Wronskians and Linear Independence: A Theorem Misunderstood by Many

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1 Introduction

Every mathematician—student, amateur, or professional—experiences a common wild emotional roller coaster occupational hazard. Discovering a proof after working on it for hours or days[1] leads to a sense of joy and achievement that can best be described as the mathematical version of a runner's high from completing a marathon.

But unlike completing a marathon or climbing a mountain, that joy is often followed by a sense of doubt. "Did I actually accomplish that?" "Did I make a mistake?" "Is there a trivial counterexample that I didn't see?" "Will the teacher or editor[2] be mean or angry if they find an error?"

While teachers may not advertise it, this happens to everyone and no student should feel embarrassed by these feelings. Indeed, many established mathematicians can believe for years that a theorem is true and proved. Yet, someone then comes along and changes everything. This is the story of just such an instance. Our protagonist noted that a theorem that had been "proved" true for decades actually had a simple counterexample! When he published his counterexample, he included the following:

> In nearly all papers, one finds the proposition: If the determinant formed with n functions of the same variable, and their derivatives of orders $1, \ldots, (n-1)$ is identically zero, there is between these functions a homogeneous linear relationship with constant coefficients.
>
> This wording is too general.

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[1]or weeks, or months, or years, …

[2]or advisor, or audience member, or referee, …

2 Linear Independence of Vectors

In linear algebra, we learn that a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is called *linearly independent* if the only solution to the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$ is when all the constants $= 0$. In other words, there is no non-trivial way to combine the given vectors to give the zero vector.

It turns out there is a very strong test to determine if the vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subset \mathbb{R}^n$ are independent or not:

> **Vector Independence Theorem.** *Form the $n \times n$ matrix where the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ form the columns of the matrix. The n vectors are linearly independent if and only if this matrix has nonzero determinant.*

What follows will be easier if we spend a few minutes reviewing the logic of the above theorem. If we define

$$p = \text{the determinant is nonzero,}$$

$$q = \text{the vectors are independent,}$$

and use the notation '$\sim p$' to mean *the negation of the statement* p, then the 'if and only if' nature of the Vector Independence Theorem places four true statements at our disposal.

> **Vector Independence Theorem.** *The following statements are all true.*

(a) $p \implies q$ — *If the determinant is nonzero, then the vectors are independent. (True by the "if" part of the theorem.)*

(b) $\sim q \implies \sim p$ — *If the vectors are dependent, then the determinant is zero. (Called the "contrapositive" of the "if" part of the theorem, it is equivalent to the original implication.)*

(c) $q \implies p$ — *If the vectors are independent, then the determinant is nonzero. (Called the "converse" of $p \implies q$, its truth is never implied by $p \implies q$, but in this case is true by the "only if" portion of the theorem.)*

(d) $\sim p \implies \sim q$ — *If the determinant is zero, then the vectors are dependent. (Called the "inverse" of $p \implies q$, its truth is never implied by $p \implies q$, but in this case is true by the "only if" part of the theorem.)*

Task 1 Are the vectors $\vec{v}_1 = (1, 2, 3)$, $\vec{v}_2 = (4, 5, 6)$ and $\vec{v}_3 = (7, 8, 9)$ linearly independent or dependent?

3 Linear Independence of Functions

It turns out there is an analogous story for the independence of *functions*.

A set of n differentiable functions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ on an interval I is called *linearly independent* if the only solution to $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$ for all $x \in I$ is when all the constants $= 0$. In other words, on the interval I, there is no non-trivial way to combine the given functions to give the zero function.

It would be useful if there were a test similar to the Vector Independence Theorem that would enable us to determine the independence of a set of functions. Given we are still talking about *linear* independence, we would hope that this might still involve the determinant of some matrix. The question is: "What matrix?"

