Solving First-Order Linear Differential Equations: Gottfried Leibniz's "Intuition and Check" Method

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In 1926, British mathematician E. L. Ince (1891–1941) described the typical evolution of solution techniques from calculus (and differential equations and science in general).[1] He discussed this in [Ince, 1926, p. 529].

> The early history of the infinitesimal calculus abounds in instances of problems solved through the agency of what were virtually differential equations; it is even true to say that the problem of integration, which may be regarded as the solution of the simplest of all types of differential equations, was a practical problem even in the middle of the sixteenth century. Particular cases of the inverse problem of tangents, that is the problem of determining a curve whose tangents are subjected to a particular law, were successfully dealt with before the invention of the calculus.[2]
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> But the historical value of a science depends not upon the number of particular phenomena it can present but rather upon the power it has of coordinating diverse facts and subjecting them to one simple code.

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[1] Ince himself is part of at least one such story within differential equations. He developed the so-called *Ince Equation* (in about 1923),

$$(1 + a\cos(2t))y''(t) + (b\sin(2t))y'(t) + (\lambda + d\cos(2t))y(t) = 0,$$

which generalized at least two other well-known equations from about 1868 and 1914, respectively. Letting $a = b = 0$ and $d = -2q$, we obtain *Mathieu's equation* (which models elliptical drumheads),

$$y''(t) + (\lambda - 2q\cos(2t))y(t) = 0,$$

and letting $a = 0, b = -4q$, and $d = 4q(\nu - 1)$, we obtain the *Whittaker-Hill equation* (with applications to lunar stability and quantum mechanics),

$$y''(t) - 4q(\sin(2t))y'(t) + (\lambda + 4q(\nu - 1)\cos(2t))y(t) = 0.$$

Ince's equation then is itself a special case of *generalized Ince equations* (studied in [Moussa, 2014]),

$$(1 + \epsilon A(t))y''(t) + \epsilon B(t)y'(t) + (\lambda + \epsilon D(t))y(t) = 0.$$

[2] The reader should note that this use of the word "tangents" has nothing to do with the trigonometric function $\tan(x)$, but rather the "problem of tangents" mentioned here is the task of constructing tangent lines to a given curve.

This is exactly the evolution of solution methods for first-order linear ordinary differential equations. First, particular problems were solved with "one-off" methods that didn't have general applications beyond that specific problem. But then those results were combined and generalized until a unified theory developed.

Task 1 In the above passage, Ince made a connection between "the solution of the simplest of all types of differential equations" and "the problem of determining a curve whose tangents are subjected to a particular law." Connect these two statements. If the differential equation is

$$\frac{dy}{dx} = f(x, y),$$

then what are the "curve," the "tangents," and the "particular law"?

Task 2 Recall that non-homogenous first-order linear ordinary differential equations have the following form

$$p(x)\frac{dy}{dx} + q(x)y = f(x), \tag{1}$$

or if made monic

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{2}$$

Explain how to make Equation (1) monic like Equation (2). In particular, why can we assume that $p(x)$ isn't identically zero? Write $P(x)$ and $Q(x)$ in terms of $p(x), q(x)$ and $f(x)$.

The theme of this project is the first "one-off" method for equations like those in Task 2, due to Gottfried Leibniz (1646–1716).[3] As time progressed, solutions to differential equations came from more general "coordinated" techniques such as variation of parameters and exact differential equations.[4].

1 Leibniz's Check

On November 27, 1694, Gottfried Leibniz wrote a letter [Leibniz, 1694] to his friend the Marquis de l'Hôpital (1661–1704).[5] It contained a method for solving non-homogenous first-order linear differential equations.

The reader should be aware of two notations that appear in the original letter. Firstly, $dy{:}dx$ or $dp{:}p$ simply means $\frac{dy}{dx}$ or $\frac{dp}{p}$, similar to how we use the colon for expressing ratios and proportions

[3]Leibniz was a German mathematician and philosopher who created (probably independently of Newton) the Calculus along with the notation that we currently use.

[4]The stories of these more general methods can be found in the two other projects of this "Solving First-Order Linear Differential Equations" series, which continue to follow the historical trail by examining works by Johann Bernoulli (1667–1748) and Leonhard Euler (1707–1783). They are also available in the MAA *Convergence* article of that same title (link in footer). Each of the three projects in the series can be completed individually or in any combination with the others.

