Topology from Analysis

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1 Introduction

Topology is often described as having no notion of distance, but possessing a notion of nearness. How can such a thing be possible? Isn't this just a distinction without a difference? In this project, we will make sense of the notion of nearness without distance by studying some work done by Georg Cantor (1845–1918), a German mathematician best known for his work in set theory. He also made contributions to number theory, philosophy, and as we will see in this module, topology. By looking at Cantor's investigation of a problem involving Fourier series, we will see that it is the relationship of points to each other, and not their distances *per se*, that is the key distinction between "distance" and "nearness." Along the way, we will see the roots of topology organically springing from the fertile soil of analysis.

2 Some background

In a calculus course covering sequences and series, you were introduced to power series; that is, the idea that a function $f(x)$ may be written as

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

where a_n is some coefficient for each n. This transforms what could be a fairly complex function into a polynomial (albeit an infinite one) which allows you to approximate the function using finitely many terms of the series. Another reason such a form is desirable is because, under reasonable hypotheses, one can integrate and differentiate the function term-by-term.

In the 19th century another kind of representation of a function was discovered, the so-called Fourier series. The Fourier series of a function $f(x)$ has the form

$$f(x) = \frac{1}{2}b_0 + \sum_{n=1}^{\infty} a_n \sin(nx) + b_n \cos(nx)$$

where b_0 and each a_n, b_n are coefficients. This is also a very important representation of a function, one often used in physics. Georg Cantor studied Fourier series in the 1870s. In his paper [Cantor, 1872], he wrote:[1]

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[1]The English translation of this and all other excerpts in this project are due to the author.

In the following, I will announce a certain extension of the theorem that trigonometric representations are unique. [I have shown] that two trigonometric representations

$$\frac{1}{2}b_0 + \sum(a_n \sin nx + b_n \cos nx) \text{ and } \frac{1}{2}b'_0 + \sum(a'_n \sin nx + b'_n \cos nx)$$

where every value of x and has the same sum, agree in their coefficients. ... I have also shown that the theorem holds if we give up either convergence or the representation for a finite number of values of x.

[Cantor, 1872, p. 123]

Task 1 Give a precise statement of what Cantor had "also shown."

Cantor continued:

The extension proposed *here* asserts that the theorem remains valid even when the assumption of the convergence of the series or the value of the limit fails for an *infinite* number of values of x in the interval $[0, 2\pi]$.

[Cantor, 1872, p. 123]

Cantor was thus interested in showing that the Fourier series representation of a function is unique even when the series is undefined or divergent for an infinite number of points on the interval $[0, 2\pi]$. But is this true for every set of infinitely many points? And, if not, for what kinds of sets of infinitely many points is it true? To follow Cantor's answer to these questions, we will let $P \subseteq [0, 2\pi]$ be the point set where f is either undefined or is divergent. Given such a function f, Cantor constructed a continuous function F which depends on f. It does not concern us here what this new function F is. What matters for our purposes is the following proposition, stated here in modern notation and language, that Cantor established holds for F.

Proposition 1. *If F is a linear function on the domain $[0, 2\pi]$, then the Fourier series for f is unique, i.e., the coefficients are uniquely determined.*

[Cantor, 1872, p. 130]

Fortunately, Cantor also gave us a practical way to show F is linear.

(A) *If there is an interval (p, q) in which only a finite number of points of the set P lie, then $F(x)$ is linear in this interval.*

[Cantor, 1872, p. 131]

This result (A) will be key below.

Task 2 Give two examples of non-empty sets P for which, if f is not defined or gives up convergence on P, then the Fourier Series for f is unique.

3 Limit points and derived sets

How can we use (A) to show that F is linear even when there is an infinite point set at which we give up convergence or the function f is not defined?

Task 3 Suppose f gives up convergence on $P := \{\frac{1}{n}+1 : n = 1, 2, 3, \ldots\} \cup \{\frac{1}{n}+2 : n = 1, 2, 3, \ldots\}$.

(a) Use result (A) to show that F is linear on all but a finite number of points of $[0, 2\pi]$. You will need to use the fact that F is the *same* linear function on each interval, a fact that Cantor proved to hold in general.

(b) Denote by P' the point set of $[0, 2\pi]$ for which we can't (yet) conclude that F is linear. Compute P'.

