in Mādhava's sine series to achieve the same accuracy as Bhāskara's formula on the interval $[0, \pi]$? Use Taylor's Error Theorem for power series approximations to verify your result.

Though it is amazing to see how many terms it takes to match Bhāskara's accuracy, one could argue that it is more appropriate to compare Bhāskara's approximation to a power series centered at $\pi/2$ rather than 0 (as Mādhava's was), since $B(x)$ was really built using symmetry about $\pi/2$.

Task 13 **Comparing to the Power Series Centered at $\pi/2$.**

Let us compare the function $\sin(x)$ to Bhāskara's approximation $B(x)$ by looking at their power series centered at $\pi/2$.

(a) Find the power series for sine centered at $\pi/2$ by using the sine angle sum identity. Specifically, add and subtract $\pi/2$:

$$\sin(x) = \sin\left((x - \pi/2) + \pi/2\right)$$

and then apply the sine angle-sum identity for $\sin(A+B)$ where $A = x-\pi/2$ and $B = \pi/2$.

(b) Use iterated derivatives to find the degree 2 power series for Bhāskara's approximation centered at $\pi/2$. That is, write

$$\frac{16x(\pi - x)}{5\pi^2 - 4x(\pi - x)} \approx a_0 + a_1(x - \pi/2) + a_2\left(x - \pi/2\right)^2$$

[14] See [Van Brummelen, 2009, pp. 114–119] for a derivation of the series given in a commentary written by one of Mādhava's students!

[15] One should note that there is currently an absence of clear evidence that Euler or others who used it extensively, such as Isaac Newton (1642–1727), learned the sine series from Mādhava or the Kerala school followers; it seems to have been discovered independently in Europe, albeit two centuries later.

and then find the coefficients a_0, a_1, and a_2 by repeatedly plugging in $x = \pi/2$ and then differentiating both sides. Note that the derivatives will be big ugly quotient rules; you are welcome to use a computer algebra system (Wolfram Alpha, Symbolab, etc.) to take those derivatives.

(c) How do the degree 0, degree 1, and degree 2 coefficients of the sine power series compare to the degree 0, degree 1, and degree 2 coefficients in the power series for Bhāskara's approximation?

4 Conclusion

Although power series is the standard framework taught in second-semester calculus for approximating a transcendental function via something algebraic (i.e., polynomials, roots, rational functions), one should be aware that it is far from the only way to do such a thing. We have seen here two prominent Indian mathematicians who both sought approximations for sine but whose techniques were drastically different from each other's!

Task 14 Come up with your own method for calculating an approximation to sine via polynomials, radicals, or rational functions! Compare the accuracy and usability of your method to the two methods we saw here from Bhāskara and Mādhava.

References

A K Bag. Mādhava's Sine and Cosine Series. *Indian Journal of History of Science*, 11:54–57, 1976.

Euclid. *Euclid: The Thirteen Books of the Elements*. c. 300 BCE. Translated with introduction and commentary by Sir Thomas L. Heath. Second edition unabridged (1925). 3 vols. Reprint, Dover Publications, Inc., 1956. ISBN 978-0-486-60089-5.

Leonhard Euler. *Institutiones calculi differentialis*. Teubner, 1755.

Radha Charan Gupta. Bhāskara I's Approximation to Sine. *Indian Journal of History of Science*, 2 (2):121–136, 1967.

Victor J Katz. *A History of Mathematics: An Introduction, Third Edition*. Pearson, 2009.

Agathe Keller. *Expounding the Mathematical Seed. Volume 1: A Translation of Bhāskara I on the Mathematical Chapter of the Āryabhaṭīya*. Birkhäuser Verlag, 2006.

Dominic Klyve. The Derivatives of the Sine and Cosine Functions: A Mini-Primary Source Project for Calculus I Students. *Convergence*, June 2017. URL www.maa.org/press/periodicals/convergence. Also available at digitalcommons.ursinus.edu/triumphs_calculus/1.

India Office of the Registrar General. Census of India, 2011. URL https://censusindia.gov.in/2011Census/C-16_25062018_NEW.pdf.

Daniel E. Otero. Teaching and Learning the Trigonometric Functions through Their Origins: Episode 4 - Varāhamihira and the Poetry of Sines. *Convergence*, March 2021. URL www.maa.org/press/periodicals/convergence. This project also appears as Section 4 of the Primary Source Project *A Genetic Context for Understanding the Trigonometric Functions*, available at https://digitalcommons.ursinus.edu/triumphs_precalc/1.

Kim Plofker. *Mathematics in India*. Princeton University Press, 2009. ISBN 978-0691120676.

K Ramasubramanian and M S Sriram. *Tantrasaṅgraha of Nīlakaṇṭha Somayājī*. Sources and Studies in the History of Mathematics and Physical Sciences. Hindustan Book Agency with Springer, 2011. ISBN 978-0-85729-035-9.

Glen Van Brummelen. *The Mathematics of the Heavens and the Earth: The Early History of Trigonometry*. Princeton University Press, 2009. ISBN 978-0-691-12973.

