Appendix: Translation of Euler's E744

On divisors of numbers contained in the form $mxx + nyy$ *

Leonhard Euler

1. We shall always assume in the following that the two numbers x and y are prime between themselves, and it is noted that numbers contained in such a form can be divided only by certain prime numbers, but always, for whatever form, certain prime numbers are excluded, the number of which is half of all prime numbers, as it were. It has thus been demonstrated that numbers contained in the form $xx + yy$ cannot be divided by other prime numbers, unless they are of the form $4N + 1$, and likewise prime numbers of the form $4N - 1$ are to be thoroughly excluded.

2. In the same way it is has been demonstrated that numbers contained in the form $2xx + yy$ do not admit other prime divisors, unless they are contained in either of these forms: $8N + 1$ or $8N + 3$. So the remaining prime numbers are contained in the forms $8N + 5$ and $8N + 7$. In the same way, all prime divisors of numbers of the form $3xx + yy$ are either of the form $12N + 1$ or $12N + 7$. Certainly, the remaining ones, which are of the form $12N + 5$ or $12N + 11$, can never be divisors, from which it is evident that all divisors are contained in the form $6N + 1$ while excluding those in the form $6N + 5$.

3. In a not dissimilar way, for something composed in the general form $mxx+ nyy$, all prime divisors of it are contained in fixed forms in this way: $4mnN + \alpha$, $4mnN + \beta$, $4mnN + \gamma$ etc., where α, β, γ, etc. are certain fixed numbers that for whatever case you wish can be easily determined, excluding however, numbers contained in the corresponding other forms $4mnN - \alpha$, $4mnN - \beta$, $4mnN - \gamma$, etc. Because it can be expressed in the following convenient way in the general

*Originally published as *De divisiborus numerorum in forma mxx + nyy contentorum*, Mémoires de l'académie des sciences de St-Pétersbourg **5** (1812), pp. 3–23. E744 in the Eneström index. Translated from the Latin by Paul R. Bialek, Department of Mathematics, Trinity International University, Deerfield, Illinois, email: pbialek@tiu.edu; and Dominic W. Klyve, Department of Mathematics, Central Washington University, Ellensburg, Washington, email: klyved@cwu.edu.

Bialek, Paul, and Dominic W. Klyve, Appendix to "Did Euler Know Quadratic Reciprocity?: New Insights from a Forgotten Work," MAA *Convergence* (February 2014), DOI:

form $mxx+nyy$, the form of the divisors is determined to be $4mnN+\alpha, \beta, \gamma, \delta, \epsilon$, etc., but the form of excluded numbers is $4mnN-\alpha, -\beta, -\gamma, -\delta, -\epsilon$, etc.

4. First, for whatever case of the numbers m and n, it is evident that the numbers $\alpha, \beta, \gamma, \delta$ etc. should be prime with respect to the number $4mn$, because otherwise prime numbers cannot be produced. Therefore one can also easily understand that unity and all square numbers prime to $4mn$ are contained among the numbers α, β, γ, etc. Additionally, in the class of these numbers $\alpha, \beta, \gamma, \delta$, etc., all perfect powers also occur, then also all products of two or three, such as $\alpha\beta, \alpha\beta\gamma$, to the extent, of course, that they do not exceed the number $4mn$. Finally, it will also help to have observed that in any case these numbers $\alpha, \beta, \gamma, \delta$ depend so much on the product mn, that these two general forms $mxx+nyy$ and $m'xx+n'yy$, have the same forms of divisors only if $m'n'=mn$.

