Solving First-Order Linear Differential Equations: Leonard Euler's Integrating Factor

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In 1926, British mathematician E. L. Ince (1891–1941) described the typical evolution of solution techniques from calculus (and differential equations and science in general).[1] He discussed this in [Ince, 1926, p. 529].

> The early history of the infinitesimal calculus abounds in instances of problems solved through the agency of what were virtually differential equations; it is even true to say that the problem of integration, which may be regarded as the solution of the simplest of all types of differential equations, was a practical problem even in the middle of the sixteenth century. Particular cases of the inverse problem of tangents, that is the problem of determining a curve whose tangents are subjected to a particular law, were successfully dealt with before the invention of the calculus.
>
> But the historical value of a science depends not upon the number of particular phenomena it can present but rather upon the power it has of coordinating diverse facts and subjecting them to one simple code.

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[1] Ince himself is part of at least one such story within differential equations. He developed the so-called *Ince Equation* (in about 1923),

$$(1 + a\cos(2t))y''(t) + (b\sin(2t))y'(t) + (\lambda + d\cos(2t))y(t) = 0,$$

which generalized at least two other well-known equations from about 1868 and 1914, respectively. Letting $a = b = 0$ and $d = -2q$, we obtain *Mathieu's equation* (which models elliptical drumheads),

$$y''(t) + (\lambda - 2q\cos(2t))y(t) = 0,$$

and letting $a = 0, b = -4q$, and $d = 4q(\nu - 1)$, we obtain the *Whittaker-Hill equation* (with applications to lunar stability and quantum mechanics),

$$y''(t) - 4q(\sin(2t))y'(t) + (\lambda + 4q(\nu - 1)\cos(2t))y(t) = 0.$$

Ince's equation then is itself a special case of *generalized Ince equations* (studied in [Moussa, 2014]),

$$(1 + \epsilon A(t))y''(t) + \epsilon B(t)y'(t) + (\lambda + \epsilon D(t))y(t) = 0.$$

This is exactly the evolution of solution methods for first-order linear ordinary differential equations. First, particular problems were solved with "one-off" methods that didn't have general applications beyond that specific problem. But then those results were combined and generalized until a unified theory developed.

Task 1 In the above passage, Ince made a connection between "the solution of the simplest of all types of differential equations" and "the problem of determining a curve whose tangents are subjected to a particular law." Connect these two statements. If the differential equation is

$$\frac{dy}{dx} = f(x, y),$$

then what is the "curve," the "tangents," and the "particular law"?

Task 2 Recall that non-homogenous first-order linear ordinary differential equations have the following form

$$p(x)\frac{dy}{dx} + q(x)y = f(x), \tag{1}$$

or if made monic

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{2}$$

Explain how to make Equation (1) monic like Equation (2). In particular why can we assume that $p(x)$ isn't identically zero? Write $P(x)$ and $Q(x)$ in terms of $p(x), q(x)$ and $f(x)$.

The theme of this project is a method due to Leonard Euler (1707–1883) in which a first-order linear differential equations is considered as being "almost" exact.[2] Prior to Euler's contribution, Gottfried Leibniz (1646–1716) published a paper in 1694 in which he solved first-order linear ordinary differential equations by intuiting a solution then checking that it worked.[3] That method wasn't general or part of a larger theory. But as Ince noted, scientists aim to combine diverse techniques into coordinated theories. Johann Bernoulli (1667–1748) did exactly this.[4] He considered first-order

[2]I think all that needs to be said about the influence of the Swiss mathematician Leonhard Euler compared to all the mathematicians in history is expressed by the following ordering:

10. You can't
9. Rank them
8. Because the
7. Importance of
6. Their contributions
5. Is
4. Relative to
3. Their respective
2. Fields
1. Leonhard Euler

[3]Leibniz was a German mathematician and philosopher who created (probably independently of Newton) the calculus along with the notation that we currently use for it.

[4]Johann Bernoulli was a very talented Swiss mathematician and third son of Niklaus. His unpleasant personality and desire for fame eventually ruined his relationships with both his brother Jacob, and his son, Daniel.

linear differential equations as special cases of what are now called "Bernoulli differential equations" and used a method similar to "variation of parameters" to solve them.[5] The work of Euler that we consider in this project made even greater strides in terms of presenting a unified theory.