[5]Guillaume Francois Antoine, Marquis de l'Hôpital was a French mathematician credited with the first textbook on differential calculus.

today. Secondly, the symbol $\int\overline{mpdx}$ may look like a square root symbol but is actually two different symbols; the integral $\int$ and the overline $\overline{mp\,dx}$. The overline acts as parentheses indicating what (today we would say) is the integrand. So $\int\overline{mp\,dx} = \int(mp\,dx) = \int mp\,dx$.

Let $m + ny + dy{:}dx = 0$, where m and n signify rational or irrational formulas which depend only on the indeterminate x; [then] I say that one can resolve it generally as $\int\overline{mp\,dx} + py = 0$, *I suppose that* $\int\overline{dp{:}p} = \int\overline{n\,dx}$. *For by finding differences, it becomes* $mp\,dx + y\,dp + p\,dy = 0$, *but* $dp = pn\,dx$, *whence it becomes* $mp\,dx + npy\,dx + p\,dy = 0$ *or* $m\,dx + ny\,dx + \ dy = 0$, *just as had been desired.*[6]

When reading the above passage, we find that Leibniz was working backwards. As is so often the case, *finding* the solution to a differential equation, or any problem for that matter, is much harder than *checking* that something is a solution. In this passage, Leibniz did the latter. He asserted that $\int\overline{mp\,dx} + py = 0$ is a solution to $m + ny + dy : dx = 0$ if we were to define the function p by the equation $\int\overline{dp : p} = \int\overline{n\,dx}$; then he checked that this in fact gives a solution to the given differential equation.

Task 3 Let's consider Leibniz's solution more carefully.

(a) Explain in your own words how Leibniz went from $\int\overline{mp\,dx} + py = 0$ to $mp\,dx + y\,dp + p\,dy = 0$.

(b) Explain in your own words how Leibniz went from $\int\overline{dp : p} = \int\overline{n\,dx}$ to $dp = pn\,dx$.

(c) Explain in your own words how Leibniz combined (a) and (b) above to obtain $m\,dx + ny\,dx + \ dy = 0$.

(d) Leibniz concluded by saying "just as had been desired." Why exactly is this the "desired" result?

We also see that $\int\overline{mp\,dx} + py = 0$ is an *implicit* solution to the differential equation. In general, when a solution technique returns an *implicit* solution it will be impossible to solve for y to make it an *explicit* solution. Luckily, this is not the case here.

Task 4 Turn Leibniz's implicit solution into an explicit one by solving $\int\overline{mp\,dx} + py = 0$ for y.

Leibniz knew that his technique was an extension of what was known previously. Perhaps the very first differential equation ever written was (essentially) a first-order linear differential equation! In a 1638 letter to French philosopher and scientist René Descartes (1596–1650), French jurist and mathematician Florimond de Beaune (1601–1652) asked for a geometric solution to an equation that today we would write as

[6] All translations of Leibniz excerpts in this project were prepared by Daniel E. Otero of Xavier University, 2020.

$$\frac{dy}{dx} = \frac{\alpha}{(y-x)}. \tag{3}$$

While this equation is not linear, it can be made linear by following the steps outlined in Task 5 [Lenoir, 1979].

Task 5 Consider Equation (3).

(a) Switch the variables x and y.

(b) Solve for $\frac{dy}{dx}$.

(c) Compare this to Equations (1) and (2). What are $p(x), q(x), f(x)$ (and, respectively, $P(x), Q(x)$)?

Allowing m and n to be rational or irrational functions of x in $m+ny+dy{:}dx = 0$, was certainly an improvement over restricting them to be the values found in Task 5. But Leibniz was also well aware that his improvement was only one step towards more universal theories. The following statement immediately preceded his technique.

> I believe that with proper applications we may finally come to the inverse of tangents;[7] I have made some beginnings which seem all the more considerable as they encompass these [results] in fairly general terms and can be extended further . . .

2 Examples

In this section, we will work through a couple of examples using Leibniz's technique.

Task 6 Consider the example

$$x\frac{dy}{dx} + y = 3x^2. \tag{4}$$

(a) Rewrite Equation (4) in the form that Leibniz used to begin his process.[8] What are the functions m and n?

(b) Leibniz then defined a new function p that satisfies the condition

$$\frac{dp}{p} = n\,dx.$$

Using n from part (a), solve for p.

[7] See footnote 2 above for clarification on this terminology.

[8] Notice that Leibniz started with a monic equation, so if your given equation isn't monic, you'll need to make it so.

