English Translation of Servois' "Memoir on Quadratures"

Abstract

François-Joseph Servois (1767–1847) was a French priest, artillery officer, professor of mathematics, and museum curator. He conducted research in many areas of mathematics, including geometry and the differential calculus. In 1817, after a three-year period with no publications, Servois published his "Memoir on Quadratures," where he tackled a new area of mathematics: numerical integration. We give here an English translation of that memoir, in which Servois addressed a debate on numerical integration techniques among the mathematicians Christian Kramp (1760–1826), Joseph-Diez Gergonne (1771–1859), and Joseph-Balthazard Bérard (1763–1844?).

We provide an analysis of Servois' paper and a guide to reading it in our article, "Servois' 1817 'Memoir on Quadratures'," available in MAA's online journal *Convergence* at www.maa.org/press/periodicals/convergence/servois-1817-memoir-on-quadratures.

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Appendix to "Servois' 1817 Mermoir on Quadratures"
MAA *Convergence* (May 2019)

Memoir on Quadratures‡

By Mr. Servois, curator of the Artillery Museum

[73][1] Quadratures are the ultimate elements by which all of the questions that arise from the integral calculus are finally resolved, and so consequently the most important problems in geometry and mechanics. On the other hand, it is generally agreed that even today we still desire a completely satisfactory method for the integration of functions of a single variable in every case, even when we are willing to settle for an approximation. Thus, it is natural that the announcement of a discovery of new methods, or even of simple improvements added to known methods, produces a great sensation among analysts, and is eagerly received by some, received with defiance and caution by others, but with a curious interest by everyone. For my part, I confess that I read with great satisfaction, in the *Annales de mathématiques*, the detailed expositions of three new methods of approximation that came from good sources, because they belong to professors *Dobenheim*,[2] *Kramp*,[3] and *Bérard*.[4] I have participated, with more or less the aptitude of an interested party, in the debates that they have initiated (*Annales*, Vol. VI, pp. 283, 304, 372, and Vol. VII, pp. 101 and 241).[5] From my diligence, more active [74] than passive, that resulted in a series of observations that I do not hesitate to communicate to the public. These are the reconciliation of these methods, as much amongst themselves as with those that were previously known. These are attempts at perfection in their technical procedures. Finally, there are theoretical insights, relating to the extent and the effectiveness of the approximative methods that they provide. What results will probably be new discussions which, by providing new insights, will bring us closer to the fulfillment of so much effort; I mean the acquisition of a method of approximation that leaves nothing to be desired.

I. Methods of approximation are ordinarily based on infinite series. Now we know how to express the integral $\int ydx$, the general type of quadrature, as a series in many forms. I begin by recalling the principal ones, with a summary of the proofs, in order to dispense with the need to send a skeptical reader to other works. To this end, I use the principal theorems on the analogy among powers, differences, and differentials–theorems now well-known–in the expression of

‡Originally published as "Mémoire sur les quadratures," an article in *Annales des Mathématiques pures et appliquées* **8** (1817–1818), pp. 73–115. In some citations, the title begins with the words "Analise Transcendante," because the headline of a title page in Gergonne's *Annales* is the editorial category to which the article was assigned. Translated from the French by Robert E. Bradley and Salvatore J. Petrilli, Jr. Department of Mathematics & Computer Science, Adelphi University, Garden City, NY 11530

[1] Numbers in square brackets represent the original page numbers of the article in Gergonne's *Annales*.

[2] Alexandre-Magnier (Magnus) d'Obenheim (1753–1840).

[3] Christian Kramp (1760–1826).

[4] Joseph-Balthazard Bérard (1763–1844?).

[5] [Kramp 1815a], [Gergonne 1815], [Kramp 1815b], [Bérard 1816], [Kramp 1816].

which I use the notation of Arbogast[6] (*Calcul des dérivations*)[7] to represent the varied state of a function.[8] Thus, assuming the increment of the variable x to be constant, and supposing

$$y = Fx \qquad \text{and} \qquad \omega = \Delta x = dx,$$

we have the following definitions and theorems:[9]

$$E^n y = F(x + n\omega) = e^d y, \tag{1}$$

$$\Delta y = Ey - y = (E-1)\, y, \tag{2}$$

[75][10]

$$dy = \log(1+\Delta)\, y \tag{3}$$

$$\begin{aligned}\Delta^{-1} y = \sum y &= (E-1)^{-1} y \\ &= E^{-1}y + E^{-2}y + E^{-3}y + \ldots + K = \left(e^d - 1\right)^{-1} y + K,\end{aligned} \tag{4}$$

$$d^{-1} y = \frac{1}{\omega}\int y dx = \{\log(1+\Delta)\}^{-1} y + K, \tag{5}$$

where e is, as usual, the base of the system of the Naperian logarithms and K is a quantity subject only to the condition $\Delta K = 0$.

From (4) we immediately conclude, by the expansion of the expression $\left(e^d - 1\right)^{-1}$, the series[11]

$$\begin{aligned}\sum y &= \left\{d^{-1} - \frac{1}{2}d^0 + \frac{B_1}{1\cdot 2}d - \frac{B_2}{1\cdot 2\cdot 3\cdot 4}d^3 + \ldots\right\} y + K \\ &= \frac{1}{\omega}\int y dx - \frac{1}{2}y + \frac{\omega B_1}{1\cdot 2}\frac{dy}{dx} - \frac{\omega^3 B_2}{1\cdot 2\cdot 3\cdot 4}\frac{d^3y}{dx^3} + \frac{\omega^5 B_3}{1\cdot 2\cdot\cdots\cdot 6}\frac{d^5y}{dx^5} - \ldots + K,\end{aligned} \tag{6}$$

where the coefficients

$$B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, B_4 = \frac{1}{30}, \ldots$$

[6] Louis François Antoine Arbogast (1759–1803).

[7] See [Arbogast 1800].