5. I hardly think it will be out of place to adjoin the following table, which, for simple numbers mn, shows the forms of the prime divisors:

Values of the product mn	Form of the divisors
1	$4N+1$
2	$8N+1$, 3
3	$12N+1$, 7
5	$20N+1$, 3, 7, 9
6	$24N+1$, 5, 7, 11
7	$28N+1$, 9, 11, 15, 23, 25
10	$40N+1$, 7, 9, 11, 13, 19, 23, 37
11	$44N+1$, 3, 5, 9, 15, 23, 25, 27, 31, 37
13	$52N+1$, 7, 9, 11, 15, 17, 19, 25, 29, 31, 47, 49
14	$56N+1$, 3, 5, 9, 13, 15, 19, 23, 25, 27, 39, 45
15	$60N+1$, 17, 19, 23, 31, 47, 49, 53

6. Certainly in these forms are all numbers which can be divisors of any numbers contained in the form $mxx+nyy$, and are made by dividing by $4mn$ and reduced to a value under the limit $4mn$. But if, however, we want to admit negative numbers, then all those numbers can be reduced to less than $2mn$; and in this way all whole numbers prime to $4mn$ and less than $2mn$ occur, written with either the + or − sign. And since the complements of these numbers to $4mn$ produce the excluded numbers, it is necessary only to permute the signs of these forms, in order to obtain all the excluded numbers, which can never be divisors of any numbers of the form $mxx+nyy$.

7. But if in this way we arrange the divisors of the form $mxx+nyy$ so that all whole numbers prime to and less than $2mn$ occur, exceptional properties in them will be discerned if for each of these numbers we write below it its complement to $2mn$; in this way the upper row will proceed only up to mn,

while those greater than mn up to $2mn$ are written under these. For various numbers mn, however, the property is well known that a pair of complements, one written under the other, have the same sign or opposite signs. Certainly they have the same signs in the cases when either $mn = 4i + 1$ or $mn = 4i + 2$. However, in the remaining two cases, $mn = 4i + 3$ or $mn = 4i + 4$, that pair of complements will have opposite signs.

8. In this way, therefore, we represent the above formulas for the divisors of numbers contained in the form $mxx + nyy$, and we continue further in this way:

mn	Form of the divisors								
1	$4N$	$+1$							
2	$8N$	$+1$							
		$+3$							
3	$12N$	$+1$							
		-5							
4	$16N$	$+1,$	-3						
		$-7,$	$+5$						
5	$20N$	$+1,$	$+3$						
		$+9,$	$+7$						
6	$24N$	$+1,$	$+5$						
		$+11,$	$+7$						
7	$28N$	$+1,$	$-3,$	-5					
		$-13,$	$+11,$	$+9$					
8	$32N$	$+1,$	$+3,$	$-5,$	-7				
		$-15,$	$-13,$	$+11,$	$+9$				
9	$36N$	$+1,$	$+5,$	-7					
		$+17,$	$+13,$	-11					
10	$40N$	$+1,$	$-3,$	$+7,$	$+9$				
		$+19,$	$-17,$	$+13,$	$+11$				
11	$44N$	$+1,$	$+3,$	$+5,$	$-7,$	$+9$			
		$-21,$	$-19,$	$-17,$	$+15,$	-13			
12	$48N$	$+1,$	$-5,$	$+7,$	-11				
		$-23,$	$+19,$	$-17,$	$+13$				
13	$52N$	$+1,$	$-3,$	$-5,$	$+7,$	$+9,$	$+11$		
		$+25,$	$-23,$	$-21,$	$+19,$	$+17,$	$+15$		
14	$56N$	$+1,$	$+3,$	$+5,$	$+9,$	$-11,$	$+13$		
		$+27,$	$+25,$	$+23,$	$+19,$	$-17,$	$+15$		
15	$60N$	$+1,$	$-7,$	$-11,$	-13				
		$-29,$	$+23,$	$+19,$	$+17$				
16	$64N$	$+1,$	$-3,$	$+5,$	$-7,$	$+9,$	$-11,$	$+13,$	-15
		$-31,$	$+29,$	$-27,$	$+25,$	$-23,$	$+21,$	$-19,$	$+17$
17	$68N$	$+1,$	$+3,$	$-5,$	$+7,$	$+9,$	$+11,$	$+13,$	-15
		$+33,$	$+31,$	$-29,$	$+27,$	$+25,$	$+23,$	$+21,$	-19