(c) Use the fact that F is continuous on all of $[0, 2\pi]$ along with (A) to conclude that F is linear on *all* of $[0, 2\pi]$.

The result you showed in Task 3 is the basic idea behind Cantor's main result. Even though P was infinite, it was relatively easy to apply result (A) to prove that F is linear on all of $[0, 2\pi]$. But what *exactly* was the property of P which made it a "well-behaved" kind of infinite set? Cantor abstracted away the particulars of the example and defined the essence of what it is that makes such an infinite set P susceptible to this kind of argument.

For the sake of brevity, I call a given finite or infinite number . . . of points of a line, a point set.

If a point set is given in a finite interval, a second point set is generally given, and with these generally a third, etc., which are essential for the conception of the nature of the first point set. . . . To define these derived point sets, we must begin with the term *limit point*[2] of a point set.

By a limit point of a point set P, I mean a point of the line such that there are infinitely many points P in every neighborhood of it, and it may happen that it also belongs to the set itself. The neighborhood of a point means here any interval which has the point in its interior. Therefore, it is easy to prove that a [bounded] point set consisting of an infinite number of points always has at least *one* limit point.

[Cantor, 1872, p. 127]

Task 4 Prove that a bounded subset of $\mathbb{R}$ consisting of an infinite number of points has at least one limit point.

[2] Cantor used the term "Grenzpunkt" ("boundary point") in his 1872 paper; the French translation of that paper (published as [Cantor, 1883]) used the term "point-set limite". The term "limit point" has become standard in English.

Task 5 Give an example of an unbounded point set of $\mathbb{R}$ that does not have a limit point.

Cantor next defined these other point sets "which are essential to understanding the first set."

> Every point of the line is now in a definite relation to a given set P, either being a limit point of P or not, and thereby along with the point set P the set of limit points of P is a set which I wish to denote by P' and call the *first derived point set of* P.
>
> Unless the point set P' contains only a finite number of points, it also has a derived set P'', which I call the *second derived point set of* P. By ν such transitions[3] one obtains the concept of the νth derived set $P^{(\nu)}$ of P.
>
> It may happen—and this is the case we are exclusively interested in at present—that after ν transitions the set $P^{(\nu)}$ consists of a finite number of points, and hence has no derived set; in this case we wish to call the original point set P a set *of type* ν, so that $P', P'', \ldots$ are of types $\nu - 1$, $\nu - 2$, ...
>
> [Cantor, 1872, p. 128]

Note that, according to this quote, Cantor did not seem to think that a finite set has a derived set. Today, we would say that the derived set of a finite set is the empty set.

Task 6 Let us practice computing the derived set. Find all derived sets of P when

(a) $P = [0, 1]$

(b) $P = (0, 1]$

(c) $P = (0, 1)$

(d) $P = \{.2, .3\}$

(e) $P = \mathbb{Q} \cap [0, 1]$

(f) $P = \{\frac{1}{n} : n = 1, 2, 3 \ldots\}$

(g) $P = \{m + \frac{1}{n+1} : m, n \in \mathbb{Z}^+\}$

(h) $P = \{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{Z}^+\}$

Task 7 If P has finitely many elements, show that $P' = \emptyset$.

[3] In his 1872 paper, Cantor restricted the value of ν to finite integers only. At the time he wrote that paper, however, he had already realized that, in the case where P^ν is a non-empty set for every finite integer ν, he could extend the notion of "type" for derived sets beyond the finite. To do this, he set $P^\infty = \bigcap_{\nu=1}^{\infty} P^\nu$ to obtain a derived set of type ∞. He then continued the iterative process to obtain $P^{\infty+1} = (P^\infty)'$, $P^{\infty+2} = (P^{\infty+1})'$, and so on. This process could be extended to even higher orders of derived sets, such as $P^{\infty^{\infty+1}}$, $P^{\infty n^\infty}$, $P^{\infty^{\infty^n}}$, $P^{\infty^{\infty^\infty}}$, etc. Cantor later substituted the symbol 'ω' for the '∞' symbol, to distinguish the actually-infinite ordinals $\omega, \omega + 1, \omega + 2, \ldots$ from the concept of potential infinity associated with the ∞ symbol in calculus.