[8] The following footnote was given in [Servois 1817]: "Also, see a previous memoir on this subject by M. Servois, Vol. V, page 93. J.D.G." The initials "J.D.G." are those of the editor of the *Annales*, Joseph-Diez Gergonne (1771–1859). Here, Gergonne was referring to [Servois 1814a]; for a reader's guide to and English translation of [Servois 1814a], see [Bradley and Petrilli 2010b].

[9] The following line is not correct as given. Perhaps it was intended to be given as:

$$E^n y = F(x + n\omega) \text{ and } Ey = e^d y.$$

[10] In [Servois 1817], the notation Log.x was used; however, we will use the modern notation of $\log(x)$.

[11] In [Servois 1817], the term $+K$ was omitted from the end of the first line.

are known as the *Bernoulli Numbers*. They are also the coefficients of the equation of the identity

$$1-\frac{1}{2}\omega\cot\left(\frac{1}{2}\omega\right)=\frac{\omega^2 B_1}{1\cdot 2}+\frac{\omega^4 B_2}{1\cdot 2\cdot 3\cdot 4}+\frac{\omega^6 B_3}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}+\dots. \tag{7}$$

I suppose x to be positive, and that one of its antecedent values is a, to which corresponds $v = Fa$. I make $x-a = n\omega$, from which $a = x-n\omega$. Let x and a be two positive extreme abscissas of a plane curve having corresponding rectangular ordinates y and v. Dividing the interval between these two ordinates into n parts, each equal to ω, and assuming at each point of division, the intermediate equidistant ordinates [76], the system of our $n+1$ ordinates may be expressed by the double sequence[12]

$$\left.\begin{array}{ccccccc} y, & E^{-1}y, & E^{-2}y, & E^{-3}y, & \dots, & E^{-(n-1)}y, & E^{-n}y; \\ E^{n}v, & E^{n-1}v, & E^{n-2}v, & E^{n-3}v, & \dots, & Ev, & v; \end{array}\right\} \tag{8}$$

where the corresponding terms, upper and lower, express the same ordinate. Given this, after substituting v for y in (6), and subtracting the result from (6); if, to abridge,[13] we make

$$Z=\int ydx-\int vda \qquad \text{and} \qquad T=\omega\left(\sum y-\sum v+\frac{1}{2}y-\frac{1}{2}v\right), \tag{9}$$

we will have the series[14]

$$\begin{aligned} Z=T-\frac{\omega^2 B_1}{1\cdot 2}\left(\frac{dy}{dx}-\frac{dv}{da}\right)+\frac{\omega^4 B_2}{1\cdot 2\cdot 3\cdot 4}\left(\frac{d^3y}{dx^3}-\frac{d^3v}{da^3}\right) \\ -\frac{\omega^6 B_3}{1\cdot 2\cdots 6}\left(\frac{d^5y}{dx^5}-\frac{d^5v}{da^5}\right)+\dots, \end{aligned} \tag{10}$$

in which Z is clearly the integral $\int ydx$, taken between the limits a and x, or rather the plane area terminated by the ordinates v and y, the interval $x-a$, and the intercepted arc of the curve. On the other hand, because of (4 and 8), we have

$$T=\frac{1}{2}\omega\left(y+E^{-1}y\right)+\frac{1}{2}\omega\left(E^{-1}y+E^{-2}y\right)+\dots+\frac{1}{2}\omega\left(Ev+v\right).$$

That is to say, that T is the sum of the areas of a sequence of rectilinear trapezoids each taken between two consecutive ordinates, the x-axis and the chord of the intercepted arc; and this, along the entire extent between the limits v and y. For the same reasons, the expression $\omega\left(\sum y-\sum v\right)$ is the sum, taken between the same limits, of the rectangles having successive heights

[12] In [Servois 1817], the ellipses in the second sequence were missing.

[13] In [Servois 1817], the coefficient of y was given as $\frac{1}{4}$.

[14] Servois has given the Composite Trapezoidal Rule here, which includes an expression for the error term.

$E^{-1}y, E^{-2}y, \ldots, v$, and the same base ω, which sum is evidently smaller than the area Z, if the preceding sequence decreases continuously.[15] Under the same hypothesis, this other expression $\omega\left(\sum y - \sum v + y - v\right)$, which is the sum of rectangles having as heights the ordinates[16] $y, E^{-1}y, E^{-2}y, \ldots Ev$, [77] will be larger[17] than Z. It will be entirely the contrary under the opposite hypothesis, that is to say, if from y to v the intermediate ordinates are greater and greater. Now, (9) is precisely the arithmetic mean of the two preceding sums, and must, consequently, under our hypothesis, better approach the area Z more closely. Moreover, we see what we would have needed in order to have introduced one of these sums of rectangles in the place of T in the series (10), because if we designate the first by T^{-1} and the second by T^{1}, we have the relations

$$T^{-1} = T - \frac{\omega}{2}\left(y - v\right) \quad \text{and} \quad T^{1} = T + \frac{\omega}{2}\left(y - v\right). \tag{11}$$

We also have, from formula (4),

$$\sum F\left(x + \frac{1}{2}\omega\right) = \sum E^{\frac{1}{2}}y = E^{\frac{1}{2}}\sum y = E^{-\frac{1}{2}}y + E^{-\frac{3}{2}}y + E^{-\frac{5}{2}}y + \ldots + K,$$

where by letting

$$R = \omega\left\{\sum F\left(x + \frac{1}{2}\omega\right) - \sum F\left(a + \frac{1}{2}\omega\right)\right\},$$

we immediately conclude,[18]

$$R = \omega E^{-\frac{1}{2}}y + \omega E^{-\frac{3}{2}}y + \ldots + \omega E^{\frac{1}{2}}v, \tag{12}$$

where the right-hand side is clearly the expression of the sum, taken between the same limits, of a series of rectangles, each taken between two consecutive ordinates, and giving for its height the intermediate equidistant ordinate. Now, from (1 and 4), we have

$$\sum E^{\frac{1}{2}}y = \left(e^{d} - 1\right)^{-1} e^{\frac{1}{2}d}y = \frac{1}{\omega}\int y dx - \omega A\frac{dy}{dx} + \omega^{3}B\frac{d^{3}y}{dx^{3}} - \ldots + K, \tag{13}$$

in which the coefficients $A, B, C, \ldots$ are also those of the equation of the identity [78]

$$\frac{1}{e^{\frac{1}{2}\omega} - e^{-\frac{1}{2}\omega}} = \frac{1}{\omega} - A\omega + B\omega^{3} - C\omega^{5} + \ldots,$$

[15] Servois seems tacitly to be assuming that the function F is increasing on the interval $[a, x]$, although he only stipulates that $F(x-\omega), F(x-2\omega), \ldots, F(a)$ is a decreasing sequence.