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18	$72N$	$+1,$	$-5,$	$-7,$	$+11,$	$-13,$	$+17$					
		$+35,$	$-31,$	$-29,$	$+25,$	$-23,$	$+19$					
19	$76N$	$+1,$	$-3,$	$+5,$	$+7,$	$+9,$	$+11,$	$-13,$	$-15,$	$+17$		
		$-37,$	$+35,$	$-33,$	$-31,$	$-29,$	$-27,$	$+25,$	$+23,$	-21		
20	$80N$	$+1,$	$+3,$	$+7,$	$+9,$	$-11,$	$-13,$	$-17,$	-19			
		$-39,$	$-37,$	$-33,$	$-31,$	$+29,$	$+27,$	$+23,$	$+21$			
21	$84N$	$+1,$	$+5,$	$+11,$	$-13,$	$+17,$	$+19$					
		$+41,$	$+37,$	$+31,$	$-29,$	$+25,$	$+23$					
22	$88N$	$+1,$	$-3,$	$-5,$	$-7,$	$+9,$	$+13,$	$+15,$	$-17,$	$+19,$	$+21$	
		$+43,$	$-41,$	$-39,$	$-37,$	$+35,$	$+31,$	$+29,$	$-27,$	$+25,$	$+23$	
23	$92N$	$+1,$	$+3,$	$-5,$	$-7,$	$+9,$	$-11,$	$+13,$	$-15,$	$-17,$	$-19,$	-21
		$-45,$	$-43,$	$+41,$	$+39,$	$-37,$	$+35,$	$-33,$	$+31,$	$+29,$	$+27,$	$+25$
24	$96N$	$+1,$	$+5,$	$+7,$	$+11,$	$-13,$	$-17,$	$-19,$	-23			
		$-47,$	$-43,$	$-41,$	$-37,$	$+35,$	$+31,$	$+29,$	$+25$			
25	$100N$	$+1,$	$-3,$	$-7,$	$+9,$	$-11,$	$+13,$	$+17,$	$-19,$	$+21,$	-23	
		$+49,$	$-47,$	$-43,$	$+41,$	$-39,$	$+37,$	$+33,$	$-31,$	$+29,$	-27	

9. And if we contemplate these examples rightly, we will be able to glean outstanding theorems from them, which merit all the more attention, because the principles, from which the desired demonstration would follow, are still most entirely unknown, so that such consideration may uncover for us a vast area of study: investigating more deeply the nature of the numbers.

Theorem 1

10. Let p denote an arbitrary number prime to $2mn$. If $4mna+p$ is a divisor of some number contained in the form $mxx + nyy$, then all prime numbers contained in the formula $4mnz + p$ will certainly be divisors of our proposed form. Conversely, certainly all whole numbers of the form $4mnz - p$ will be excluded from this class of divisors.

Theorem 2

11. Let p denote a number prime to $2mn$. If $4mna + p$ is a prime number which is not a divisor of any number contained in the form $mxx + nyy$, then all whole numbers contained in the form $4mnz+p$, whether prime or composite, will be excluded from this class of divisors. Conversely, certainly all prime numbers of the form $4mnz - p$ will be divisors of some number contained in the form $mxx + nyy$.

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Theorem 3

12. Let p denote a number prime to $2mn$. If the number $2mna - p$ is a divisor of the proposed form $mxx + nyy$, then all prime numbers contained in the form $4mnz - p$ will certainly be divisors of the proposed form.[1] Conversely, certainly all whole numbers contained in the form $4mnz + p$ will be excluded from this class of divisors.

Theorem 4

13. Let p denote a number prime to $2mn$. If $4mna - p$ is a prime number which is not a divisor of any number contained in the form $mxx + nyy$, then all whole numbers contained in the form $4mnz - p$, whether prime or composite, will be excluded from this class of divisors. Conversely, certainly all prime numbers of the form $4mnz + p$ will be divisors of some number contained in the form $mxx + nyy$.