(c) With these functions, use Leibniz's method to verify that

$$\int \overline{mp\,dx} + py = 0$$

solves the original differential equation for the form of the equation in part (a).

(d) At this point we know m, n, and p so the only unknown is y. Solve for y and show it solves Equation (4).

Task 7 Solve

$$\frac{dy}{dx} - y = xe^x$$

using the method from the previous task.[9]

3 Leibniz's Intuition

"Tricks" for solving specific differential equations that were similar to Leibniz's "guess and check" method proliferated in the literature for decades. As in the case of Leibniz's method, these tricks often appeared to come from nowhere. In his work on the history of differential equations, Dick Jardine has noted that mathematicians of the day apparently immersed themselves in practice that allowed them to gain the intuition needed to create those methods [Jardine, 2011, p. 211]:

> Students initially are bewildered at how anyone "observed" or "noted" such relationships. My best explanation is that Leibniz, Bernoulli, and Euler spent many hours determining those and many other useful results with the calculus. Because of their effort, they developed useful mathematical intuition about such relationships.

It was that same intuition that eventually allowed mathematicians to organize their tricks into a general theory. Jardine concluded the above quote by stating, "With similar effort, our students can obtain similar intuition." Perhaps you won't quite develop Leibniz's intuition if you put in Leibniz's effort, but everyone can develop intuition about which integrals might use integration by parts, which proofs might use contradiction, or even what trick would allow you to solve a first-order linear non-homogenous differential equation.

It is likely that your ordinary differential equations class utilizes an integrating factor μ to solve first-order linear differential equations:

$$\frac{dy}{dx} + P(x)y = Q(x), \quad \mu = e^{\int P(x)dx} \tag{5}$$

While μ was not derived by Leibniz (remember that he technically only checked an answer), it is interesting that our modern μ formula is equivalent to his p function:[10]

$$m + ny + dy : dx = 0, \quad \int \frac{dp}{p} = \int ndx. \tag{6}$$

[9] This example is not historically accurate as Leibniz did not deal with functions of the form e^x.

[10] It should be noted that this equivalency requires that the given differential equation is made monic, as that is the form that Leibniz starts from.

Task 8 Show that μ in Equation (5) is the same as Leibniz's p function in Equation (6).

We may not know exactly where Leibniz's trick came from, but we know where it ended up—in your textbooks!

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Notes to Instructors

PSP Content: Topics and Goals

This Primary Source Project (PSP) is one of a set of three PSPs that share the name *Solving Linear First Order Differential Equations*, designed to show three solution methods for non-homogenous first-order linear differential equations, each from a different context. Recall that a non-homogenous first-order linear differential equation has the form

$$a(x)\frac{dy}{dx} + p(x)y = q(x).$$

- The PSP subtitled *Gottfried Leibniz's "Intuition and Check" Method* explains how in 1694 Leibniz solved these equations using a one-off method applicable only to this specific problem. Strictly speaking, Leibniz didn't solve the equation, but rather asserted a solution and then showed it worked. Part of his proposed solution will be familiar to the students: it is the standard integrating factor method we teach today.

- The PSP subtitled *Johann Bernoulli's (Almost) Variation of Parameters Method* explains how in 1697 Bernoulli provided a method for solving Bernoulli differential equations that reduces to variation of parameters when applied to first-order linear equations. This was decades before Lagrange received credit for the technique. Again, part of Bernoulli's solution will be the standard integrating factor.

- The PSP subtitled *Leonard Euler's Integrating Factor Method* explains how in 1763 Euler solved these equations as a special case of exact differential equations by finding an integrating factor. His integrating factor is the same as the one that the students would have seen. This PSP is a bit longer than the others, and may require a bit more time or advance preparation.

All three of these PSPs are designed for use in a course on ordinary differential equations. While they work best after at least presenting the standard integrating method of solution found in modern textbooks, they can be used in three different ways at that point.

- Since the *type* of equation (first-order linear) has been introduced, all three projects can be done immediately after presenting the standard integrating method of solution. This would require the instructor to "preview" techniques (e.g., variation of parameters, exactness) that will be introduced more fully later. While this is somewhat awkward, it does mimic how these techniques were actually developed.

- The "Gottfried Leibniz's 'Intuition and Check' Method" project can be done immediately after presenting the standard integrating method of solution, and the other projects done after the respective technique (e.g., variation of parameters, exactness) is first introduced. Showing how those techniques can solve first-order linear differential equations makes a great first example of each technique. This is typically the way that I utilize these projects.