Cantor eventually connected his study of the series of "transfinite ordinals" $\omega, \omega + 1, \omega + 2, \ldots$ associated with an ordered iterative process to his use of one-to-one correspondences between two sets as a means to measure their relative sizes, or cardinalites. This led him to introduce an unbounded sequence of "transfinite cardinals," denoted (by Cantor and today) as $\aleph_0$, $\aleph_1$, $\aleph_2$, ..., $\aleph_\omega$, $\aleph_{\omega+1}$, ... Here, $\aleph_0$ is the cardinality of the set of natural numbers $\mathbb{N}$, and also that of the equally-large set of rational numbers $\mathbb{Q}$. The problem of determining the cardinality of the set of real numbers $\mathbb{R}$ preoccupied Cantor throughout much of his later life. The conjecture that $\mathbb{R}$ has cardinality $\aleph_1$, known as Cantor's Continuum Hypothesis, continues to be of interest in set theory today. In this way, a question in analysis led not only to the development of point set topology, but also to today's modern set theory.

For more information about Cantor's development of set theory, see [Dauben, 1979].

4 Main result

Let's now go back to the question introduced in Section 2 and let f be a function represented by a Fourier series and F the associated continuous function that Cantor defined based on f. In general, we would like to show that, no matter what finite value we assign to n, the function F is linear on $[0, 2\pi]$ even when we give up convergence on a point set P of the n^{th} kind; that is, when $P^{(n)}$ is a non-empty finite set and $P^{(n+1)} = \emptyset$. Notice that for $n = 0$, this assumption tells us that f gives up convergence or is undefined only on a finite number of points; that is, $(0, 2\pi)$ contains a finite number of points of P, so that by (A), F is linear on $(0, 2\pi)$. Because F is continuous on all of $[0, 2\pi]$, it then follows that F is linear on all of $[0, 2\pi]$.

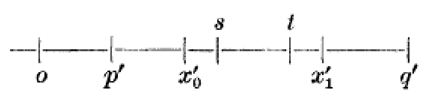
Cantor also established the $n = 1$ case in his 1872 paper. To do so, he first showed the following:

> **(A′)** *If (p', q') is any interval in which only a finite number of points of the set P' lie, then $F(x)$ is linear in this interval.*

In this case, P' is finite by supposition, so that any subinterval (p', q') of (p, q) contains a finite number of points $x'_0, x'_1, \ldots, x'_v \in P'$, where $x'_0 < x'_1 < \ldots < x'_v$. We now quote Cantor's argument:

> Each of these subintervals generally contains infinitely many points of P so that result (A) does not directly apply; however each interval (s, t) that falls within $(x'_0, \ldots, x'_1)$ contains only a finite number of points from P (otherwise another point of the set P' would fall between x'_0 and x'_1), and the function is also linear on (s, t) because of (A). The endpoints s and t can be made arbitrarily close to the points x'_0 and x'_1 so that the continuous function $F(x)$ is also linear in $(x'_0, \ldots, x'_1)$.
>
> [Cantor, 1872, p. 131]

Cantor illustrated this situation with the following picture:



He then noted that the argument in the preceding excerpt together with the continuity of F implies that F is linear over all of (p', q'). *(Make sure you see why this is true! Task 3, part (a), may be useful here.)* In the case where $(p', q') = (0, 2\pi)$, invoking the continuity of F once more allows us to then conclude that F must be linear over all of $[0, 2\pi]$.

Cantor recognized that this strategy could easily be adapted to a proof for an arbitrary finite integer n. The key step in doing this is to show that the following holds for every finite integer n:

> **($\mathbf{A}^{(n)}$)** *If $(p^{(n)}, q^{(n)})$ is any interval in which only a finite number of points of the set $P^{(n)}$ lie, then $F(x)$ is linear in this interval.*

Task 8 Using an argument similar to Cantor's for the $n = 1$ case, assume the inductive hypothesis for $(A^{(n)})$ and prove the inductive step.

Together with the proof for $n = 1$ given earlier, this last task completes the proof by induction that ($\mathrm{A}^{(n)}$) holds for every finite integer n. In the case where $(p^{(n)}, q^{(n)}) = (0, 2\pi)$, combining ($\mathrm{A}^{(n)}$) with Cantor's Proposition 1 (from page 2 of this project) now allows us to immediately see that $f(x)$ has a unique trigonometric representation on $[0, 2\pi]$ even when convergence is given up on an infinite point set P of the n^{th} kind for a finite value of n.