Theorem 5

14. If mn is a number of the form $4i + 1$ or $4i + 2$, and $4mna + p$ is a divisor of the form $mxx + nyy$, so that all prime numbers contained in the form $4mnz + p$ are divisors of the proposed form, then all prime numbers contained in the form $4mnz + 2mn - p$ will also be divisors of the proposed form. Conversely, certainly all numbers of the form $4mnz - 2mn + p$ or $4mnz + 2mn + p$ will be excluded from this class of divisors.

Theorem 6

15. If mn is a number of the form $4i + 1$ or $4i + 2$, and $4mna - p$ is a divisor of the form $mxx + nyy$, so that all prime numbers contained in the form $4mnz - p$ are divisors of the proposed form, then all prime numbers contained in the form $4mnz - 2mn + p$ will also be divisors of the proposed form. Conversely, certainly all numbers of the form $4mnz - 2mn - p$ or $4mnz + 2mn - p$ will be excluded from this class of divisors.

[1] Translator: This is not always true, so it may be a typo. For example, when $mn = 13$, $a = 1$, and $p = 19$, then $2mna - p = 7$, which can be a prime divisor of $mxx + nyy$, since it is of the form $52N + 7$. However, $4mnz - p$ is the congruence class $52N - 19$, which has no prime divisors of $mxx + nyy$. Theorem 6 tells us that Theorem 3 fails for numbers mn of the form $4i + 1$ or $4i + 2$, like $mn = 13$. A correct statement would be as follows: If mn is a number of the form $4i$ or $4i - 1$, and the number $2mna - p$ is a divisor of the proposed form $mxx + nyy$, then all prime numbers contained in the form $4mnz - p$ will certainly be divisors of the proposed form.

Theorem 7

16. If mn is a number of the form $4i$ or $4i-1$, and $4mna+p$ is a divisor of the form $mxx+nyy$, so that all prime numbers contained in the form $4mnz+p$ are divisors of the proposed form, then all prime numbers contained in the form $4mnz-2mn+p$ will also be divisors of the proposed form. Conversely, however, all numbers of the form $4mnz-2mn-p$ will be excluded from this class of divisors.

Theorem 8

17. If mn is a number of the form $4i$ or $4i-1$, and $4mna-p$ is a divisor of the form $mxx+nyy$, so that all prime numbers contained in the form $4mnz-p$ are divisors of the proposed form, then all prime numbers contained in the form $4mnz+2mn-p$ will also be divisors of the proposed form. Conversely, certainly all numbers contained in the form[2] $4mnz-2mn+p$ will be excluded from this class of divisors.

Corollary

18. Therefore, let p denote any number prime to $2mn$. All prime numbers contained in the form $4mnz+p$ or $4mnz-p$ will certainly be divisors of the proposed form.

Theorem 9

19. If in the general formula for divisors of the proposed form, as shown above, whether parts f and g occur as positive or negative, then also in that very place their product fg will occur, and precisely in the class not only any of their powers f^μ and g^ν but also all products $f^\mu g^\nu$ of the pairs, by means of the rule of the signs rightly considered, namely when these numbers, divided by $4mn$, will be reduced to less than the limit $2mn$.

Corollary

20. Hence, it is clear that if p denotes any number prime to $2mn$, then all prime numbers contained in the form $4mnz+pp$ will always be divisors of the proposed form. Conversely, certainly all numbers contained in the form $4mnz-pp$ are excluded from this class of divisors.

[2] Translator: Original version has $4mnz-2mn-p$.

Note

21. Concerning the general form, which here we show for divisors of the proposed form, it does not rightly hold for all numbers contained in that form that they can be confirmed to be divisors; truly this is valid only for prime numbers, since cases can occur in which composite numbers contained in this form consist of factors excluded from this class of divisors; conversely, certainly all whole numbers in the form given for excluded numbers, whether they are prime or composite, are always excluded. Now one may give another far more reliable method, condensing formulas for divisors of all the proposed forms easily, which we will show in the following problem.