The answer is a special matrix called the *Wronskian*[3] of the n functions. This is an $n \times n$ matrix where the n functions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ form the first row, their first derivatives $\{f_1'(x), f_2'(x), \ldots, f_n'(x)\}$ form the second row, their second derivatives $\{f_1''(x), f_2''(x), \ldots, f_n''(x)\}$ form the third row, etc.[4] The Scottish mathematician Thomas Muir (1844–1934) first named these matrices in his 1882 *Treatise on the theory of determinants* [Muir, 1882, p. 224], in which he stated:

194. If there be n functions of one and the same variable x, the determinant[5] which has in every case the element in its i^{th} row and s^{th} column the $(r-1)^{\text{th}}$ differential coefficient of the s^{th} function may be called the *WRONSKIAN* of the functions with respect to x. Thus, the Wronskian of y_1, y_2, y_3 with respect to x is

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \frac{dy_3}{dx} \\ \frac{d^2y_1}{dx^2} & \frac{d^2y_2}{dx^2} & \frac{d^2y_3}{dx^2} \end{vmatrix} \quad \text{or,} \quad \begin{vmatrix} y_1 & \frac{dy_2}{dx} & \frac{d^2y_3}{dx^2} \end{vmatrix}$$

It may be shortly denoted by

$$W_x(y_1, y_2, y_3) \quad \text{or} \quad |y_{1(0)}, y_{2(1)}, y_{3(2)}|,$$

the enclosed suffixes referring to differentiations.

Task 2
(a) Compute $\det(W(x^2, x^3))$.
(b) Compute $\det(W(\cos x, \sin x, x))$.

These matrices were named after the supremely determined yet frequently incorrect Polish thinker Józef Maria Hoene-Wroński[6] (1776–1853). In his *Réfutation de la Théorie des Fonctions Analytiques*

[3]The title of this project was almost "Wronskian: A Great Name for a Metal Band." I stand by that claim.

[4]Of course, the first $n-1$ derivatives of the functions must all be defined to form this matrix.

[5]It is not standard whether "Wronskian" refers to the matrix or the determinant of said matrix. It will hopefully be clear from context which is intended. In this project, we generally use W to refer to the Wronskian as a matrix, and $\det(W)$ to denote its determinant.

[6]Wroński may simply have been ahead of his time with a coarse personality that prevented acceptance of his ideas. His entry in the *Dictionary of Scientific Biography* notes that Wroński's interactions "indicate a marked psychopathic tendency: grandiose exaggeration of the importance of his own research, violent reaction to the slightest criticism, and repeated recourse to nonscientific media as allies against a supposed conspiracy. His aberrant personality, as well as the thesis of his esoteric philosophy (based on a revelation received on 15 August 1803 or, according to his other writings, 1804), tempt one to dismiss his work as the product of a gigantic fallacy engendered by a troubled and deceived mind. Later investigation of his writings, however, leads to a different conclusion. Hidden among the multitude of irrelevancies are important concepts that show him to have been a highly gifted mathematician whose contribution, unfortunately, was overshadowed by the imperative of his all-embracing absolute philosophy. [Dobrzycki, 2008]"

de Lagrange (Refutation of Lagrange's Theory of Analytic Functions),[7] Wroński developed certain "combinatorial sums" which ended up being the determinants of Wronskians [Hoene-Wroński, 1812, p.14], thereby justifying Muir's naming of the matrix.

The analogous theorem to the Vector Independence Theorem is:

Function Independence Theorem. *If* $\det(W(f_1(x), f_2(x), \ldots, f_n(x))) \neq 0$ *for some* $x_0 \in I$, *then* $f_1(x), f_2(x), \ldots, f_n(x)$ *are linearly independent on the interval* I*.*

As above, a quick analysis of the logic of the Function Independence Theorem will be helpful. Consider the following statements:

$$p = \text{the determinant of the Wronskian is nonzero for some } x_0 \in I$$
$$q = \text{ the functions are independent on } I$$

Task 3

(a) With the above definitions of p and q, what is the statement of the Function Independence Theorem, both symbolically and in words?

(b) With the above definitions of p and q, what is the contrapositive of the Function Independence Theorem, both symbolically and in words? (Be careful to correctly negate the "for some" statement in p.)

