The Logarithm of -1 *

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Introduction

Logarithms were introduced to the world by John Napier (1550–1617) in 1614, in a work entitled *A Description of the Wonderful Table of Logarithms* (or, in the Latin original, *Mirifici logarithmorum canonis descriptio*), as a bridge between an arithmetic sequence and a geometric one. Four hundred years later, logarithms are ubiquitous in mathematics and science, and it is difficult to imagine a world without them.

The eighteenth century was a time of transition in the story of logarithms (as it was in so many fields of mathematics). The elementary properties of logs were well understood, but many questions about them remained. This project will focus on one of these questions—namely, how to extend the domain of the log function to negative numbers.

Recall that $\log_b(x) = a$ is equivalent to $b^a = x$, so, for example, $\log_{10}(1000) = 3$. Given any function, it is convenient for its domain to be as large as possible, and for mathematical reasons (the applications would come many years later), eighteenth-century mathematicians wished to find a way to do this for the logarithm function. As it turns out, deciding the best way to do so was not easy, and would lead to a serious disagreement in the mathematical community.

Task 1 Why is it difficult to find the logarithm of a negative number? For example, why can't you easily find a value for $\log_{10}(-100)$?

Part 1: Functions with Several Values?

You have likely seen a formal definition of a function in the past. Although there are small differences between texts, the definition usually looks something like this:

Definition 1. *Let X and Y be sets. A* function *f, $f : X \to Y$, defined on X is a rule that assigns to each element $x \in X$ an element $y \in Y$, in which case we write $y = f(x)$.*

Task 2 Check the definition of function in your current textbook. Does it differ in any significant way from the definition above? If so, how?

*The author is grateful to Robert Bradley, whose article "Euler, D'Alembert and the Logarithm Function" [Bradley, 2007] provided the central ideas for much of this project.

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Task 3 A very important property of functions seems to be that for each x, $f(x)$ has only one value. (This is related to the "vertical line test" that you have seen in Calculus.) Can you think of any "function" (possibly expanding or modifying the definition a bit) for which $f(x)$ has more than one value? Try to give at least one example—or more if you can!

Part 2: Extending the Domain of the Logarithm

One of the strongest voices in the effort to expand the scope of the log function was Leonhard Euler (1707–1783), the leading mathematician and physicist of the eighteenth century. He had been thinking about logarithms as early as 1727, in correspondence with his mentor Johann Bernoulli (1667–1748). Bernoulli had written in the past that $\log(-x) = \log(x)$ (so, in particular, $\log(-1) = 0$). Euler was unconvinced, though he recognized that there were arguments in favor of this position. Indeed, in a letter to Bernoulli, he suggested one such argument himself.[1]

> If $\log(xx) = z$, then $\frac{1}{2}z = \log\sqrt{xx}$, but $\sqrt{xx}$ is as much $-x$ as x, so $\frac{1}{2}z$ is $\log x$ and $\log -x$. ... but whoever claims two [values of log] ought to claim an infinite number.

Task 4 Explain Euler's argument in your own words.

Task 5 Does Euler's argument convince you (at this point) that $\log(-x) = \log(x)$? Why or why not?

Task 6 What argument do you think Euler may have had in mind when he stated that if the logarithmic function has two values, then it ought to have infinitely many? Write down as many ideas as you can come up with for this. Do you agree with the claim at this point? Why or why not?

Part 3: The Beginning of a Conflict

By the end of 1746, Euler was much more mature, both as a mathematician in general and in his understanding of logarithms. Indeed, he had come to strongly suspect that the values of logs of negative values would involve imaginary and complex numbers, although he had yet to publish these ideas. In that year, Euler wrote a letter to Jean le Rond d'Alembert (1717–1783), a French philosopher and thinker who had just submitted a paper to the Berlin *Memoires* (a mathematics journal edited by Euler) in which he tried to prove the Fundamental Theorem of Algebra. In his paper, d'Alembert wrote that for any positive number x, $\log(-x) = \log(x)$. Euler wrote privately to d'Alembert in part to alert his colleague that this part of his paper may be in error. Regarding the logarithms of negative numbers, Euler wrote:

[1] All translations of excerpts from Euler's correspondence in this project were taken from [Bradley, 2007], with some minor modification by the project author following consultation of the original sources published in [Juskevic and Taton, 1980]. The translations in [Bradley, 2007] were prepared by Robert Bradley and John Glaus.