Problem

Given any form of numbers $mxx + nyy$, to find the general formula which includes the divisors of all numbers contained in it.

Solution

22. Here, before all the results are to be obtained, the letters x and y should always be interpreted as numbers prime between themselves, because otherwise all whole number divisors could exist. Then also it is immediately evident that among the divisors themselves, the numbers m and n and their factors can occur, from which our problem is restricted only to divisors which are prime to the numbers m and n, and also, so that 2 is excluded, divisors which which are prime to $2mn$ should be sought.

23. Above however we obtained the principal formula $4mnN$ to incorporate all numbers prime to $2mn$ and less than mn, by which numbers the series above is obviously constituted, and here all the work was reduced to this: putting the appropriate sign in front of each of these numbers; this certainly is freely evident in the first of these numbers, namely unity, always with a + sign attached, and because the signs of composite numbers follow the rules of multiplication, it suffices to redirect this investigation to prime numbers.

24. Therefore let p be any prime number less than mn and different from both m and n and this sign + will be put before this number. When the given number of the form $mxx + nyy$ is divisible by p, then such a number $mn + yy$ which likewise is also divisible by p can also be given, and precisely so that $y < \frac{1}{2}p$. Therefore when $p < mn$, [3] nothing else is required except that all values from 1 to $\frac{1}{2}mn$ be assigned to the letter y in order, and all prime divisors of the resulting numbers less than mn and different from m and n be written, since the sign + will be put before them.

[3] Translator: First edition has $p > mn$.

25. To these signed numbers, therefore, it is necessary to give the opposite sign $-$ to the remaining numbers prime to mn; when done, the appropriate signs appear before composite numbers occurring in the same series above, by reason of multiplication.

26. After, however, the upper series of numbers is obtained in this way, for the lower series, which contains the complements of the upper numbers to $2mn$, either the same sign or a different sign appears before; certainly the former when the number mn will be of the form $4i+1$ or $4i+2$, certainly the latter when it will be of the form $4i$ or $4i-1$, and in this way all formulas for the divisors will be complete.

Example 1

27. Assume $mn = 24$; since the number is of the form $4i$, opposite signs are to be given to the lower series. Now the numbers prime to $2mn$ and less than 24 are 1, 5, 7, 11, 13, 17, 19, and 23, which are all also prime, from which in the formula $24+yy$, y itself is assigned in order the values 1, 2, 3, etc. up to 12, and thus arises a progression of numbers arising according to the odd numbers 1, 3, 5, 7, etc., whose terms are noted to be prime divisors less than 24, excluding 3, which most conveniently in the following manner will make:

24		Divisors
1	25	5
3	28	7
5	33	11
7	40	5
9	49	7
11	60	5
13	73	–
15	88	11
17	105	7
19	124	–
21	145	–
23	168	7

From this, therefore, it is evident that only the prime numbers 1, 5, 7 and 11 are to have the sign $+$ affixed, the others indeed having the sign $-$; from this, the formula for divisors with the series of complements being written below will be:

$$\begin{array}{lllllllll} 96N & +1, & +5, & +7, & +11, & -13, & -17, & -19, & -23 \\ & -47, & -43, & -41, & -37, & +35, & +31, & +29, & +25 \end{array}$$

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Example 2

28. Let $mn = 26$; since the number is of the form $4i + 2$, the lower series should have the same signs as the upper. Now the formula $26 + yy$, y itself to be assigned the values 1, 2, 3, up to 13, will produce for us prime divisors less than 26, excluding 13, so that the following calculation shows:

26		Divisors	
1	27	3	
3	30	3	5
5	35	5	7
7	42	3	7
9	51	3	17
11	62	–	
13	75	5	3
15	90	3	5
17	107	–	
19	126	3	7
21	147	3	7
23	170	5	17
25	195	3	5