(c) What two statements (both symbolically and in words) are NOT necessarily true because of the Function Independence Theorem?

(d) What statement (both symbolically and in words) would it make sense to call the "Converse Function Independence Theorem"?

We want to concentrate on the statements coming from Task 3 above: the converse and inverse of the Function Independence Theorem, which, remember, are NOT implied true by the theorem itself. In other words, the converse and inverse might still be true—they just aren't automatically true from the implication $p \implies q$.

Indeed this is exactly what happened. The Converse Function Independence Theorem appeared in many textbooks *with proof* in the mid- to late-1800s. These included Charles Hermite's 1873 *Cours d'analyse de lécole polytechnique* [Hermite, 1873], Hermann Laurent's 1885 *Traité d'analyse* [Laurent, 1885], Camille Jordan's 1887 *Cours d'analyse de l'école polytechnique, Tome Troisième* [Jordan, 1887], and more.

The problem is … the Converse Function Independence Theorem is not true.

[7] According to one of his biographers, Wroński submitted this paper to the French Academy of Sciences, but later withdrew it after the committee assigned to review it issued a negative report; the members of that committee included Lagrange himself [Pragacz, 2007, p.9].

4 "This wording is too general"

In 1889, Guiseppe Peano[8] opened his *Mathesis*[9] paper "Sur le déterminant Wronskien" with the following observation about the Converse Function Independence Theorem[10] [Peano, 1889a].

In nearly all papers, one finds the proposition: If the determinant formed with n functions of the same variable, and their derivatives of orders $1, \ldots, (n-1)$ is identically zero, there is between these functions a homogeneous linear relationship with constant coefficients.

This wording is too general.[11]

Task 4 This is not the converse of the Function Independence Theorem. What is the proposition that Peano stated above? And why doesn't that matter?

We will now take a short aside to provide the logic that Peano used when he approached disproving the theorem.

The key observation is that it is impossible for a statement and its negation to be true at the same time. So, suppose we have a general universal implication[12]:

$$\forall x,\, a(x) \implies b(x).$$

The negation of the above statement is

$$\exists\ x \text{ s.t. } a(x) \text{ and } \sim b(x).$$

Thus, if we find just *one* x so that $a(x)$ is true and $b(x)$ is false, we know the negation is true. And by the key observation, the initial statement *can't* be true. Such an x is called a *counterexample*, and this is exactly how Peano approached the problem.

[8]Peano (1858–1932) was an Italian mathematician who made essential contributions to foundational mathematics. He was an extraordinarily precise and rigorous mathematician. Indeed, "Peano had a great skill in seeing that theorems were incorrect by spotting exceptions" [O'Connor and Robertson, 1997], and he developed the Peano axioms with the goal of proving that $1 + 1 = 2$. Maybe we shouldn't be surprised that he found the statement "too general."

[9]*Mathesis* was a Belgian elementary mathematics journal founded in 1881 by Paul Mansion and Joseph Neuberg. It became the official journal of the Belgian Mathematical Society and was published until 1965, with a break for WWI.

[10]Peano translations by Susannah Engdahl, Wittenberg University class of 2013. See also [Engdahl and Parker, 2011].

[11]What a nice way to say, "You all are wrong."

[12]In what follows, we use the following notation: $\forall$ means "for all," $\exists$ means "there exists," and s.t. means "such that."

We offer, in fact,

$$x = t^2[1 + \Phi(t)] \qquad\qquad y = t^2[1 - \Phi(t)]$$

where $\Phi(t)$ designates a function equal to zero for $t = 0$, equal to 1 for positive t, to -1 for negative t.[‡]

Task 5

(a) Plot $\Phi(t)$ on the $(t, \Phi(t))$ plane.

(b) Fill in the following Table.

t	$\Phi(t)$	$x(t)$	$y(t)$
-3			
-2			
-1			
0			
1			
2			
3			

Table 1

(c) Verify the following passage from Peano.