1 Exactly What Background Do We Need?

Recall that a differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{3}$$

is *exact* if there exists an equation $f(x,y) = c$ with c a constant, such that the proposed differential equation is the total differential[6] of both sides of $f(x,y) = c$. In other words, there exists a function $f(x,y)$ such that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M(x,y)\,dx + N(x,y)\,dy = 0.$$

Under reasonable assumptions, a necessary and sufficient condition to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. When this condition holds, the process for finding f is well understood; indeed, Euler solved this as Problem I in his paper [Euler, 1763]. This process may also already be familiar to you if you have taken a multivariable calculus class that included conservative vector fields and their potentials. To see how it goes, we start with the left side of Equation (3). I stress that understanding the following *process* is far more important than memorizing the formula.

- First notice that $\frac{\partial f}{\partial x} = M(x,y)$. *Integrating* both sides with respect to x then gives

  $$f(x,y) = \int_x \frac{\partial f}{\partial x} = \int_x M + \text{ "constant" } = \int_x M + g(y)$$

  because the variable y is a constant when integrating with respect to x.

- Now notice that $\frac{\partial f}{\partial y} = N(x,y)$. *Differentiating* the previous equation with respect to y, gives

  $$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int_x M + \frac{dg}{dy} = N.$$

- Now *rearrange* the previous expression to isolate $\frac{dg}{dy}$. Indeed if the original differential equation was exact, $\frac{dg}{dy}$ will have no x's in it and we will *solve* for $g(y)$ by integrating with respect to y.

- Now that we have an expression for $g(y)$ we substitute into the first expression for $f(x,y)$ and thus get the solution:[7]

  $$f(x,y) = \int_x M + g(y) = c.$$

[5] The story of Leibniz's and Bernoulli's methods can be found in the two other projects in this "Solving First-Order Linear Differential Equations" series. They are also available in the MAA *Convergence* article of that same title (link in footer). Each of the three projects in the series can be completed individually or in any combination with the others.

[6] The *total differential* of a function $f(x,y)$ is $\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$, which immediately gives $dc = 0$.

[7] Remember when I stressed that understanding the *process* is far more important than memorizing the solution formula? I meant that.

This solution will be *implicit* and it may be impossible to solve for y to make it *explicit*. Thus, in order to check that your answer is correct you will need think back to implicit differentiation from Calculus 1.

Task 3 Here is Euler's Example 1 in [Euler, 1763]. Solve this equation using the above method, then check it with implicit differentiation.

EXAMPLE 1

§12. To integrate this differential equation

$$2axy\,dx + axx\,dy - y^3\,dx - 3xyy\,dy = 0.$$

Task 4 If you need additional practice, you can try Euler's Example 2 using the above method.

EXAMPLE 2

§13. To integrate this differential equation:

$$\frac{y\,dy + x\,dx - 2y\,dx}{(y-x)^2} = 0.$$

Even if the differential equation is not exact, it may become exact if we multiply it by an *integrating factor* $L(x, y)$. This is the first theorem in [Euler, 1763].

THEOREM

§16. If in the differential equation

$$M\,dx + N\,dy = 0$$

is not

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right)$$

always a multiplicator is given, multiplied by which the formula $M\,dx + N\,dy$ become integrable.

In other words $(LM)dx + (LN)dy = 0$ becomes exact. Equivalently, by the exactness test,

$$\frac{\partial(LM)}{\partial y} = \frac{\partial(LN)}{\partial x},$$

which means after applying the product rule that L solves the following partial differential equation (PDE). Again, I stress that it is far more important to understand the process than to memorize this formula.[8]

$$L\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) + M\frac{\partial L}{\partial y} - N\frac{\partial L}{\partial x} = 0. \tag{4}$$

Task 5 Euler assumed in the above theorem that the differential equation was not exact. If it were, then there is a very easy solution to Equation (4). What is that L? Keep in mind that if the differential equation were exact we'd just do the technique at the beginning of this section.