5 Conclusion

In this short project, we have seen how a problem in Fourier series led Cantor to define topological concepts that allow us to characterize different kinds of infinite point sets. In order to apply a result that only worked for finite sets, Cantor had to control how the points in these infinite sets behaved. The points could not be too "bunched up" or "close together." This notion was made precise in the definition of the limit point and the derived set, two concepts which, although having their origins in Fourier series, are now fundamental in topology. This transition from real numbers to a more general "closeness" or "nearness" was just the beginning of point set topology.

References

G. Cantor. Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen. *Math. Ann.*, 5(1):123–132, 1872.

G. Cantor. Extension d'un théorème de la théorie des séries trigonométriques. *Acta Mathematica*, 2:336–348, 1883.

J. W. Dauben. *Georg Cantor: His mathematics and philosophy of the infinite.* Princeton University Press, Princeton, NJ, 1979.

Notes to Instructors

Primary Source Project Content: Topics and Goals

This project is intended as a transition from more familiar mathematics to the ideas found in topology. It is best utilized on either the first or second day of an introduction to topology course. Part of the idea is that students can see in Fourier series a kind of math that is familiar to them, even if they have never worked with Fourier series *per se*. The question of uniqueness for Fourier series is again something that they should be able to appreciate. From this question, concepts like limit points and derived sets which are purely point set notions arise quite naturally by the end of the PSP. In this way, it is hoped that the students will appreciate where these more abstract definitions came from.

Student Prerequisites

As mentioned above, the hope is that this project can be done on the first day, or perhaps first week, of an introduction to topology course. Hence, the necessary student background is a course in proof writing or the equivalent exposure to proofs. Analysis may be helpful, but not strictly necessary. As mentioned in the next section of these Notes, however, the conceptual level of this PSP is high, and it is recommended that the students either have good mathematical maturity when it comes to proof writing or that the instructor is willing to provide a good bit of hints and support during implementation of the project.

PSP Design, and Task Commentary

It should be noted that this PSP will be challenging for most students, so that providing scaffolding, hints, and strong guidance is important for its success. Especially critical for student understanding is Task 3, where students first meet the main idea behind Cantor's argument. This task asks students to reduce the question of the uniqueness of a Fourier series which is not defined or gives up convergence on an *infinite* set to certain behavior on a *finite* set. It is recommended that students work on this task in groups, then share out answers as a class. The instructor can ensure that the students stay on track, and are guided to the correct answer. One way I do this is by drawing out from students what we know about the function F. For instance, I might say to my students, "We know this function F is linear. Where is it linear? At what points is it linear?" A student may choose any open interval of $[0, 2\pi]$ not containing 1 or 2, say $(\frac{1}{3}, \frac{1}{2})$. You can then draw a linear function just on $(\frac{1}{3}, \frac{1}{2})$. "Is it linear anywhere else?" you might ask. If no one offers an initial interval, you can ask "Is F linear at $\frac{1}{2}$?" If there are again no responses, you can prompt them further by asking, "How can we tell if it is linear? Do we have a way of knowing? Is there any theorem or result that tells us when F is linear?" This is all to get the students to think about Theorem (A). Once they have done this, you can coax them to offer an interval containing $\frac{1}{2}$ but not 0 or 1. "Can we show that F is linear at $\frac{2}{3}$? $\frac{4}{3}$?" etc. Hopefully students will start to see that F is linear everywhere except possibly 1 and 2. Again, as students are offering points or sets where F is linear, draw a line on the board at those points or sets. In the end, you should have a line which is defined on all of $[0, 2\pi]$ with a hole at 1 and 2. Now the question becomes "If F is continuous on all of $[0, 2\pi]$ and is linear everywhere, is F linear at the two points 1 and 2?" To prove that this is indeed the case, it is necessary to use the fact that F is the *same* linear function on each interval; as noted in the instructions for Task 3(a), Cantor proved that this fact holds in general. Though students may not know how to prove this fact (and are not expected to do so!), they should be able to see how it (together with F's continuity) implies that F must be linear at the points 1 and 2 as well.