One has for

$$t < 0 \qquad \Phi(t) = -1 \qquad x = 0 \qquad y = 2t^2;$$

for

$$t = 0 \qquad \Phi(t) = 0 \qquad x = 0 \qquad y = 0;$$

for

$$t > 0 \qquad \Phi(t) = 1 \qquad x = 2t^2 \qquad y = 0.$$

(d) Plot $x(t)$ and $y(t)$. Don't let the names of the variables scare you. This is just the normal plotting of two functions $f(x)$ and $g(x)$ where $x = t$, $f = x$ and $g = y$.

[‡] Peano provided a footnote in the paper here that provided two closed form functions that behave identically to the piecewise defined Φ.

Task 5 - continued

(e) Prove that both $x(t)$ and $y(t)$ are differentiable at $t = 0$. In particular, use the limit definition of the derivative and show that the left and right limits defining $x'(t)$ and $y'(t)$ agree.

(f) Give an argument as to why $x(t)$ and $y(t)$ are linearly independent. This argument can be geometric, or analytic. For example, if the functions were dependent, what would the constants c_1 and c_2 have to be to simultaneously solve

$$c_1\, x(1) + c_2\, y(1) = 0 \qquad \text{and} \qquad c_1\, x(-1) + c_2\, y(-1) = 0?$$

(g) Fill in the following table.

Table 2

t	$W(x(t), y(t))$	$\det\left(W(x(t), y(t))\right)$
-3		
-2		
-1		
0		
1		
2		
3		
$t > 0$		
$t < 0$		

At this point, it is important to highlight what has been accomplished. You have found two functions $f_1(t)$ and $f_2(t)$ such that: $f_1(t)$ and $f_2(t)$ are linearly independent on I and $\det\left(W(f_1(t), f_2(t))\right)$ is identically 0 for all $x \in I$. Using the symbolic statements p and q from page 4, this is an example of "q and $\sim p$," which is the negation of "$q \implies p$." In other words: the Converse of the Function Independence Theorem doesn't hold.

Task 6

(a) For the p and q we defined on page 4, write $\sim p \implies \sim q$ in words. Remember this is NOT always true, even when the implication $p \implies q$ holds.

(b) What is the inverse of the Function Independence Theorem? What can you say about the validity of that statement?

Peano concluded his short note with the following statement, in some sense challenging other mathematicians to continue examining this problem.

> The proposal is true if one supposes that there exists no value of t which cancels all the minors of the last line of the proposed determinant, and perhaps in other cases which would be interesting to investigate.

Probably the most common case that makes "the proposal true" is due to Maxime Bôcher.[13] In his 1901 "The Theory of Linear Dependence" [Bôcher, 1900/1901, p.91], he stated:

> The identical vanishing of the Wronskian of n analytic[14] functions is a necessary and sufficient condition for their linear dependence.

But many people discovered other conditions that would guarantee that an identically zero Wronskian meant that the functions were dependent. (See, for example, [Engdahl and Parker, 2011]). Even the *Mathesis* editor Pierre Mansion[15] placed a footnote at the end of Peano's paper giving his own condition.

> (*) One can correct the wording of the expressed theorem at the beginning of this note, as we have indicated in our *Résume d'analyse*, by adding the words: or one of the functions is identically zero. (P.M.)

It is not clear how this possibly corrects the theorem, as neither of Peano's functions $x(t)$ nor $y(t)$ are identically zero on I. Mansion's additional condition doesn't eliminate Peano's counterexample. This observation absolutely didn't escape Peano.

[13] Bôcher (1867–1918) was an American mathematician. He published foundational work in differential equations and series.

[14] *Analytic* functions are a class of functions which includes most functions (polynomial, exponential, trigonometric, logarithmic, etc.) with which we are familiar. There are also infinitely many differentiable functions, such as $x|x|$, that are not analytic.

[15] Mansion (1844–1919) was a Belgian mathematician who spent his entire educational and professional career at the University of Ghent.