In general it is difficult to solve Equation (4), but if we can, we could make the initial differential equation exact and solve it using the above method. This general technique was first published by Alexis-Claude Clairaut (1713–1765) [Clairaut, 1739].[9] As is so often the case, Euler independently discovered the technique [Euler, 1740], which was published in 1740 though written in 1734.

2 Euler's Method (Not That Euler's Method)

In 1763, Leonard Euler published the paper "De integratione aequationum differentialium" [Euler, 1763].[10] In typical Eulerian form, the publication consists of Problems and Solutions. We are interested in his "Problema 4."[11]

Problem 4.

§34. Suppose the differential equation

$$P\,dx + Qy\,dx + R\,dy = 0$$

is proposed, where P, Q and R denote functions of x of any sort, and so that the other variable y has no more than one dimension; to find the factor which allows it to be integrated.

[8] Remember when I stressed that understanding the *process* is far more important than memorizing the solution formula? I meant that.

[9] Clairaut was an extremely talented French mathematician and astronomer. Unfortunately he never reached his potential as "he maintained an active social life" [Judson, 2000]. In 1734, he went to Basel where he studied with Johann Bernoulli. Euler wrote that Clairaut's work on the three body problem was "the most important and profound discovery that has ever been made in mathematics" (as quoted in [O'Connor and Robertson, 1998]).

[10] Or, in English, "On the Integration of Differential Equations." According to the Euler Archive (`https://scholarlycommons.pacific.edu/euler-works/269/`), Euler actually wrote this paper as early as 1755

[11] The translation of this excerpt from [Euler, 1763] was prepared by Daniel E. Otero, Xavier University, 2020. A minor adjustment to Euler's notation has been made by the project author in the interest of clarity for today's readers.

Solution.

Comparing this equation with the form $M\,dx + N\,dy = 0$ we get $M = P + Qy$ and $N = R$, whence

$$\left(\frac{dM}{dy}\right) = Q \qquad \text{and} \qquad \left(\frac{dN}{dx}\right) = \frac{dR}{dx}$$

Now let L stand for the required factor, and so $dL = pdx + qdy$, and whence it is necessary that it satisfy this equation:

$$\frac{Np - Mq}{L} = Q - \frac{dR}{dx} = \frac{Rp - (P + Qy)q}{L}. \tag{5}$$

Since now $Q - \frac{dR}{dx}$ is a function only of x we should also take for L a function only of x, so that $q = 0$ and $dL = p\,dx$; whence:

$$Q - \frac{dR}{dx} = \frac{Np}{L}, \qquad \text{or} \qquad Qdx - dR = \frac{RdL}{L},$$

and therefore, $\frac{dL}{L} = \frac{Qdx}{R} - \frac{dR}{R}$. Wherefore, integrating, we obtain $\log L = \int \frac{Qdx}{R} - \log R$, and assuming that e is the number whose hyperbolic logarithm is unity,[12] it yields

$$L = \frac{1}{R} e^{\int \frac{Q\,dx}{R}}.$$

In addition with this factor, the integral equation becomes:

$$\int \frac{P}{R} e^{\int \frac{Q\,dx}{R}}\,dx + ye^{\int \frac{Q\,dx}{R}} = \text{ Constant}$$

Let's examine this passage more closely.

Euler started with the differential equation

$$P\,dx + Qy\,dx + R\,dy = 0 \tag{6}$$

and then rewrote it in the starting form of every problem from this text

$$M\,dx + N\,dy = 0,$$ [13]

with the intent to apply the exactness test:

$$\frac{\partial M}{\partial y} \stackrel{?}{=} \frac{\partial N}{\partial x}.$$

[12] Notice that the notation of e is so new that Euler felt the need to explain it (and referred to "hyperbolic logarithm" instead of "natural logarithm"). The first use of the letter e to denote the constant $2.71828\ldots$ appeared in a letter from Euler to Goldbach in 1731 [O'Connor and Robertson, 2001].

[13] Indeed the very first sentence in the text is, "§1 Here, I consider differential equations of first degree, which involve only two variables and which therefore can be represented in this general form $M\,dx + N\,dy = 0$ if M and N denote any arbitrary function of the two variables x and y."