Notes to Instructors

PSP Content: Topics and Goals

This Primary Source Project (PSP) is intended to enrich an second-semester Calculus student's understanding of the process of approximating a transcendental function (sine) by an algebraic one (rational in the case of Bhāskara and polynomial in the case of Mādhava). The key competencies that come up in this project are as follows:

- Geometric definition of sine
- Power series manipulations
- Taylor's Error Theorem for power series approximations

Student Prerequisites

In this project, we assume the student has already been exposed to the basics of power series, including Taylor's Formula, Taylor's Error Theorem, and the power series for sine and cosine.

PSP Design, and Task Commentary

This PSP will expose the student to the first attempts at approximating sine by algebraic functions. The first section and corresponding primary sources aim to remind students that the sine function really does compute something very geometric (hence all the twiddling with chords and arcs). The second section ultimately presents the same formula that students see in a standard calculus textbook, but gives proper credit to Mādhava, who found it long before any Europeans. The second and third sections both aim to give the student practice with very standard second-semester Calculus power series techniques, but within the context of a historically important example as opposed to an arbitrary example constructed for sake of practice alone.

Note that Task 11 is a bit fiddly. The instructor may wish to give the following hint: rewrite n^2+n as $n(n+1)$. It will make it more clear how to follow Mādhava's instructions in order to recover the sine power series formula in its more familiar form.

Suggestions for Classroom Implementation

The author strongly suggests the instructor work through the entire project before using it in class. In particular, it is easy to make simple indexing errors when applying Taylor's Error Theorem.

The reading and tasks up to and including Section 1.1 make an ideal class preparation assignment. One could then start in-class work at the more complicated Section 1.2.

The author is happy to provide LaTeX code for this project. It was created using Overleaf which makes it convenient to copy and share projects and can allow instructors to adapt this project in whole or in part as they like for their course.

Sample Implementation Schedule (based on a 50-minute class period)

This following outline allows for completion of this project in two class periods. For the first class period, one could try the following:

- Assign students to read and complete tasks through the end of Section 1.1 as a class preparation assignment.
- Begin class with 10 minutes to have students share their observations and/or difficulties.
- Allow them to work through the PSP for the next 35 minutes in small groups as you and/or learning assistants assist.
- In the last 5 minutes, it is sometimes nice to call the students together to regroup for a brief discussion. See if anyone had common difficulties.

Class preparation for the second class period could involve completion of all of the Bhāskara portions of the project. The second class session could then have similar structure to the first, but with focus on the Mādhava series, as well as the comparison of the two methods.

Connections to other Primary Source Projects

The following additional projects based on primary sources are also freely available for use in teaching standard topics in the calculus sequence. The PSP author name of each is given (together with the general content focus, if this is not explicitly given in the project title). With the exception of the final two projects in the list (which require 4 and 6 days respectively for full implementation), each of these can be completed in 1–2 class days. Classroom-ready versions of these projects can be downloaded from `https://digitalcommons.ursinus.edu/triumphs_calculus`.

- Investigations Into d'Alembert's Definition of Limit (calculus version), Dave Ruch
- L'Hôpital's Rule, Danny Otero
- The Derivatives of the Sine and Cosine Functions,[16] Dominic Klyve
- Fermat's Method for Finding Maxima and Minima, Kenneth M Monks
- Beyond Riemann Sums: Fermat's Method of Integration, Dominic Klyve
- How to Calculate π: Buffon's Needle (calculus version), Dominic Klyve (integration by parts)
- Gaussian Guesswork: Elliptic Integrals and Integration by Substitution, Janet Heine Barnett
- Gaussian Guesswork: Polar Coordinates, Arc Length and the Lemniscate Curve, Janet Heine Barnett
- Gaussian Guesswork: Infinite Sequences and the Arithmetic-Geometric Mean, Janet Heine Barnett
- How to Calculate π: Machin's Inverse Tangents, Dominic Klyve (infinite series)
- Euler's Calculation of the Sum of the Reciprocals of Squares, Kenneth M Monks (infinite series)
- Fourier's Proof of the Irrationality of e, Kenneth M Monks (infinite series)
- Braess' Paradox in City Planning: An Application of Multivariable Optimization, Kenneth M Monks
- Stained Glass, Windmills and the Edge of the Universe: An Exploration of Green's Theorem, Abe Edwards
- The Radius of Curvature According to Christiaan Huygens, Jerry Lodder

[16] This PSP makes a great follow-up to the current project; it guides the reader through the work by Euler discussed in Section 2.

Recommendations for Further Reading

If an advanced student finishes quickly and wants a follow-up, there is an excellent article by V. N. Krishnachandran titled "Where Do the Terms of the Power Series Expansions of Sine and Cosine Functions Come from? Involutes!" which shows a beautiful geometric proof (due originally to a Russian school teacher, Y. S. Chaikovsky) of the sine power series. This is certainly outside the scope of this project, but it provides a perfect sneak peek into the ideas of vector calculus and parametric curves for the curious second-semester calculus student. Find it at arxiv.org/abs/1610.04825. For expositions on how Mādhava likely found his series that are more historically plausible than Chaikovsky's derivation, refer the student to [Van Brummelen, 2009, pp. 114–119] or [Katz, 2009, pp. 255–259].

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For more information about TRIUMPHS, visit http://blogs.ursinus.edu/triumphs/.