[16] In [Servois 1817], the variable y was missing from the term $E^{-1}y$.

[17] In [Servois 1817], this was given as "smaller." In these two sentences, Servois is presenting the situation that when you have an increasing function, the area under the curve is greater than the left-hand Riemann Sum and smaller than the right-hand Riemann Sum.

[18] In [Servois 1817], the second term on the right-hand side was missing a y.

or else this other equation

$$\frac{\frac{1}{2}\omega}{\sin\left(\frac{1}{2}\omega\right)} = 1 + A\omega^2 + B\omega^4 + C\omega^6 + \ldots.$$

Now, substituting v for y in (13), subtracting the result from (13), and taking note of (12), we immediately have, between the limits a and x,[19]

$$Z = R + \omega^2 A\left(\frac{dy}{dx} - \frac{dv}{da}\right) - \omega^4 B\left(\frac{d^3y}{dx^3} - \frac{d^3v}{da^3}\right) + \omega^6 C\left(\frac{d^5y}{dx^5} - \frac{d^5v}{da^5}\right) - \ldots. \quad (14)$$

This is the series given in *Exercices de calcul intégral*, (part III, p. 311).[20]

In the spirit of the series (14), and immediately following, the author of the excellent work that we just cited gives himself over to researching the formula for determining the rectangular coordinates of a curve whose equation is given only between the arc and the angle it makes, at its extremity, with the x-axis, such as, in particular, the equation of the ballistic curve following Newton's Law. He arrives quite happily at his goal, but by a route for which he does not hide his embarrassments, because in speaking of his result, he says:

> The state of simplicity to which we have reduced this formula makes it seem that it is possible to achieve it by a more direct and less laborious route, however, without abandoning this research... (*Ibid*, p. 327).

Indeed, we arrive quite simply to the formula in question by the following path.

In formula (6), I write θ in place of x, and[21] $s \cdot \sin\theta$ in place of y. After a slight transformation, we find[22]

$$\begin{aligned}\omega\left\{\sum(s\cdot\sin\theta) + \frac{1}{2}s\cdot\sin\theta\right\} &= \omega\sum\left\{s\cdot\sin\theta + \frac{1}{2}\Delta(s\cdot\sin\theta)\right\}\\ &= \int(s\cdot\sin\theta)\,d\theta + \frac{\omega^2B_1}{1\cdot2}\frac{d(s\cdot\sin\theta)}{d\theta} - \frac{\omega^4B_2}{1\cdot2\cdot3\cdot4}\frac{d^3(s\cdot\sin\theta)}{d\theta^3} + \ldots + K.\end{aligned} \quad (15)$$

[79] I then suppose that s, a function of θ, is an arc of a plane curve, determined by the rectangular coordinates x and y and making the angle θ with the x-axis, at its extremity; this hypothesis is expressed by the relations

$$dx = ds\cdot\cos\theta \qquad \text{and} \qquad dy = ds\cdot\sin\theta. \quad (16)$$

[19] Servois has given the Composite Midpoint Rule here, which includes an expression for the error term.

[20] See [Legendre 1811].

[21] Servois used "Sin." for sine, "Cos" for cosine, and "Cot." for cotangent. We will consistently use sin, cos and cot in place of those.

[22] In [Servois 1817], the last term of this series (15) was k.

First of all, we have, as we know[23]

$$
\begin{aligned}
s\cdot\sin\theta+\frac{1}{2}\Delta\left(s\cdot\sin\theta\right) &= s\cdot\sin\theta+\Delta\left(s\cdot\sin\theta\right)-\frac{1}{2}\Delta\left(s\cdot\sin\theta\right)\\
&= \left(s+\Delta s\right)\sin\left(\theta+\omega\right)-\frac{1}{2}\Delta\left(s\cdot\sin\theta\right)\\
&= \left(s+\Delta s\right)\cos\left(\frac{1}{2}\omega\right)\sin\left(\theta+\frac{1}{2}\omega\right)\\
&\quad+\left(s+\Delta s\right)\sin\left(\frac{1}{2}\omega\right)\cos\left(\theta+\frac{1}{2}\omega\right)-\frac{1}{2}\Delta\left(s\cdot\sin\theta\right).
\end{aligned}
$$

Then, because

$$
\begin{aligned}
\Delta\left(s\cdot\sin\theta\right) &= \Delta s\cdot\sin\theta+2\left(s+\Delta s\right)\sin\left(\frac{1}{2}\omega\right)\cos\left(\theta+\frac{1}{2}\omega\right),\\
\Delta\left(s\cdot\cos\theta\right) &= \Delta s\cdot\cos\theta-2\left(s+\Delta s\right)\sin\left(\frac{1}{2}\omega\right)\sin\left(\theta+\frac{1}{2}\omega\right),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\left(s+\Delta s\right)\cos\left(\frac{1}{2}\omega\right)\sin\left(\theta+\frac{1}{2}\omega\right) &= -\frac{1}{2}\cot\left(\frac{1}{2}\omega\right)\Delta\left(s\cdot\cos\theta\right)+\frac{1}{2}\cot\left(\frac{1}{2}\omega\right)\Delta s\cdot\cos\theta,\\
\left(s+\Delta s\right)\sin\left(\frac{1}{2}\omega\right)\cos\left(\theta+\frac{1}{2}\omega\right) &= \frac{1}{2}\Delta\left(s\cdot\sin\theta\right)-\frac{1}{2}\Delta s\cdot\sin\theta.
\end{aligned}
$$