Task 6 Consider Euler's example $P\,dx + Qy\,dx + R\,dy = 0$.

(a) Verify his answers to the following questions. What is M? N? $\frac{\partial M}{\partial y}$? $\frac{\partial N}{\partial x}$?

(b) Is this differential equation exact in general? Under what very restrictive condition will it be exact?

Euler next defined the "required factor" L, which is today called an "integrating factor." He then immediately wrote $dL \equiv p\,dx + q\,dy$. This simply meant that

$$p = \frac{\partial L}{\partial x} \qquad \text{and} \qquad q = \frac{\partial L}{\partial y}.$$

Task 7 Use Euler's definitions of p and q to show that Equation (5) is just a restatement of Equation (4) with the values of Task 6.

As we noted earlier, solving Equation (4) is typically hard. But for first-order linear differential equations, Equation (5) is actually a separable ordinary differential equation (ODE) and can be solved. Euler arrived at this by noticing that L can be chosen to be "a function only of x."

Task 8 Let's unpack how Euler's observation about L leads to a solution of his original differential equation.

(a) As Euler asserted, L can be chosen as a function of only x. Explain why this means that $q = 0$.

(b) Following Euler, derive the formula for the integrating factor L.

(c) Using the method described in Euler's solution of his Problem 4 above, find a solution to Equation (6).

By construction, multiplication by L creates an exact differential equation. At this point in his solution method, Euler used exact techniques and integrated both sides of Equation (6).

Task 9 For what follows, it may be simpler to not use the explicit form of L but leave it as L and simply note that $\frac{dL}{dx} = \frac{L\left(Q - \frac{dR}{dx}\right)}{R}$ and $\frac{dL}{dy} = 0$, until the end of the problem.

(a) Verify that

$$(LM)dx + (LN)dy = 0$$

is exact for the values of L, M and N from Problem 4.

(b) Show the solution you found in Task 8 is equivalent to Euler's.

3 Examples

In this section, we will work through a couple of examples using Leibniz's technique.

Task 10 Consider the example

$$x\frac{dy}{dx} + y = 3x^2. \tag{7}$$

(a) Write Equation (7) in the form of

$$M\,dx + N\,dy = P\,dx + Qy\,dx + R\,dy = 0.$$

(b) What is Equation (5) for this example?

(c) Solve the above equation to find an integrating factor L. What would that indicate about our original differential equation?

(d) Verify that $(LM)\,dx + (LN)\,dy = 0$ is exact.

(e) Solve this exact differential equation.

Task 11 Solve

$$\frac{dy}{dx} - y = xe^x$$

using the method from the previous task.

4 Conclusion

We have come to the end of the story, and Ince predicted our path. At the beginning, Leibniz presented his results with no explanation and no generalization to other problems. But, soon Bernoulli wrote a solution in a bit more generality with more explanation. Then came Euler, who presented clearly the theory of integrating factors with lots of explanation and examples along the way.

It is likely that your ordinary differential equations class utilizes an integrating factor μ to solve first-order linear differential equations:

$$\frac{dy}{dx} + P(x)y = Q(x), \quad \mu = e^{\int P(x)\,dx} \tag{8}$$

Euler derived his integrating factor by noticing the PDE

$$M(x,y)\,dx + N(x,y)\,dy = 0, \quad L\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) + M\frac{\partial L}{\partial y} - N\frac{\partial L}{\partial x} = 0 \tag{9}$$

is a separable ODE for first-order linear differential equations. We conclude by showing that the factor L that Euler defined is equivalent to today's μ.

Task 12 Show that μ from Equation (8) is the same as Euler's L from Equation (9).

References

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Notes to Instructors

PSP Content: Topics and Goals

This Primary Source Project (PSP) is one of a set of three PSPs that share the name *Solving Linear First-Order Differential Equations*, designed to show three solution methods for non-homogenous first-order linear differential equations, each from a different context. Recall that a non-homogenous first-order linear differential equation has the form

$$a(x)\frac{dy}{dx} + p(x)y = q(x).$$

- The PSP subtitled *Gottfried Leibniz's "Intuition and Check" Method* explains how in 1694 Leibniz solved these equations using a one-off method applicable only to this specific problem. Strictly speaking, Leibniz didn't solve the equation, but rather asserted a solution and then showed it worked. Part of his proposed solution will be familiar to the students: it is the standard integrating factor method we teach today.