5 "I must return to the subject"

Just a few pages later, in the same issue of *Mathesis* in which the original note appeared, an excerpt of correspondence between Peano and Mansion was published under the title "Sur les Wronskiens" [Peano, 1889b]. It began:

> The note added to the end of my brief article on the Wronskians (*Mathesis*, p. 75–76), if it contributes to the clarity, it adds nothing to the rigor of the contested proposition;[16] and I must return to the subject.

He then attacked the content of Mansion's footnote:

> If one of the functions, for example x_1, is identically zero, these functions are connected by the linear relation $x_1 + 0x_2 + \cdots + 0x_n = 0$. Thus the conclusion of the proposition in question is not at all modified if one adds the words: or one of the functions is identically zero.[17]

Peano next noted (as we did above) that even *if* Mansion's additional condition did make the theorem true, it wouldn't apply to Peano's examples as neither is "identically zero." He then gave a second counterexample—the one that is probably in your textbook.

> Neither one nor the other of the two functions is identically zero, because $x > 0$ if $t > 0$ and $y > 0$ if $t < 0$. But these two functions present this unusual characteristic that, for all values of t, one or the other is zero. To make this anomaly disappear, we propose
>
> $$X = t^2, \qquad\qquad Y = t \text{ mod } t.$$
>
> These two functions of t satisfy the condition $XY' - X'Y = 0$; they only cancel for $t = 0$, and between them there is no homogeneous linear relation. The sum and difference of these functions are the functions of my first example …

If you are familiar with modular arithmetic, you may recognize "mod" as an abbreviation for "modulo." However, Peano used it in this example to mean "modulus," or size, of the independent variable. Today we might write these functions as $X = t^2$ and $Y = t|t|$.

[16] What a nice way to say, "Mansion was wrong."

[17] Ouch … quoting Mansion back to Mansion.

Task 7 (a) Fill in the following table.

t	$X(t)$	$Y(t)$
-3		
-2		
-1		
0		
1		
2		
3		

Table 3

(b) Plot $X(t)$ and $Y(t)$. Don't let the names of the variables scare you. This is just the normal plotting of $f(x)$ and $g(x)$ where $x = t$, $f = X$ and $g = Y$.

(c) Verify Peano's statement that, "The sum and difference of these functions are the functions of my first example."

(d) Give an argument as to why $X(t)$ and $Y(t)$ are linearly independent. This argument can be geometric, or analytic. For example, if the functions were dependent, what would the constants c_1 and c_2 have to be to simultaneously solve

$$c_1\, X(1) + c_2\, Y(1) = 0 \qquad \text{and} \qquad c_1\, X(-1) + c_2\, Y(-1) = 0$$

thus showing there is no "homogeneous linear relation."

(e) Fill in the following table

t	$W(X(t), Y(t))$	$\det(W(X(t), Y(t)))$
-3		
-2		
-1		
0		
1		
2		
3		
$t > 0$		
$t < 0$		

Table 4

At this point, it is important (again) to highlight what has been accomplished. You have two functions $f_1(t)$ and $f_2(t)$ that are independent yet have identically zero Wronskian, and are only zero at one point. While the above might be the canonical example of independent functions giving zero Wronskian, they are in no way the only ones. Try these problems:

Task 8 American mathematician Maxime Bôcher (1867–1918) gave the following example of two functions that are linearly independent yet have a Wronskian that is identically zero [Bôcher, 1900, p. 120]. Prove this is true.

$$\mathsf{u}_1 = \begin{cases} e^{-\frac{1}{x^2}} & (x \neq 0), \\ 0 & (x = 0), \end{cases} \qquad \mathsf{u}_2 = \begin{cases} e^{-\frac{1}{x^2}} & (x > 0), \\ 0 & (x = 0), \\ 2e^{-\frac{1}{x^2}} & (x < 0), \end{cases}$$

Task 9 Building from Peano's examples, construct an example of three functions that are not linearly dependent, but have a Wronskian that is identically zero.