Therefore, we will finally have

$$
\begin{aligned}
s\cdot\sin\theta+\frac{1}{2}\Delta\left(s\cdot\sin\theta\right) &= -\frac{1}{2}\cot\left(\frac{1}{2}\omega\right)\Delta\left(s\cdot\cos\theta\right)+\frac{1}{2}\cot\left(\frac{1}{2}\omega\right)\Delta s\cdot\cos\theta-\frac{1}{2}\Delta s\cdot\sin\theta\\
&= -\frac{1}{2}\cot\left(\frac{1}{2}\omega\right)\Delta\left(s\cdot\cos\theta\right)+\frac{1}{2}\Delta s\cdot\frac{\cos\left(\theta+\frac{1}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}.
\end{aligned}
\tag{17}
$$

Moreover, because

$$
d\left(s\cdot\cos\theta\right)=ds\cdot\cos\theta+s\cdot d(\cos\theta)=ds\cos\theta-s\cdot\sin\theta d\theta,
$$

we have by (16) [80]

$$
\int\left(s\cdot\sin\theta\right)d\theta=\int\left(ds\cdot\cos\theta\right)-s\cdot\cos\theta=x-s\cos\theta.
$$

[23]In [Servois 1817], the last term of the first line was given as $-\frac{1}{2}\left(s\cdot\sin\theta\right)$.

I substitute this expression and the previous one (17) into the series (15), and I have[24]

$$\frac{\frac{1}{2}\omega}{\sin\left(\frac{1}{2}\omega\right)}\sum\left[\Delta s\cdot\cos\left(\theta+\frac{1}{2}\omega\right)\right]+\left(1-\frac{1}{2}\omega\cot\left(\frac{1}{2}\omega\right)\right)s\cdot\cos\theta$$

$$=x+\frac{\omega^2 B_1}{1\cdot 2}\frac{d\,(s\cdot\sin\theta)}{d\theta}-\frac{\omega^4 B_2}{1\cdot 2\cdot 3\cdot 4}\frac{d^3\,(s\cdot\sin\theta)}{d\theta^3}+\ldots+K,$$

where, in substituting its expansion (7) in place of $1-\frac{1}{2}\omega\cdot\cot\left(\frac{1}{2}\omega\right)$, we find immediately that

$$x=\frac{\frac{1}{2}\omega}{\sin\left(\frac{1}{2}\omega\right)}\sum\left[\Delta s\cdot\cos\left(\theta+\frac{1}{2}\omega\right)\right]-\frac{\omega^2 B_1}{1\cdot 2}\left\{\frac{d\,(s\cdot\sin\theta)}{d\theta}-s\cdot\cos\theta\right\}$$

$$+\frac{\omega^4 B_2}{1\cdot 2\cdot 3\cdot 4}\left\{\frac{d^3\,(s\cdot\sin\theta)}{d\theta^3}+s\cdot\cos\theta\right\}+\ldots+K. \tag{18}$$

In determining K in such a way that x is the integral beginning where $\theta=\alpha$, and noting that $\frac{B_1}{1\cdot 2},\frac{B_2}{1\cdot 2\cdot 3\cdot 4},\ldots$ are respectively the same as the $A^o, B^o, \ldots$ of the *Exercises*, we will see series (18) coinciding perfectly with the one in the cited work (p. 328). If we wish to have $y=\int(ds\cdot\sin\theta)$, it suffices to change x to y in (18), θ to $90^o-\theta$, and ω to $-\omega$; this is clear. In any case, it is clear that $\sum\left\{\Delta s\cdot\cos\left(\theta+\frac{1}{2}\omega\right)\right\}$ is the approximation given for x, by the ingenious method that Euler gave in his famous memoir (*Academie de Berlin* 1753),[25] which since then has so occupied the authors of ballistics. That is to say, it is the expression of the sum of the projections on the x-axis, of a series of rectified arcs, all of which have between their extremities the same difference in polar angle[26] ω, and taking for the angle of projection the mean inclination of each arc.

II. [81] The series (10, 14, 18) belong to the class of formulas that express the integral $\int ydx$ by means of the finite integral $\sum y$ and the successive differentials $dy, d^2y,\ldots$. It is very easy to obtain $\int ydx$ by differentials only. Indeed, in supposing again that $x-a=n\omega$, or more simply $x-a=n$, which amounts to taking ω as a unit, we have by *Taylor's Theorem*,

$$y=E^n v=v+\frac{n}{1}\frac{dv}{da}+\frac{n^2}{1\cdot 2}\frac{d^2v}{da^2}+\frac{n^3}{1\cdot 2\cdot 3}\frac{d^3v}{da^3}+\ldots.$$

Regarding n as continuous, multiplying by dn, then integrating with respect to n between the limits 0 and n, we immediately find

[24] In [Servois 1817], the second term of the left-hand side of the following equation was given as $+\left(1-\frac{1}{2}\omega\right)s\cdot\cos\theta$.

[25] Vol IX, pp. 321-352; this is Euler's paper E217.