- Any one of the three projects can stand on their own as they don't necessarily build on the others. However, they do create a richer experience when used together in a course. Additionally, students gain confidence as they proceed through the three projects.

Student Prerequisites

This PSP requires some algebraic manipulation of differentials along with differentiation up to the product rule. It also calls upon knowledge of separable differential equations. The first Fundamental Theorem of Calculus makes an appearance, but other techniques of integration needed are typically dictated by the examples used. Finally, the project benefits from the students being aware of the modern integrating factor method.

PSP Design and Task Commentary

This PSP consists of four sections.

- The initial (unnumbered) section contains a short introduction to what first-order linear differential equations are, along with a description of the way that mathematics often evolves. Mathematicians might first solve a specific problem using any tool at their disposal. They then attempt to see if they could find a class of problems (to which the initial one belongs) that can also be attacked using that technique. This closely mimics the evolution of how first-order linear differential equations were solved.

- Section 1 is devoted to Leibniz's method of solution. A translation is provided along with a few tasks to explain his method. Strictly speaking, students may notice that Leibniz doesn't "solve" the equation. Rather, he asserts a solution and then shows it "works." Since the solution isn't derived, I refer to it as "one-off" or a "trick."

- Section 2 consists of two tasks that prompt students to solve specific examples of first-order linear differential equations with Leibniz's method. The first task is broken into steps, while the second requires students to solve it on their own. These can be swapped with any examples you wish, in particular so that the integrations utilize techniques your students are comfortable with.

- Section 3 reiterates what we saw in the first section. These first solutions appear to come "fully formed from the heads" of the great mathematicians. It is only with extensive practice that they developed the necessary intuition to find those methods. While the origins may have been dependent on intuition, later developments were more satisfactory and there is a task to show that Leibniz's trick utilizes the modern integrating factor method.

This project is typically done in groups. One site tester warned that, "The groups often want to take a divide and conquer approach, which is just utterly useless for these documents, because the only person who is going to make any progress is the person who is working on Intro/Section 1. These are all designed to be read top to bottom in slow careful detail, and the later parts of the PSP rarely make any sense unless you've seen the earlier parts."

Suggestions for Classroom Implementation

Please see the student prerequisites section above and implementation schedule below for suggestions.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Sample Implementation Schedule (based on a 50-minute class period)

The actual number of class periods spent on each section naturally depends on the instructor's goals and on how the PSP is actually implemented with students. The following schedule allows its completion in one 50-minute class period.

The introductory section (including Tasks 1–2) should be completed out of class, as advance preparation. Tasks 3 and 6 shouldn't be skipped, but the remainder of the tasks in the project are stand-alone and what is completed (either in-class or as homework) can be dictated by the interests of the instructor and time available. Task 8 is useful for integrating this PSP into the material the student sees in their textbook. Also, Task 7 can be assigned as homework after class.

Connections to other Primary Source Projects

As described above, this mini-PSP is part of a series of three, all of which are intended for use in an Ordinary Differential Equations (ODE) course:

- *Solving Linear First-Order Differential Equations: Gottfried Leibniz's "Intuition and Check" Method*
- *Solving Linear First-Order Differential Equations: Johann Bernoulli's (Almost) Variation of Parameters Method*
- *Solving Linear First-Order Differential Equations: Leonard Euler's Integrating Factor Method*

The following additional projects based on primary sources are also freely available for use in teaching standard topics in an ODE course. Each of these can be completed in 1–2 class days, with the exception of the two projects marked by an asterisk (which require 3 and 4 days for implementation, respectively). Classroom-ready versions of all projects in the list can be downloaded at https://digitalcommons.ursinus.edu/triumphs_differ/.

- *Fourier's Heat Equation and the Birth of Fourier Series*,* Kenneth M Monks
  (This is an extended version of the PSP *Fourier's Heat Equation and the Birth of Modern Climate Science* listed below.)
- *Fourier's Heat Equation and the Birth of Modern Climate Science*, Kenneth M Monks
- *Leonhard Euler and Johann Bernoulli Solving Homogenous Higher Order Linear Differential Equations With Constant Coefficients*, by Adam E. Parker
- *Wronskians and Linear Independence: A Theorem Misunderstood by Many*, by Adam E. Parker
  (Also suitable for use in Linear Algebra and Introduction to Proof courses.)
- *Runge-Kutta 4 (and Other Numerical Methods for ODEs)*,* by Adam E. Parker

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For more information about TRIUMPHS, visit https://blogs.ursinus.edu/triumphs/.