The prime numbers, which here receive the sign +, are 1, 3, 5, 7, 17; the sign − is put before the remaining ones, obviously, 11, 19, 23; however, signs arising from the rule of multiplication are to be given to composite numbers, from which the formula for divisors will be formed in the following way:

$$\begin{array}{lcccccccccccc} 104N & +1, & +3, & +5, & +7, & +9, & -11, & +15, & +17, & -19, & +21, & -23, & +25 \\ & +51, & +49, & +47, & +45, & +43, & -41, & +37, & +35, & -33, & +31, & -29, & +27 \end{array}$$

Example 3

29. Assume $mn = 27$; and since this number is of the form $4i - 1$, the complements to be written below should be marked with opposite signs. Now if we examine the formula $27 + yy$, the prime divisors less than 27, those divisible by 3 excluded, will be 1, 7, 13, and 19, marked with the sign +, so that the remaining primes with sign −, will be 5, 11, 17, 23, on account of which the general formula for divisors will be

$$\begin{array}{llllllllll} 108N & +1, & -5, & +7, & -11, & +13, & -17, & +19, & -23, & +25 \\ & -53, & +49, & -47, & +43, & -41, & +37, & -35, & +31, & -29 \end{array}$$

Example 4

30. Assume $mn = 28$; and since this number is of the form $4i$, the complements to be written below should be marked with opposite signs. Now if we examine the formula $28 + yy$, the prime divisors less than 28, those divisible by 7 excluded, will be 1, 11, and 23, marked with the sign $+$; obviously the remaining primes will be 3, 5, 13, 17, and 19, marked with the sign $-$; therefore, the general formula for divisors will be

$112N$	$+1,$	$-3,$	$-5,$	$+9,$	$+11,$	$-13,$	$+15,$	$-17,$	$-19,$	$+23,$	$+25,$	-27
	$-55,$	$+53,$	$+51,$	$-47,$	$-45,$	$+43,$	$-41,$	$+39,$	$+37,$	$-33,$	$-31,$	$+29$

Example 5

31. Assume $mn = 30$; and since this number is of the form $4i + 2$, the numbers to be written below should be marked with the same signs as those above. Now the numbers prime to $2mn$ and less than 30 are 1, 11, 13, 17, 23, and 29, marked with the sign $+$; obviously the remaining prime numbers are 7 and 19, marked with the sign $-$; therefore, the general formula for divisors will be

$120N$	$+1,$	$-7,$	$+11,$	$+13,$	$+17,$	$-19,$	$+23,$	$+29$
	$+59,$	$-53,$	$+49,$	$+47,$	$+43,$	$-41,$	$+37,$	$+31$

Example 6

32. Let us assume $mn = 50$; and since this number is of the form $4i + 2$, the complements to be written below should be marked with the same signs. Now the formula $50 + yy$ produces a sequence of prime divisors less than 50, except 5, which, marked with the sign $+$, are 1, 3, 11, 17, 19, 41, 43; obviously the remaining primes are 7, 13, 23, 29, 31, 37, and 47, marked with the sign $-$; therefore, the general formula for divisors will be

$200N$	$+1,$	$+3,$	$-7,$	$+9,$	$+11,$	$-13,$	$+17,$	$+19,$	$-21,$	-23
	$+99,$	$+97,$	$-93,$	$+91,$	$+89,$	$-87,$	$+83,$	$+81,$	$-79,$	-77
	$+27,$	$-29,$	$-31,$	$+33,$	$-37,$	$-39,$	$+41,$	$+43,$	$-47,$	$+49$
	$+73,$	$-71,$	$-69,$	$+67,$	$-63,$	$-61,$	$+59,$	$+57,$	$-53,$	$+51$

Example 7

33. Finally, let us assume $mn = 60$; and since this number is of the form $4i$, the complements to be written below should be marked with opposite signs. Now if we examine the formula $60 + yy$, the prime divisors less than 60, except 3 and 5, obtained are 1, 17, 19, 23, 31, 47, 53, marked with the sign +; so that the remaining primes with sign −, will be 7, 11, 13, 29, 37, 41, 43, and 59; therefore, the general formula for divisors will be