- The PSP subtitled *Johann Bernoulli's (Almost) Variation of Parameters Method* explains how in 1697 Bernoulli provided a method for solving Bernoulli differential equations that reduces to variation of parameters when applied to first-order linear equations. This was decades before Lagrange received credit for the technique. Again, part of Bernoulli's solution will be the standard integrating factor.

- The PSP subtitled *Leonard Euler's Integrating Factor Method* explains how in 1763 Euler solved these equations as a special case of exact differential equations by finding an integrating factor. His integrating factor is the same as the one that the students would have seen. This PSP is a bit longer than the others, and may require a bit more time or advance preparation.

All three of these PSPs are designed for use in a course on ordinary differential equations. While they work best after at least presenting the standard integrating method of solution found in modern textbooks, they can be used in three different ways at that point.

- Since the *type* of equation (first-order linear) has been introduced, all three projects can be done immediately after presenting the standard integrating method of solution. This would require the instructor to "preview" techniques (e.g., variation of parameters, exactness) that will be introduced more fully later. While this is somewhat awkward, it does mimic how these techniques were actually developed.

- The "Gottfried Leibniz's 'Intuition and Check' Method" project can be done immediately after presenting the standard integrating method of solution, and the other projects done after the respective technique (e.g., variation of parameters, exactness) is first introduced. Showing how those techniques can solve first-order linear differential equations makes a great first example of each technique. This is typically the way that I utilize these projects.

- Any one of the three projects can stand on their own as they don't necessarily build on the others. However, they do create a richer experience when used together in a course. Additionally, students gain confidence as they proceed through the three projects.

Student Prerequisites

This PSP requires some algebraic manipulation of differentials along with knowledge of partial differentiation and integration. The differential equation defining the integrating factor is separable, so knowledge of separable techniques would be helpful. Other techniques of integration needed are dictated by the examples used, where instructors can modify the project itself in order to substitute any example they wish in place of those included in the tasks. Knowledge of exact differential equations (or conservative vector fields) will make this project go much faster, though it is not necessary based on the background provided.

PSP Design and Task Commentary

This PSP consists of five sections.

- The initial (unnumbered) section contains a short introduction to what first-order linear differential equations are, along with a description of the way that mathematics often evolves. Mathematicians might first solve a specific problem using any tool at their disposal. They then attempt to see if they could find a class of problems (to which the initial one belongs) that can also be attacked using that technique. This closely mimics the evolution of how first-order linear differential equations were solved. Much of this section is the same for all three projects so if either of the first two has been covered, this section can be skipped.[14]

- Section 1 includes some background on exact differential equations, including
  - the definition of exact differential equations;
  - the test for exactness;
  - a discussion of how to solve exact differential equations; and
  - the PDE defining the integrating factor.
- Section 2 is devoted to Euler's method of solution. A translation is provided along with tasks to explain his method. The main take-away is that for a first-order linear differential equation, the PDE defining the integrating factor is actually a separable ODE that can be solved and therefore the proposed equation can be made exact. There are a fair amount of numbered equations which are referred to later which can be confusing. Thus, I always encourage students to learn the process and not the formulas. I do that all the time but I think it is especially helpful here.
- Section 3 consists of two tasks that prompt students to solve specific examples of first-order linear differential equations with Euler's method. The first is broken into steps that mimic the primary source, while the second requires the student to solve it on their own. These can be swapped with any examples you wish—in particular, so that the integrals utilize techniques with which your students are comfortable.

  One of the TRIUMPHS reviewers of this project correctly observed that these examples could be chosen better. They are not historical when there are plenty of primary source examples available. They may utilize techniques not known at the time. And, the first example given is already exact so that Euler's integrating factor is not necessary (though it still works). I am thus very open to suggestions on improving these examples.
- The final section concludes the three project series. Students are asked to show that the modern integrating factor μ can be found in Euler's method. If all three PSPs in the series are used, this is the third time that students are asked to do this, once for each of the three methods that are studied in these PSPs. However, Euler's is the only one that derives it as what we know as an integrating factor. The section does briefly reference the historical episodes that are treated in the first two projects so if you've not assigned them, just a word of clarification will be useful.

