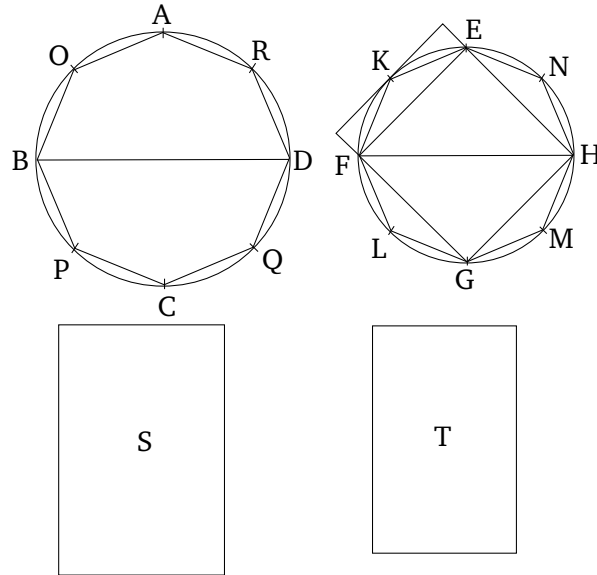


Book 12

Proposition 2

Circles are to one another as the squares on (their) diameters.

Let $ABCD$ and $EFGH$ be circles, and [let] BD and FH [be] their diameters. I say that as circle $ABCD$ is to circle $EFGH$, so the square on BD (is) to the square on FH .



For if the circle $ABCD$ is not to the (circle) $EFGH$, as the square on BD (is) to the (square) on FH , then as the (square) on BD (is) to the (square) on FH , so circle $ABCD$ will be to some area either less than, or greater than, circle $EFGH$. Let it, first of all, be (in that ratio) to (some) lesser (area), S . And let the square $EFGH$ have been inscribed in circle $EFGH$ [Prop. 4.6]. So the inscribed square is greater than half of circle $EFGH$, inasmuch as if we draw tangents to the circle through the

points E , F , G , and H , then square $EFGH$ is half of the square circumscribed about the circle [Prop. 1.47], and the circle is less than the circumscribed square. Hence, the inscribed square $EFGH$ is greater than half of circle $EFGH$. Let the circumferences EF , FG , GH , and HE have been cut in half at points K , L , M , and N (respectively), and let EK , KF , FL , LG , GM , MH , HN , and NE have been joined. And, thus, each of the triangles EKF , FLG , GMH , and HNE is greater than half of the segment of the circle about it, inasmuch as if we draw tangents to the circle through points K , L , M , and N , and complete the parallelograms on the straight-lines EF , FG , GH , and HE , then each of the triangles EKF , FLG , GMH , and HNE will be half of the parallelogram about it, but the segment about it is less than the parallelogram. Hence, each of the triangles EKF , FLG , GMH , and HNE is greater than half of the segment of the circle about it. So, by cutting the circumferences remaining behind in half, and joining straight-lines, and doing this continually, we will (eventually) leave behind some segments of the circle whose (sum) will be less than the excess by which circle $EFGH$ exceeds the area S . For we showed in the first theorem of the tenth book that if two unequal magnitudes are laid out, and if (a part) greater than a half is subtracted from the greater, and (if from) the remainder (a part) greater than a half (is subtracted), and this happens continually, then some magnitude will (eventually) be left which will be less than the lesser laid out magnitude [Prop. 10.1]. Therefore, let the (segments) have been left, and let the (sum of the) segments of the circle $EFGH$ on EK , KF ,

FL , LG , GM , MH , HN , and NE be less than the excess by which circle $EFGH$ exceeds area S . Thus, the remaining polygon $EKFLGMHN$ is greater than area S . And let the polygon $AOBPCQDR$, similar to the polygon $EKFLGMHN$, have been inscribed in circle $ABCD$. Thus, as the square on BD is to the square on FH , so polygon $AOBPCQDR$ (is) to polygon $EKFLGMHN$ [Prop. 12.1]. But, also, as the square on BD (is) to the square on FH , so circle $ABCD$ (is) to area S . And, thus, as circle $ABCD$ (is) to area S , so polygon $AOBPGQDR$ (is) to polygon $EKFLGMHN$ [Prop. 5.11]. Thus, alternately, as circle $ABCD$ (is) to the polygon (inscribed) within it, so area S (is) to polygon $EKFLGMHN$ [Prop. 5.16]. And circle $ABCD$ (is) greater than the polygon (inscribed) within it. Thus, area S is also greater than polygon $EKFLGMHN$. But, (it is) also less. The very thing is impossible. Thus, the square on BD is not to the (square) on FH , as circle $ABCD$ (is) to some area less than circle $EFGH$. So, similarly, we can show that the (square) on FH (is) not to the (square) on BD as circle $EFGH$ (is) to some area less than circle $ABCD$ either.

So, I say that neither (is) the (square) on BD to the (square) on FH , as circle $ABCD$ (is) to some area greater than circle $EFGH$.

For, if possible, let it be (in that ratio) to (some) greater (area), S . Thus, inversely, as the square on FH [is] to the (square) on DB , so area S (is) to circle $ABCD$ [Prop. 5.7 corr.]. But, as area S (is) to circle $ABCD$, so circle $EFGH$ (is) to some area less than circle $ABCD$ (see lemma). And, thus, as the (square) on FH (is) to

the (square) on BD , so circle $EFGH$ (is) to some area less than circle $ABCD$ [Prop. 5.11]. The very thing was shown (to be) impossible. Thus, as the square on BD is to the (square) on FH , so circle $ABCD$ (is) not to some area greater than circle $EFGH$. And it was shown that neither (is it in that ratio) to (some) lesser (area). Thus, as the square on BD is to the (square) on FH , so circle $ABCD$ (is) to circle $EFGH$.

Thus, circles are to one another as the squares on (their) diameters. (Which is) the very thing it was required to show.

Lemma

So, I say that, area S being greater than circle $EFGH$, as area S is to circle $ABCD$, so circle $EFGH$ (is) to some area less than circle $ABCD$.

For let it have been contrived that as area S (is) to circle $ABCD$, so circle $EFGH$ (is) to area T . I say that area T is less than circle $ABCD$. For since as area S is to circle $ABCD$, so circle $EFGH$ (is) to area T , alternately, as area S is to circle $EFGH$, so circle $ABCD$ (is) to area T [Prop. 5.16]. And area S (is) greater than circle $EFGH$. Thus, circle $ABCD$ (is) also greater than area T [Prop. 5.14]. Hence, as area S is to circle $ABCD$, so circle $EFGH$ (is) to some area less than circle $ABCD$. (Which is) the very thing it was required to show.