[26] In [Servois 1817], the term *courbure* was used here, literally "curvature."

$$Z = nv + \frac{n^2}{1 \cdot 2}\frac{dv}{da} + \frac{n^3}{1 \cdot 2 \cdot 3}\frac{d^2v}{da^2} + \frac{n^4}{1 \cdot 2 \cdot 3 \cdot 4}\frac{d^3v}{da^3} + \dots. \tag{19}$$

If, in this series, we change a to x, v to y, and n to $-n$, which amounts to taking the base of the ordinate y as the origin of the n's, and passing from a to v in the negative sense of n, we will easily see that we obtain in absolute value an area equal to Z, but with the opposite sign. Thus, we have this other series

$$Z = ny - \frac{n^2}{1 \cdot 2}\frac{dy}{dx} + \frac{n^3}{1 \cdot 2 \cdot 3}\frac{d^2y}{dx^2} - \frac{n^4}{1 \cdot 2 \cdot 3 \cdot 4}\frac{d^3y}{dx^3} + \dots. \tag{20}$$

This latter series is appropriately the one that bears the name of JEAN BERNOULLI, who published it in the *Acta eruditorum* in the year 1694.[27]

By simply changing n to $-n$, in (19), we have the area between v and $E^{-n}v$; or between Fa and $F(a-n)$ and, in subtracting the result from (19), we will clearly have the area contained between $E^n v$ and $E^{-n}v$, or between $F(a+n)$ and $F(a-n)$. Thus, denoting this area by W, we have a third series [82]

$$W = 2nv + \frac{2n^3}{1 \cdot 2 \cdot 3}\frac{d^2v}{da^2} + \frac{2n^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}\frac{d^4v}{da^4} + \dots. \tag{21}$$

Here, W becomes equal to Z, provided that we change $2n$ to n, and v to $E^{\frac{1}{2}n}v$, that is to say, we also have

$$Z = nE^{\frac{1}{2}n}v + \frac{n^3}{1 \cdot 2 \cdot 3}\frac{d^2E^{\frac{1}{2}n}v}{2^2da^2} + \frac{n^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}\frac{d^4E^{\frac{1}{2}n}v}{2^4da^4} + \dots. \tag{22}$$

$E^{\frac{1}{2}n}v$ will be one of the equidistant ordinates, when n is an even number.

III. The series (19, 20, 21, 22) are in differentials only. We will have series in differences only, by the same procedure if, in place of *Taylor's Theorem*, we use the *Theorem of Differences*[28] to expand y or $E^n v$. Thus, we have[29]

$$y = E^n v = v + \frac{n}{1}\Delta v + \frac{n}{1}\frac{n-1}{2}\Delta^2 v + \dots.$$

Multiplying by dn, integrating with respect to n, between the limits 0 and n, we immediately find[30]

$$Z = nv + \frac{1}{2}n^2\Delta v + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 v}{1 \cdot 2} + \left(\frac{n^4}{4} - \frac{3n^3}{3} + \frac{2n^2}{2}\right)\frac{\Delta^3 v}{1 \cdot 2 \cdot 3} + \dots. \tag{23}$$

[27] In [Servois 1817], the year given was 1674. See [Bernoulli 1694].

[28] This formula is usually referred to as Newton's Forward Difference formula. See [Burden 2016, p. 126]. We note that this is a finite series terminating with the $\Delta^n v$ term.

[29] In [Servois 1817], the equation given below was presented as

$$y = E^n v = v = \frac{n}{1}\Delta^2 v + \frac{n}{1}\frac{n-1}{2}\Delta^2 v + \dots.$$

[30] In [Servois 1817], the following equation was not numbered (23), but it was later referenced as (23).

This is the series given by Mr. Kramp (*Annales*, Vol. VI, p. 372ff.).

IV. Here I end my exposition of series by means of which one can express the integral $\int ydx$. We must now see what we may draw from it.

1. All of these series, like those of the theorem of differences and Taylor's Theorem, of which the first ones are fundamentally nothing more than modifications or immediate consequences [83] of these, terminate when the function Fx is of a kind that leads to differences of zero. This is the case, as we know, for all whole rational functions[31] of x, or of all parabolic curves.[32]

2. Only with respect to the numerical coefficients, these series are not convergent enough. They only acquire sufficient convergence when the differences or differentials, in passing to the higher orders, diminish in value; that is to say, when they tend to become zero. It is therefore only under this hypothesis that they can be used to *directly* solve quadrature problems by approximation, that is to say, taking for the approximate value of $\int ydx$, a certain number of their first terms.

Under the same hypothesis–that is to say, we suppose that the difference $\Delta^{n+1}Fx$, for example, and the following ones are zero or taken as such–we draw from the same series other very remarkable approximation formulas, which offer to the calculator the great advantage of making the approximation depend only on a number $n+1$ of equidistant ordinates, combined linearly with coefficients that, calculated once and for all and kept in permanent tables, can be retrieved without need of work. I turn to the examination of these methods.

The first one goes straight to the point by substituting in place of $\sum$, Δ, and d in the above series, their expressions in terms of the varied states $E, E^2, \ldots$, given by the formulas (1, 2, 3, 4), expressions that are all finite and linear, when we suppose all differences beyond a certain order are zero. Indeed, when we suppose

$$0 = \Delta^{n+1}y = \Delta^{n+2}y = \ldots, \quad \text{we also have} \quad 0 = d^{n+1}y = d^{n+2}y = \ldots,$$

because, according to (3)

$$d^{n+1}y = \left(\Delta - \frac{1}{2}\Delta^2 + \ldots \pm \frac{1}{n}\Delta^n\right)^{n+1} y = \Delta^{n+1}y - \frac{n+1}{2}\Delta^{n+2}y + \ldots.$$

[84] Then the same theorems generally give, for a positive integer k,

$$\Delta^k y = (E-1)^k y = E^k y - kE^{k-1}y + \frac{k}{1}\frac{k-1}{2}E^{k-2}y - \ldots \pm y, \qquad (24)$$

$$d^k y = \left\{(E-1) - \frac{1}{2}(E-1)^2 + \ldots \pm \frac{1}{k}(E-1)^k\right\}^k y,$$

[31] I.e., polynomial functions.