This project is typically done in groups. One site tester warned that, "The groups often want to take a divide and conquer approach, which is just utterly useless for these documents, because the only person who is going to make any progress is the person who is working on Intro/Section 1. These are all designed to be read top to bottom in slow careful detail, and the later parts of the PSP rarely make any sense unless you've seen the earlier parts."

[14] It is my hope to soon produce a PSP that uses primary sources to derive the equality of mixed partials, the test for exactness, gives the definitions of an exact differential equation and conserved vector field, and shows primary sources for the solution method that I've given. Much of this can be found in [Euler, 1740] and [Clairaut, 1739]; indeed references to specific sections and examples from [Euler, 1740] are provided in the present PSP. When produced, the planned PSP could be used to expand upon the cursory descriptions that are provided in this section.

Suggestions for Classroom Implementation

Please see the student prerequisites section above and implementation schedule below for suggestions.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Sample Implementation Schedule (based on a 50-minute class period)

The actual number of class periods spent on each section naturally depends on the instructor's goals and on how the PSP is actually implemented with students.

Depending on students' background, the PSP is a doable activity in one 50-minute class period. The introductory section (including Tasks 1–2) should be assigned as out-of-class advance preparation homework (or it could be skipped entirely if either of the previous projects has been assigned). If you have already covered a "modern" method for exact differential equations, then Section 1 can be skipped entirely. Sections 2 and 3 are required, as is encouraging students to understand process rather than memorizing the formulas. The task in the final section is the "point" of the series as it shows that Euler derived the modern integrating factor. But, that can be assigned as homework as well. To be safe completing this in 50 minutes, I would also drop Task 9.

If you have not covered a "modern" method for exact differential equations, then Section 1 is necessary. In this case, it may be best just to be safe and schedule one and half periods for implementation. Otherwise, it can be very rushed for students unfamiliar with exact differential equations or for whom Section 1 doesn't make sense. Again, the introductory section (including Tasks 1–2) should be assigned as out-of-class advance preparation homework, with Sections 2 and 3 forming the core of students' in-class work. If needed, I'll have students work more slowly through the first task in Section 3 and then leave the second example as homework. The task in the final section is once more the "point" of the series, but can be assigned as homework as well.

Connections to other Primary Source Projects

As described above, this mini-PSP is part of a series of three, all of which are intended for use in an Ordinary Differential Equations (ODE) course:

- *Solving Linear First-Order Differential Equations: Gottfried Leibniz's "Intuition and Check" Method*
- *Solving Linear First-Order Differential Equations: Johann Bernoulli's (Almost) Variation of Parameters Method*
- *Solving Linear First-Order Differential Equations: Leonard Euler's Integrating Factor Method*

The following additional projects based on primary sources are also freely available for use in teaching standard topics in an ODE course. Each of these can be completed in 1–2 class days, with the exception of the two projects marked by an asterisk (which require 3 and 4 days for implementation, respectively). Classroom-ready versions of all projects in the list can be downloaded at https://digitalcommons.ursinus.edu/triumphs_differ/.

- *Fourier's Heat Equation and the Birth of Fourier Series*,* Kenneth M Monks
  (This is an extended version of the PSP *Fourier's Heat Equation and the Birth of Modern Climate Science* listed below.)

- *Fourier's Heat Equation and the Birth of Modern Climate Science*, Kenneth M Monks
- *Leonhard Euler and Johann Bernoulli Solving Homogenous Higher Order Linear Differential Equations With Constant Coefficients*, by Adam E. Parker
- *Wronskians and Linear Independence: A Theorem Misunderstood by Many*, by Adam E. Parker (Also suitable for use in Linear Algebra and Introduction to Proof courses.)
- *Runge-Kutta 4 (and Other Numerical Methods for ODEs)*,* by Adam E. Parker

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For more information about TRIUMPHS, visit `https://blogs.ursinus.edu/triumphs/`.