Task 10 In [Bôcher, 1901, p. 143], Bôcher gave the following example of three functions that are linearly independent yet have a Wronskian that is identically zero. Prove this is true.

$$\mathsf{u}_1 = \begin{cases} 1 + e^{-\frac{1}{x^2}} & (x \neq 0), \\ 1 & (x = 0), \end{cases} \qquad \mathsf{u}_2 = \begin{cases} 1 + e^{-\frac{1}{x^2}} & (x > 0), \\ 1 & (x = 0), \\ 1 - e^{-\frac{1}{x^2}} & (x < 0), \end{cases} \qquad \mathsf{u}_3 = 1$$

Task 11 As mentioned above, in response to Peano's challenge of finding additional conditions that could be included so that a zero Wronskian would be sufficient for independence, Bôcher showed that requiring the functions to be analytic would work [Bôcher, 1900/1901, p. 91]. Another theorem in the same publication [Bôcher, 1900/1901, p. 93] is:

Theorem 1. *Let $u_1, u_2, \ldots, u_n$ be functions of a real variable x, which at every point of I have finite derivatives of the first $n-1$ orders and an identically zero Wronskian $W(u_1, u_2, \ldots, u_n)$. In addition assume that $n-1$ of the functions can be selected so that their Wronskian and its first derivative do not vanish together at any point of the interval. Then $u_1, u_2, \ldots, u_n$ are linearly dependent.*

American mathematician David Raymond Curtiss[18] (1878–1953) later presented the following theorem[19] in [Curtiss, 1908, p. 292].

Theorem 2. *Let $u_1, u_2, \ldots, u_n$ be functions of a real variable x, which at every point of I have finite derivatives of the first $n-1$ orders and an identically zero Wronskian $W(u_1, u_2, \ldots, u_n)$. Form the $n \times (n-1)$ matrix M where the $n-1$ functions $\{u_1, u_2, \ldots, u_{n-1}\}$ form the first row, their first derivatives $\{u_1', u_2', \ldots, u_{n-1}'\}$ form the second row, their second derivatives $\{u_1'', u_2'', \ldots, u_{n-1}''\}$ form the third row, etc. Assume that all of the $(n-1)$-rowed determinants of M do not simultaneously vanish at any point of I. Then $u_1, u_2, \ldots, u_n$ are linearly dependent.*

Curtiss then gave the following example of four functions that have a Wronskian that is zero on any interval containing the origin and are linearly dependent.

$$\mathsf{u}_1 = -(\mathsf{x}^4 + \mathsf{x}^3 + \mathsf{x} + 1), \qquad \mathsf{u}_2 = \mathsf{x} + 1,$$

$$\mathsf{u}_3 = \begin{cases} \mathsf{x}^2 + \mathsf{x} + 1 & (\mathsf{x} > 0), \\ \mathsf{x}^4 + \mathsf{x}^2 + \mathsf{x} + 1 & (\mathsf{x} \le 0) \end{cases} \qquad \mathsf{u}_4 = \begin{cases} \mathsf{x}^4 + \mathsf{x}^3 - \mathsf{x}^2 - \mathsf{x} - 1 & (\mathsf{x} > 0), \\ \mathsf{x}^3 - \mathsf{x}^2 - \mathsf{x} - 1 & (\mathsf{x} \le 0) \end{cases}$$

Show that these four functions satisfy the additional hypotheses of Curtiss' Theorem 2, but not those of Bôcher's Theorem 1. A computer algebra system will be helpful.

6 Conclusion

For years, "nearly all papers" contained a Theorem that wasn't true. Hermite, Jordan, and Laurent are some of the most influential mathematicians of the end of the 19th century and they fell into this category. Peano was able to give an elementary counterexample that we still teach in differential equations today. The value of this project isn't just the example itself. Rather, it is more important to truly understand what has actually been proved, how the counterexample engages with the logic of the statement, and how hypotheses can be added to make a theorem true.

[18] Curtiss was one of Bôcher's students at Harvard.

[19] This is the $k = n-1$ case.

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Notes to Instructors

This set of notes accompanies the Primary Source Project "Wronskians and Linear Independence: A Theorem Misunderstood by Many" written as part of the TRIUMPHS project. (See end of notes for details about TRIUMPHS.)