I believe that I have shown [in another work, not yet published] that it [the value of $\log(-1)$] is imaginary and that it is $= \pi(1 \pm 2n)\sqrt{-1}$ where π indicates the circumference of a circle whose diameter $= 1$, and n is any whole number.[2]

Because, I have shown just as every sine responds to an infinity of arcs of the circle, so the logarithm of every number have an infinite amount of different values, among which there is only one that is real when the number is positive, but when the number is negative all the numbers are imaginary. Therefore $\log 1 = \pi(0 \pm 2n)\sqrt{-1}$, n being any whole number, and letting $n = 0$, we will obtain the natural logarithm $\log 1 = 0$. In the same way we have $\log a = \log a + \pi(0 \pm 2n)\sqrt{-1}$, where $\log a$ indicates the ordinary logarithm of a. Now $\log(-a) = \log(a) + \pi(1 \pm 2n)\sqrt{-1}$, all of whose values are imaginary numbers. This all follows from the formula $\log(\cos\theta + \sin\theta\sqrt{-1})^k = (k\theta \pm 2mk\pi \pm 2n\pi)\sqrt{-1}$, where m and n are any whole numbers, the truth of which is easy enough to prove.[3]

Task 7 What did Euler mean he wrote "sine responds to an infinity of arcs of the circle"? Use this idea to give an example of a function with infinitely many values for one input. Does this function match the definition given on Page 1?

Task 8 Assuming that Euler is correct, give three different values of $\log 1$.

The next two tasks ask you to investigate Euler's formula

$$\log(\cos\theta + \sin\theta\sqrt{-1})^k = (k\theta \pm 2mk\pi \pm 2n\pi)\sqrt{-1}.$$

Task 9 Using this formula, pick values of k, m, and n that allow you to calculate $\log(-1)$. What do you find?

Task 10 Now set $\theta = \pi, n = 0, m = 0$, and $k = 1$. Then exponentiate both sides of the equation and rewrite the equation so that you have 0 on the right hand side, and all other terms on the left. What do you get?

Part 4: The Fight Escalates

D'Alembert was unconvinced by Euler's explanation, and he sent back no fewer than six arguments in his next letter concerning why Euler was wrong. Some of these arguments were quite weak (amounting to little more than "I can't really imagine logs being imaginary"), but one was quite interesting. D'Alembert wrote

[2] Euler would eventually use the symbol i to represent $\sqrt{-1}$, but not until 1777.

[3] Euler may have been understating the difficulty in establishing this equation. It didn't appear until the final page of a paper that he was working on at the time on the logarithms of negative and imaginary numbers. We will take a look at part of this paper in the final section of this project.

> All difficulties reduce, it seems to me, to knowing the value of $\log(-1)$. Now why may we not prove it by the following reasoning? $-1 = 1/-1$, so $\log(-1) = \log 1 - \log(-1)$. Thus $2\log(-1) = \log 1 = 0$. Thus $\log(-1) = 0$.

Task 11 Explain why d'Alembert can claim that $\log(-1) = \log 1 - \log(-1)$.

Task 12 Can you find any flaw in d'Alembert's reasoning? Are you convinced? Why or why not?

Euler, as it turns out, was not convinced. He was thinking more about the earlier work of Johann Bernoulli. In an investigation of the logarithms of imaginary numbers [Bernoulli, 1702], Bernoulli had demonstrated a rather non-obvious result (restated here in modern notation):

Theorem 2 (Johann Bernoulli)**.** *Given a circle of radius a, the area of a sector of the circle with boundaries of the x-axis and the line from the origin to the point (x, y) is given by*

$$\frac{a^2}{4\sqrt{-1}} \log\left(\frac{x + y\sqrt{-1}}{x - y\sqrt{-1}}\right).$$

Task 13 Test Bernoulli's theorem. Use his expression (and possibly a Computer Algebra System) to find the area of a sector with a central angle of 45°. Now find the answer using geometry. Do the answers match?