$240N$	$+1,$	$-7,$	$-11,$	$-13,$	$+17,$	$+19,$	$+23,$	$-29,$	$+31$
	$-119,$	$+113,$	$+109,$	$+107,$	$-103,$	$-101,$	$-97,$	$+91,$	-89
	$-37,$	$-41,$	$-43,$	$+47$	$+49,$	$+53,$	-59		
	$+83,$	$+79,$	$+77,$	$-73,$	$-71,$	$-67,$	$+61$		

Addition
to the discussion about the divisors of numbers contained in the form
$mxx + nyy$

From this, there will not be just a single observation added here about the last sections and their formulas for divisors, which of course one may assign in general if only the six cases are distinguished from one another in turn.

Observation 1

If $mn = 4i$, then in the given formula for divisors in the upper series, the last term will always be $-(4i - 1)$, and its complement to $8i$, itself written below, will be $+(4i + 1)$, since it should have the opposite sign. So when $mn = 4i$, then $mn + 1 = 4i + 1$, a number which, because it is prime to mn, clearly ought to have the sign +, and therefore its complement, written above, has the opposite sign −.

Observation 2

If $mn = 4i + 2$, in which case the complements have the same signs, then $mn + 1 = 4i + 3$, a number which, because it is prime to mn, should have the sign +, and therefore its complement, written above, similarly will have the sign +.

Observation 3

If $mn = 8i+1$, in which case the pair of complements have the same signs, then in the upper series, the last term will always be $8i-1$, and its complement will be $8i+3$, and the sign $-$ should be put placed in front of both, and thus in this case the last terms will be

$$\begin{array}{c} -(8i-1) \\ -(8i+3). \end{array}$$

Moreover, in this case one may attach signs to the penultimate terms. So when $mn+4 = 8i+5$, the sign $+$ will be put before this number, and therefore the same sign will also be put before its complement $8i-3$; therefore it is evident from these cases in which $mn = 8i+1$ that the last pairs of terms should be

$$\begin{array}{cc} +(8i-3) & -(8i-1) \\ +(8i+5) & -(8i+3). \end{array}$$

Observation 4

If $mn = 8i+3$, the last term in the upper series will be $8i+1$, and its complement, marked with the opposite sign, will be $8i+5$. However, because the upper number can be a square, it should have the sign $+$, and therefore, its complement, the sign $-$; clearly the penultimate numbers will be $8i-1$, $8i+7$, of which the lower, because $mn+4 = 8i+7$, receives the sign $+$, and therefore the upper, the opposite sign $-$; from this, for the cases in which $mn = 8i+3$, the last pairs will be

$$\begin{array}{cc} -(8i-1) & +(8i+1) \\ +(8i+7) & -(8i+5). \end{array}$$

Observation 5

If $mn = 8i+5$, the last forms of the divisors will be $8i+3$, $8i+7$ and marked with equal signs, which are found to be $+$; however, the penultimate terms will be $8i+1$, $8i+9$, of which the lower, because it is $mn+4$, will have the sign $+$ and therefore also the one written above. Consequently, for the case in which $mn = 8i+5$, the last pairs of terms in the formula for divisors will be

$$\begin{array}{ll} +(8i+1) & +\,(8i+3) \\ +(8i+9) & +\,(8i+7). \end{array}$$

Observation 6

If $mn = 8i + 7$, in the formula of divisors the last terms are $8i + 5$ and $8i + 9$, of which the lower should certainly have the sign $+$, because it can take on square values; certainly the upper has sign $-$; indeed, the penultimate terms will be $8i + 3$ and $8i + 11$, of which the lower, inasmuch as it equals $mn + 4$, certainly has the sign $+$ and therefore, the upper has the sign $-$. On account of this, in cases for which $mn = 8i + 7$, in the formula obtained for divisors the last pairs of terms will be

$$\begin{array}{ll} -(8i+3) & -(8i+5) \\ +(8i+11) & +(8i+9). \end{array}$$