PSP Content: Topics and Goals

This mini-Primary Source Project (mini-PSP) is designed for use in a differential equations course, a linear algebra course, or a bridge proofs course. A bit of each class's material is needed regardless of how you use it (for example some predicate logic is needed even if used in a linear algebra course), but the emphasis can definitely be different.

1. If being used in a differential equations course, the emphasis can be on the Wronskian itself. Students will need to find fundamental sets of solutions for higher order linear equations, and this theorem is essential for doing that. The Wronskian also appears when calculating the coefficient functions during variation of parameters. The hypotheses that can be added to make the theorem true are also helpful, as typically the functions being tested are analytic and thus the converse *does* hold by [Bôcher, 1900/1901]. This is how I use the project.
2. If being used in a linear algebra course, the similarities and differences between linear independence of vectors and functions can be highlighted. All students should have an understanding of linear independence of vectors and the Vector Independence Theorem, but this project can also serve as an introduction to linear combinations of objects other than vectors.
3. If being used in a proofs course, the predicate logic can be highlighted. The relationships between if and only if statements, universal and existential conditional statements, negation, contrapositives, inverses, etc., can all be found in this project and thus provides a concrete example of how all these logical statements interact with each other.

In all cases, the goal is to truly understand what the Function Independence Theorem says, how the counterexample engages with the logic of the statement, and how hypotheses can be added to make the theorem true.

I think this project is most valuable from a pedagogical standpoint. Students only see mathematics presented in textbooks where everything is correct, the notation is optimized, and results logically build on previous work. Since that is all they've ever seen, they believe that is how they should be producing mathematics. And when they don't—when they make mistakes or struggle—they are discouraged as they have never seen a mathematician struggle. The value of incorporating the history of mathematics generally, and perhaps this project specifically, is that it explicitly shows that an approachable theorem can be misunderstood for years. Counterexamples to the theorem can be both relatively simple and plentiful. And first-rate mathematicians may still struggle to "fix" the theorem even when such counterexamples are known. Seeing that some of the finest mathematicians of all time placed false theorems in their publications gives permission for a student to fail on a HW problem in class and not become unduly discouraged.

Student Prerequisites

Students will need to know how to take the determinant of at least a 2×2 matrix and larger matrices if Tasks 9, 10 or 11 are assigned. Students will also need to repeatedly take derivatives in order to construct a Wronskian. The required differentiation techniques largely depend on the examples used.

Students may benefit from a knowledge of predicate logic, though this project does explain many of the definitions and implications they would need.

PSP Design, and Task Commentary

This PSP has six sections.

The Introductory section is a short motivation for the project. It argues that all mathematicians worry they've made an error when they've "proven" something. That is a theme in this project. Fine mathematicians make mistakes, even publish them. It's ok for our students to do so as well. This section should be assigned as prior reading.

Section 2 is a quick overview of the Linear Independence of Vectors. If you are using this project in a linear algebra class, this can certainly be assigned as prior reading. That can be done for other courses as well, though this section does contain the predicate logic that will be needed to carefully unravel the logic of the Function Independence Theorem and how Peano's counterexample plays into that logic.

Section 3 shouldn't be skipped regardless of the class the project is being utilized in as it provides much of the notation and definitions used later. It defines linear independence of functions, the Wronskian, the Function Independence Theorem, along with all the permutations of that theorem.

Section 4 provides Peano's first counterexample along with tasks to help students understand the counterexample and how it actually disproves the converse of the Function Independence Theorem. The end of the section shows Peano asked for what conditions needed to be added in order to make the theorem true. The editor of the journal, Pierre Mansion, provided one answer.

Section 5 shows that Mansion's answer was wrong. Peano provided an additional counterexample to disprove Mansion's "improvement." This is the counterexample that appears in most textbooks. Task 7 is the task that walks the students toward understanding that counterexample in their texts. Several other counterexamples are included as Tasks for the students to actually show how they play out. They can be assigned or not.

Section 6 is simply a short conclusion.