[32] Some authors of this time referred to the graph of the function $y = x^n$ as a generalized parabola. Later on, Servois will refer to the graph of a polynomial of order n as a complete parabolic curve of order n.

expressions which, after expanding, contain only linear multiples of different orders of the varied states. Moreover (4), the integral $\sum$, and consequently the expressions T and R are immediately resolved in linear terms of the varied state. We see, furthermore, that these substitutions in our series must ultimately lead to the same result. The one that will make it easiest to come to this will be the one that demands the least complicated formulas for substitutions. Now, (23) is such a formula in which we will substitute the varied states for the values $\Delta v, \Delta^2 v, \ldots, \Delta^n v$, following the simple formula (24). This is exactly the procedure that Mr. Kramp followed, in the cited memoir (*Annales*, Vol. VI, p. 372) and from which he presented the table of expressions for Z in equidistant coordinates for the values of the number n (which he calls the *Divisor*) from 1 to 12 inclusive. This procedure may be subjected to analytic rules that allow us to immediately calculate the coefficients of the ordinates $v, Ev, E^2v, \ldots$ in the general case of n being any positive integer. I place these details here all the more willingly because it may be formulas of this kind that the able geometer calls for when he says (Vol. VII, p. 243):

> I would have gone further than 12, if the presumed length of the calculations did not frighten me off. I observed that there must inevitably be some method, much more abridged, in order to attain the same end in all cases.

Substituting a for x and expanding, formula (5) becomes [85]

$$\frac{1}{\omega}\int F da = \Delta^{-1}v + Av + B\Delta v + C\Delta^2 v + \ldots + K, \tag{25}$$

where the coefficients

$$A = \frac{1}{2}, B = -\frac{1}{12}, C = \frac{1}{24}, D = -\frac{19}{720}, E = \frac{3}{160}, F = -\frac{863}{60480}, \ldots,$$

are those of the identity[33]

$$\left(1 - \frac{1}{2}\omega + \frac{1}{3}\omega^2 - \ldots\right)^{-1} = 1 + A\omega + B\omega^2 + C\omega^3 + \ldots.$$

It is also easy to see that they are linked together by the following rule: A, B, C, …, L, M, N, being respectively the first, second, third …, $(n-2)$th, $(n-1)$th, nth coefficients, we have

$$N = \frac{1}{n+1} - \frac{1}{2}M - \frac{1}{3}L - \ldots - \frac{1}{n-2}C - \frac{1}{n-1}B - \frac{1}{n}A, \tag{26}$$

a formula in which we only pay attention to the absolute values of the numbers A, B, C, …, L, M, N; giving them their alternative signs $+$ and $-$ afterward.

[33] In [Servois 1817], the left-hand side of the equation was given as $\left(2 - \frac{1}{2}\omega + \frac{1}{3}\omega^2 - \ldots\right)^{-1}$.

As above, let us make $\omega = 1$, then substitute $a+n$ for a in (25), and subtract (25) from the result. We will have

$$Z = (E^n - 1)\left(\Delta^{-1} + A + B\Delta + \ldots\right) v = \Delta^{-1}(E^n - 1)\left(1 + A\Delta + B\Delta^2 + \ldots\right) v.$$

Now, according to the theorem of differences,

$$\Delta^{-1}(E^n - 1) = \Delta^{n-1} + \frac{n}{1}\Delta^{n-2} + \frac{n}{1}\frac{n-1}{2}\Delta^{n-3} + \ldots + n,$$

therefore

$$Z = \left(\Delta^{n-1} + \frac{n}{1}\Delta^{n-2} + \frac{n}{1}\frac{n-1}{2}\Delta^{n-3} + \ldots + n\right)$$
$$\left(1 + A\Delta + B\Delta^2 + \ldots + M\Delta^{n-1} + N\Delta^n\right) v.$$

Expanding this and rejecting all the differences greater than those of order n, we find an equation of the form

$$Z = \alpha\Delta^n v + \beta\Delta^{n-1} v + \gamma\Delta^{n-2} v + \delta\Delta^{n-3} v + \ldots, \tag{27}$$

in which it will be necessary to make [86]

$$\left.\begin{array}{rcllll}
\alpha &=& A+ & B\tfrac{n}{1}+ & C\tfrac{n}{1}\cdot\tfrac{n-1}{2}+ & \ldots + Nn,\\
\beta &=& 1+ & A\tfrac{n}{1}+ & B\tfrac{n}{1}\cdot\tfrac{n-1}{2}+ & \ldots + Mn,\\
\gamma &=& & \tfrac{n}{1}+ & A\tfrac{n}{1}\cdot\tfrac{n-1}{2}+ & \ldots + Ln,\\
\delta &=& & + & \tfrac{n}{1}\cdot\tfrac{n-1}{2}+ & \ldots + Kn,\\
& & & & \ldots\ldots\ldots &
\end{array}\right\} \tag{28}$$

an expression of which the rule is clear.

It may be noted here that the series (27), with its coefficients (28), is fundamentally the same as a formula given by LORGNA[34] in the *Mèmoires de la sociètè italienne* (Vol. I).

It remains to expand the differences in (27) into varied states, following formula (24). We finally obtain

$$Z = aE^n v + bE^{n-1} v + cE^{n-2} v + dE^{n-3} v + \ldots, \tag{29}$$

an equation in which we must make

$$\left.\begin{array}{rcl}
a &=& \alpha,\\
b &=& \beta - \tfrac{n}{1}\alpha,\\
c &=& \gamma - \tfrac{n-1}{1}\beta + \tfrac{n}{1}\cdot\tfrac{n-1}{2}\alpha,\\
d &=& \delta - \tfrac{n-2}{1}\gamma + \tfrac{n-1}{1}\cdot\tfrac{n-2}{2}\beta - \tfrac{n}{1}\cdot\tfrac{n-1}{2}\cdot\tfrac{n-2}{3}\alpha,\\
& & \ldots\ldots\ldots
\end{array}\right\} \tag{30}$$

[34] Antonio Maria Lorgna (1735–1796).