With Bernoulli's theorem in hand, Euler responded to d'Alembert's argument with the following reasoning.

> With the same reasoning that you use to prove $\log(-1) = 0$, you could equally well prove that $\log\sqrt{-1} = 0$, for since $\sqrt{-1}\cdot\sqrt{-1} = -1$, you would have $\log\sqrt{-1} + \log\sqrt{-1} = \log(-1)$, i.e. $2\log\sqrt{-1} = \log(-1) = \frac{1}{2}\log(+1)$ and thus $\log\sqrt{-1} = \frac{1}{4}\log 1 = 0$. If you do not approve of this reasoning, you will agree that the first [d'Alembert's argument above] is no more convincing. Now, you must at least agree that logarithms of imaginary numbers are not real, for otherwise the expression $\log\sqrt{-1}/\sqrt{-1}$ could not express the quadrature[4] of the circle.

Task 14 Explain how the reasoning Euler used above is analogous to d'Alembert's claim.

Task 15 Explain why Euler believed that Bernoulli's result implied that the $\log\sqrt{-1}/\sqrt{-1}$ is a real number, and why therefore the logarithm of imaginary numbers must not be real.

Sadly, the two mathematicians never agreed on the logarithm of -1. Indeed, this was one of the issues that broke up their friendship years later (the two would eventually attack each other in print over a series of arguments). For more on this fascinating story, see [Bradley, 2007].

[4] For Euler, "expressing the quadrature" of a circle was analogous to finding its area.

Part 5: The Formalization of the Logarithm Function

In a 1747 paper,[5] Euler definitively showed that in order to extend the domain of the logarithm function to negative and imaginary numbers, it was necessary to assume that the logarithm function took multiple (in fact, infinitely many) values for each argument. His arguments were so convincing that the mathematical community quickly accepted his definition of the logarithm as true; indeed, it's the definition we still use today. Euler summarized his work as follows:[6]

> This equation we just found, expressing the relation between the arc ϕ and the sine and cosine, will also apply for all the other arcs which have the same sine, x, and cosine, y. Consequently, we will have
>
> $$\phi \pm 2n\pi = \frac{1}{i}\log(x+iy),$$
>
> and therefore
>
> $$\log(y+ix) = i(\phi \pm 2n\pi).$$

Task 16 Using this claim, find three values of $\log(1)$.

Task 17 Return to d'Alembert's argument that $\log(-1) = 0$, and fix it using Euler's correct interpretation of the logarithm function. How would you answer someone now who asked you what the logarithm of -1 is? What about the logarithm of 1?

Task 18 You have discovered that the logarithm function is a "multi-valued function"—a useful idea that seems to contradict the standard definition of a function. Give some other examples of functions that can be thought of as having several values. Ideally, try to find examples for functions defined over both the real numbers and the complex numbers.

[5]His paper, entitled "Sur les logarithmes des nombres négatifs et imaginaires" ("On the logarithms of negative and imaginary numbers") was not published until 1862, in a two-volume publication by the St. Petersburg Academy of Sciences containing all the works of Euler that had not yet been published prior to his death in 1783. In the Eneström index that enumerates all of Euler's works, based on a comprehensive survey conducted by Swedish mathematician Gustaf Eneström between 1910–1913, Euler's 1847 paper is [E807]. The complete Eneström index includes 866 distinct works by Euler.

[6]The translation of this excerpt from [E807] was prepared by Todd Doucet, Lambent Research Corporation, 2005.

References

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Notes to Instructors

PSP Content: Topics and Goals

This Primary Source Project (PSP) is intended to be used in courses on Complex Analysis, and introduces students to a definition of the logarithm function on negative and complex numbers via an epistolary argument between Leonhard Euler and Jean d'Alembert. Its primary goal is to motivate the definition of the logarithm over $\mathbb{C}$, in particular its property of having multiple values. Students familiar with the definition of a function from a real analysis class (or even the vertical line test in calculus) have been taught that a function can have only one output for every input, a lesson belied by the behavior of natural log (along with several other functions) over the complex numbers.