Hence, if this goes according to plan, the natural next questions are: "How do we quantify or characterize the infiniteness of such a set? What *exactly* is that property of P that makes its infiniteness much more manageable than other infinite sets?" This last question is very important for students to appreciate. Even if the entire class has seen, for example, Cantor's theory of infinite numbers, most students will still be stuck in the mindset that "infinity is infinity," or "If a set is infinite, then its infinite." Part of the challenge in grasping the import of point set topology (and in particular, the content contained in this PSP) is getting students to realize that not all infinite sets are created equal. Here, the essential point is not that of transfinite cardinality, but rather the idea that some infinite sets are nicer in that they are easier to work with than others. Can we describe these "nice" properties of infinite sets? Once a student begins to appreciate and perhaps even to ask that question for herself, she has made it past a large mental barrier and is ready to work with the concepts in topology.

A couple other tasks are worth noting. Task 4 is essentially the Bolzano–Weierstrass theorem, which students may or may not have been exposed to in a real analysis course. Even for students who have seen it previously, the proof will be challenging. In Task 6(d), the answer depends on whether you use Cantor's view about the status of the derived set of a finite set (e.g., that it doesn't exist) or the modern one (e.g., that it is the empty set).

Suggestions for Classroom Implementation and Sample Schedule

The following schedule, based on a 65-minute class period, allows for completion in two days.

Day 1 (65 minutes)

- **Preassignment:** Have students read the first page and the beginning of the second page up until Task 1 and come up with an answer for Task 1 for discussion in class.
- **Whole-class discussion (15 minutes):** Begin class with a discussion of what the students read the night before. More likely than not, most students will feel that they didn't understand much from the reading. This is to be expected, so be upfront about how they are being asking to wrestle with something fairly delicate. This will also help set the tone for the rest of the PSP. Then help the class as a whole understand what it is that Cantor had done previously: what result on Fourier series did he already know and why is it important?
- **Working in groups (15 minutes):** Next turn to the new question that Cantor is trying to answer; this is the section of the PSP that starts after Task 1 up until the beginning of Section 3. Working in groups or individually, have students read the rest of Section 2 and work on Task 2, then begin Task 3.
- **Debrief (10 minutes):** As a class, discuss Proposition 1 and Theorem (A). These two results are crucial to understanding the rest of the PSP.
- **Working in groups (10 minutes):** Have students work on Task 3 in groups.
- **Debrief (15 minutes):** As a class, talk through Task 3. See the previous section of these Notes for several ideas and details about how to do this.
- **Homework:** Have students read the rest of Section 3 and do at least Task 6 for homework. Tasks 4, 5 and 7 may also be assigned for homework, or some or all of them could done together as a class or in groups on Day 2.

Day 2 (50 minutes)

- **In class discussion (20 minutes):** Depending on the instructor interests, there are some options for day 2. If they were not assigned for homework, students can work in groups on any of Tasks 4, 5 and 7. Another option is to go over a few of the parts of Task 6 as a class or to have students present some of their answers for this task. The key here is to ensure that students understand the derived set.
- **Reading (10 minutes):** Have students simply read through Section 4. If you wish, you can have students read with a partner and then discuss the reading with the partner. The key here is to understand Theorems (A') and $(A^{(n)})$.
- **Debrief (20 minutes):** Discuss what was read, helping students to understand Theorem (A') and $(A^{(n)})$. Give examples and illustrations. Assign Task 8 for homework.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Connections to other Primary Source Projects

There are several other projects in topology written by the author. Project titles along with links are given below. Those marked by an asterisk (*) are full-length projects; all others are shorter mini-PSPs, intended to be completed in 1–2 class periods.

- *The Cantor Set before Cantor*
  `https://digitalcommons.ursinus.edu/triumphs_topology/2/`
- *Connecting Connectedness*
  `https://digitalcommons.ursinus.edu/triumphs_topology/3/`
- *The Closure Operation as the Foundation of Topology*
  `https://digitalcommons.ursinus.edu/triumphs_topology/4/`
- *A Compact Introduction to a Generalized Extreme Value Theorem*
  `https://digitalcommons.ursinus.edu/triumphs_topology/5/`
- *From Sets to Metric Spaces to Topological Spaces*
  `https://digitalcommons.ursinus.edu/triumphs_topology/6/`
- *Nearness Without Distance**
  `https://digitalcommons.ursinus.edu/triumphs_topology/7/`
- *Connectedness: Its Evolution and Applications**
  `https://digitalcommons.ursinus.edu/triumphs_topology/8/`

For Further Reading

A nice source on Cantor's life and work is Joseph Dauben's *Georg Cantor: His Mathematics and Philosophy of the Infinite*, [Dauben, 1979].

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