[87] We see that if we had a large number of coefficients $A, B, C, \ldots$ in a table, whose values are independent of the number n, and that are easily calculated, using formula (26), we would quickly obtain the coefficients $a, b, c, \ldots$ (29), by means of formulas (28, 30), in which all the coefficients depending on n can be taken from a table of figurate numbers.[35] Furthermore, we know that we really only need to calculate half, or the simple majority (if n is odd), of the numbers of these coefficients, because as in the preceding, the origin of the coordinates was placed at the foot of the ordinate v, and we considered $Ev, E^2v, E^3v, \ldots$ as situated in the region of positive coordinates. However, if we transpose the origin to the foot of y, and we take as positive ordinates those that get successively further in approaching v, which is essentially the same, the differences, and consequently the area Z, which remains the same, will be expressed by y, $E^{-1}y, E^{-2}y, \ldots$ and n, as they had previously been expressed by v, $Ev, E^2v, \ldots$ and n. Therefore, the coefficients of the ordinates $(v, E^n v)\,, \left(Ev, E^{n-1}v\right), \ldots$, that is to say, the ordinates equally distant from the extremes are equal. Furthermore, we will give other formulas below for immediately calculating the coefficients of equidistant ordinates in the final expression of Z.

V. Another method is founded on the observation that, in the series expressions of Z, as with those of the varied states, the differences and differentials appear linearly and in the same way, and relate exclusively to the limits of the area. The method consists of eliminating these differences or differentials among several expressions for the same area, where we vary the number of intermediate coordinates, or else between expressions of an area with equidistant coordinates. This elimination, among equations of the first degree in several unknowns and executed by known procedures, will only introduce terms that are linear in the terms of the equation that is used into the final equation.

[88] I explain myself with a first example. To abbreviate, I put the series (10) in the form

$$Z = T + \alpha\omega^2 + \beta\omega^4 + \gamma\omega^6 + \ldots. \tag{31}$$

Preserving the limits v and y of the integral Z, if I let ω vary so that we have respectively $T', T'', \ldots$ in place of T, as ω becomes $\omega', \omega'', \ldots$, then I will have (31)

$$\begin{aligned} Z &= T' + \alpha\omega'^2 + \beta\omega'^4 + \gamma\omega'^6 + \ldots, \\ Z &= T'' + \alpha\omega''^2 + \beta\omega''^4 + \gamma\omega''^6 + \ldots, \\ &\ldots\ldots\ldots\ldots, \end{aligned}$$

in which the coefficients $\alpha, \beta, \gamma, \ldots$, which depend only on the limits, remain the same as in (31). From these equations, assumed to be n in number, I determine the same number n of coefficients of the sequence $\alpha, \beta, \gamma, \ldots$. I substitute these in (31) and take as zero the ones I have not determined, and I have for Z an approximation that is equivalent to that which would result from the hypothesis that the difference Δ^{2n+1} is zero, as well as those of the higher orders, because

[35] Specifically, binomial coefficients.

the first term neglected in (31) is that of order $n+1$, counting the terms that belong to T exclusively. Now, this term is of the form[36]

$$Q\omega^{2(n+1)}\left\{\frac{d^{2n+1}y}{dx^{2n+1}}-\frac{d^{2n+1}v}{da^{2n+1}}\right\},$$

which we recognize by a simple inspection of the series (10).

If the interval $x-a$ is divided into $2n$ equal parts, for example, with the same ordinates that were used to compose T, we may form a certain number of areas $T', T'', \ldots$ divided in a different manner, by taking for $\omega', \omega'', \ldots$, respectively, the multiples $n'\omega, n''\omega, \ldots$, where $n', n'', \ldots$ denote the divisors of $2n$. If the number $2n$ has n divisors, we will form, simply by means of the ordinates [89] that make up T a number n of other areas $T', T'', \ldots$, and consequently we will bring the approximation up to the differences of the order $2n$, inclusively. If the number of divisors $n', n'', \ldots$ is less than n we may still, with the ordinates of T, form a certain number of auxiliary areas that will give an approximation of lower order.

Those who know the method of integration of Mr. Dobenheim published in the *Ballistics* (Strasbourg 1816), a method Mr. Kramp presented, with important developments due entirely to him (*Annales*, Vol. VI, p. 281ff.), will no doubt find that they coincide with the procedure which I have just sketched out.

As a second example, I apply the method to the series (21). Taking ω to be unity, and rejecting differences of order $n+1$ and higher, by Taylor's Theorem, we have, without difficulty[37]

$$\left.\begin{aligned}
E^n v+E^{-n}v &= 2v+2\tfrac{n^2}{1\cdot 2}\tfrac{d^2v}{da^2}+2\tfrac{n^4}{1\cdot 2\cdot 3\cdot 4}\tfrac{d^4v}{da^4}+\ldots+2\tfrac{n^{2n}}{1\cdot 2\cdots 2n}\tfrac{d^{2n}v}{da^{2n}},\\
E^{n-1}v+E^{-(n-1)}v &= 2v+2\tfrac{(n-1)^2}{1\cdot 2}\tfrac{d^2v}{da^2}+2\tfrac{(n-1)^4}{1\cdot 2\cdot 3\cdot 4}\tfrac{d^4v}{da^4}+\ldots+2\tfrac{(n-1)^{2n}}{1\cdot 2\cdots 2n}\tfrac{d^{2n}v}{da^{2n}},\\
&\ldots\ldots\ldots\ldots\ldots\\
E^2v+E^{-2}v &= 2v+2\tfrac{2^2}{1\cdot 2}\tfrac{d^2v}{da^2}+2\tfrac{2^4}{1\cdot 2\cdot 3\cdot 4}\tfrac{d^4v}{da^4}+\ldots+2\tfrac{2^{2n}}{1\cdot 2\cdots 2n}\tfrac{d^{2n}v}{da^{2n}},\\
Ev+E^{-1}v &= 2v+2\tfrac{1}{1\cdot 2}\tfrac{d^2v}{da^2}+2\tfrac{1}{1\cdot 2\cdot 3\cdot 4}\tfrac{d^4v}{da^4}+\ldots+2\tfrac{1}{1\cdot 2\cdots 2n}\tfrac{d^{2n}v}{da^{2n}},\\
v &= v.
\end{aligned}\right\}\tag{32}$$

These equations, $n+1$ in number, multiplied respectively by [90] the indeterminate coefficients $\alpha, \beta, \gamma, \ldots$, then added up, give the following, denoting by V the sum of their left-hand sides:

$$V=\left\{\begin{array}{l}
\frac{2}{1}v\left(\alpha+\beta+\gamma+\ldots+\mu+\frac{1}{2}\nu\right)\\
+\frac{2}{1\cdot 2}\frac{d^2v}{da^2}\left[\alpha n^2+\beta(n-1)^2+\gamma(n-2)^2+\ldots+\mu\right]\\
+\frac{2}{1\cdot 2\cdot 3\cdot 4}\frac{d^4v}{da^4}\left[\alpha n^4+\beta(n-1)^4+\gamma(n-2)^4+\ldots+\mu\right]\\
+\ldots\ldots\ldots\ldots\ldots
\end{array}\right\}$$

[36]In [Servois 1817], the denominator of the second term was given as dx^{2n+2}.