Student Prerequisites

Ideally, students should have seen a formal definition of a function. They should have a level of mathematical maturity sufficient to look for their own examples in a few places and to be able to reach conclusions using properties of logarithms. For Task 12, it's extremely helpful for them to be able to use a Computer Algebra System to find a logarithm of a complex number. (In the absence of such a CAS, the problem can be solved using Wolfram Alpha.)

PSP Design, and Task Commentary

The biggest pedagogical hurdle that it tries to help students clear is the idea of "multi-valued function," which might seem a contradiction in terms based on their previous experience. In fact, it was this exact property of the logarithms that made extending the domain of this function to negative and natural numbers so difficult in the eighteenth century. Johann Bernoulli and Jean d'Alembert both seemed unable to make the leap to multi-valued functions, and even Euler (who eventually extended the domain of the log function in the way we think of it today) struggled for several years.

Through this project, students have an opportunity to recapitulate some of the early work on the logarithm of negative and imaginary values, and to try to solve an early paradox, proposed by d'Alembert, that seems to indicate that $\log(-1) = 0$.

Task 3 can be tricky for students; a multi-valued function doesn't exist by definition. The hope here is that they may think of $f(x) = \sqrt{x}$, which we can think of as a two-valued function, or $\arcsin x$, which could (through appropriate choices of domain and range) take more. The question of arcsine is explored in more detail in Task 7.

Task 7 is designed to help students start thinking about multi-valued functions by reminding them of sine and arcsine. As they know, there are infinitely many angles θ for which the $\sin(\theta) = \pi$, and thus $\arcsin(\pi)$ can be thought of as a multi-valued function. We usually avoid this issue in calculus and pre-calculus classes by restricting the domain and range of arcsine, but one could imagine not doing so.

Task 13 asks students to calculate the logarithm of a complex number. They should find an imaginary value of this logarithm, leading to a real result for the area of a circle. If calculating these values has not been covered in the class at this point, students should feel free to trust the result of a Computer Algebra System (Wolfram Alpha works fine) in this task.

Suggestions for PSP Implementation and Sample Implementation Schedule (based on a 50-minute class period)

One possible implementation, using just 1.5 days of class time, follows. If time allows, the homework can be eliminated and replaced with in-class time, in which case it may take 2–2.5 days to complete the PSP.

- **Day 0.** Students work in groups for about 15 minutes on Day 0 to complete Part 1.

  Day 0 Homework: Assign Part 2, the reading in Part 3 up to the Task 7, and Task 7. This is the longest section of reading in the PSP, and it may be better done as homework than during class.

- **Day 1.** Students meet in groups to discuss their answers to Task 7, and work together to complete Part 3 and Part 4 through Task 13. The remaining tasks can be completed as homework.

  Day 1 Homework: Assign Tasks 14–18.

- **Day 2.** I would encourage beginning Day 2 with a short full-class discussion of some of the meatier tasks Task 17 would be a really good one to work through, and the instructor should feel free to use Task 18 to discuss multi-valued functions in Complex Analysis. If it's appropriate for the current state of your course, this could lead to a discussion of branch cuts.

LaTeX code of the entire PSP is available from the author by request to facilitate preparation of reading guides or other assignments related to the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Connections to other Primary Source Projects

The following primary source-based projects are also freely available for use in teaching courses in complex variables. The number of class periods required for full implementation is given in parentheses. Classroom-ready versions of these five projects can be downloaded from https://digitalcommons.ursinus.edu/triumphs_complex, or obtained (along with their LaTeX code) from their authors.

- *Euler's Square Root Laws for Negative Numbers* by David Ruch (1–2 days)
- *An Introduction to Algebra and Geometry in the Complex Plane* by Nicholas A. Scoville and Diana White (5 days)
- *Argand's Development of the Complex Plane* by Nicholas A. Scoville and Diana White (5 days)
- *Riemann's Development of the Cauchy-Riemann Equations* by David Ruch 3 days)
- *Gauss and Cauchy on Complex Integration* by David Ruch (3 days)

Recommendations for Further Reading

This project was inspired by Robert Bradley's 2007 article "Euler, D'Alembert and the Logarithm Function." While I have tried to encapsulate the most important ideas from that article in this Primary Source Project, a lot of rich discussion about eighteenth-century views of logarithms were left out. Instructors (and interested students!) are strongly encouraged to read Bradley's paper.

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