Suggestions for Classroom Implementation

Please see student requirements and implementation schedule for suggestions.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Sample Implementation Schedule (based on a 60-minute class period)

If an instructor decides to use the entire mini-PSP in class and as is, it will take two class periods. However, as with most projects, some modification would be helpful. This may be particularly necessary with this PSP as it can be implemented in several different classes. For example if using this project in an introduction to proofs course, it would be essential to spend a lot of time on Task 3 and the passage after Task 4. If using this project in a linear algebra course, some of Tasks 8–11 can be skipped. In all cases, some of the early sections can be assigned as advance outside-of-class reading. This mini-PSP is also designed to be done in groups.

The author uses this in a differential equations course with the following implementation schedule:

- Advance reading / work: Read the first 2.5 Sections, up to the Function Independence Theorem. Complete Tasks 1 and 2.

- First 20 minutes of class time: cover predicate logic. Task 3 is the most important here, and I make sure that the contrapositive, converse, inverse, etc., are written carefully on the side board for reference throughout the class.

- Next 30 minutes of class time: cover Peano's first counterexample. Task 5 is key here. Specifically, part (f) where students argue $x(t)$ and $y(t)$ are linearly independent and part (g) where students calculate the Wronskian is identically zero.

- Final 10 minutes of class: reading and speaking about how Peano ended his paper. He asked for conditions that could be appended to the hypotheses to make the converse of the Function Independence Theorem true. It is possible that the student text has a version of such a theorem and that can be highlighted at this point, showing how the additional conditions (whatever they are) now exclude Peano's counterexample.

At this point, the project could end if needed, with most of the pedagogical goals described met. Rather than seeing only the polished version of a theorem, they will see the stops and starts and false starts that are necessary to produce a final project. They should expect to experience the same struggles as they produce mathematics. They will also get practice thinking about the logic of a statement and how hypotheses play into its validity.

However, I believe that the project benefits from another half day of work (or a total of 90 minutes of class time). Peano's second paper showed that Mansion's attempt at adding hypotheses didn't work as it didn't actually exclude Peano's counterexample. Peano then provided a second, even simpler, counterexample. Task 7 mimics Task 5 from earlier and should be completed to understand this second counterexample. Tasks 8–11 can then be worked in class or assigned as homework depending on the time available.

Connections to other Primary Source Projects

The author has also developed a series of three mini-PSPs that share the name "Solving Linear First Order Differential Equations," designed to show three solutions to non-homogeneous first-order linear differential equations, each from a different context.

- The mini-PSP subtitled "Gottfried Leibniz' 'Intuition and Check' Method" explains how in 1694 Leibniz solved these equations using a one-off method applicable only to this specific problem. Strictly speaking, Leibniz didn't solve the equation, but asserted a solution and then showed it worked. Part of his proposed solution will be familiar to the students: it is the standard integrating factor method we teach today.

- The mini-PSP subtitled "Johann Bernoulli's (Almost) Variation of Parameters Method" explains how in 1697, Bernoulli solved these equations using variation of parameters decades before Lagrange received credit for the technique. Again, part of Bernoulli's solution will be the standard integrating factor.

- The mini-PSP subtitled "Leonard Euler's Integrating Factor Method" explains how in 1763, Euler solved these equations as a special case of exact differential equations by finding an integrating factor. His integrating factor is the same as the one students would have seen.

All three of these mini-PSPs are designed for use in an Ordinary Differential Equations course, and can be used alone or in conjunction with each other. They are available via the TRIUMPHS website, at https://digitalcommons.ursinus.edu/triumphs_differ/.

Recommendations for Further Reading

Those who are interested in additional details, in particular the purported proofs of the Converse Function Independence Theorem, may be interested in reading the *Convergence* article "Peano on Wronskians: A translation" by Susannah M. Engdahl and Adam E. Parker. It is available at:

www.maa.org/press/periodicals/convergence/peano-on-wronskians-a-translation-introduction.

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For more information about TRIUMPHS, visit `https://blogs.ursinus.edu/triumphs/`.