[37]In [Servois 1817], the last term of the first line was given as $2\frac{n^{2n}}{1\cdot 2\cdots 2n}\frac{da^{2n}}{d^{2n}v}$.

I determine the coefficients $\alpha, \beta, \ldots$, by making V coincide with W (21), term by term. This provides the $n+1$ conditions

$$\left.\begin{array}{rcl} \alpha+\beta+\gamma+\ldots+\mu+\frac{1}{2}\nu &=& n, \\ \alpha n^2+\beta(n-1)^2+\gamma(n-2)^2+\ldots+\mu &=& \frac{n^3}{3}, \\ \alpha n^4+\beta(n-1)^4+\gamma(n-2)^4+\ldots+\mu &=& \frac{n^5}{5}, \\ \ldots\ldots\ldots\ldots\ldots & & \\ \alpha n^{2n}+\beta(n-1)^{2n}+\gamma(n-2)^{2n}+\ldots+\mu &=& \frac{n^{2n+1}}{2n+1}, \end{array}\right\} \tag{33}$$

in the same number as the coefficients, and therefore sufficient to determine them. After which, I will have

$$W=V=\alpha\left(E^n v+E^{-n}v\right)+\beta\left(E^{n-1}v+E^{-(n-1)}v\right)+\ldots\mu\left(Ev+E^{-1}v\right)+\nu v. \tag{34}$$

This process conforms perfectly with that of the method given by Mr. Bérard (*Annales*, vol. VII, p. 101 and following). [91] Indeed, suppose $n=12$, in the table of equations (33), and you will have identically the thirteen equations relative to this case, given on page 108 of the cited volume. There is, moreover, no resemblance between the metaphysics of the learned author and that which brought us here; however, that is not what we are concerned with at the present.

We must point out that the equations (33) possess a particular method of very expeditious solution, which even allows us to arrive at simple enough formulas for expressing the coefficients $\alpha, \beta, \gamma, \ldots$.

We eliminate α from the equations (33) by subtracting from each one of the first n, multiplied by n^2, the one that immediately follows it. Now, this clearly amounts to multiplying, term by term, and in order, the left-hand sides of the first n equations, respectively, by the sequence

$$n^2-n^2=0, n^2-(n-1)^2=b, n^2-(n-2)^2=c, n^2-(n-3)^2=d, \ldots, n^2-1, n^2,$$

then giving to each result, on the right-hand side, that of the corresponding equation, multiplied by n^2, and then diminished by the right-hand side of the equation immediately following it. We thus obtain the n equations without α:

$$\left.\begin{array}{rcl} b\beta+c\gamma+d\delta+\ldots+\frac{n^2\nu}{2} &=& n^3 p, \\ b\beta(n-1)^2+c\gamma(n-2)^2+d\delta(n-3)^2+\ldots &=& n^5 p', \\ b\beta(n-1)^4+c\gamma(n-2)^4+d\delta(n-3)^4+\ldots &=& n^7 p'', \\ \ldots\ldots\ldots\ldots\ldots, & & \end{array}\right\} \tag{35}$$

where we have

$$p=\frac{2}{1\cdot 3}, p'=\frac{2}{3\cdot 5}, p''=\frac{2}{5\cdot 7}, p'''=\frac{2}{7\cdot 9}, \ldots;$$

these coefficients being independent of n.

[92] Now, it is clear that in the left-hand sides of the equations in table (35), $(n-1)^2$ plays the same role as n^2 in equations (33). Thus, we will form a second table of $n-1$ equations without β by multiplying the left-hand sides of the first $n-1$ equations (35), term by term and in order, by the sequence

$$(n-1)^2-(n-1)^2=0,(n-1)^2-(n-2)^2=c',(n-1)^2-(n-3)^2=d',\\ \ldots,(n-1)^2-1,(n-1)^2,$$

and taking for the right-hand side of each its original, multiplied by $(n-1)^2$, and then diminished by the one that follows it. In this way we have,

$$\left.\begin{array}{rcl} cc'\gamma+dd'\delta+\ldots+\frac{n^2(n-1^2)\nu}{2} & = & n^3q, \\ cc'\gamma(n-2)^2+dd'\delta(n-3)^2+\ldots & = & n^5q', \\ cc'\gamma(n-2)^4+dd'\delta(n-3)^4+\ldots & = & n^7q'', \\ \ldots\ldots\ldots\ldots\ldots, & & \end{array}\right\} \quad (36)$$

where we have

$$q=(n-1)^2p-n^2p',q'=(n-1)^2p'-n^2p'',q''=(n-1)^2p''-n^2p''',\ldots.$$

Here $(n-2)^2$ has taken the place of $(n-1)^2$ in (35), and of n^2 in (33). Additionally, one notices, without it being necessary to insist on it, how one will pass to a sequence of tables of $n-2,n-3,\ldots$ equations each comprising one fewer unknown; and finally how one will arrive at a single equation of the form

$$\frac{n^2(n-1)^2(n-2)^2\cdots 2^2\cdot 1}{2}\nu=n^3w,$$

which immediately gives

$$\nu=\frac{2n^3w}{n^2(n-1)^2(n-2)^2\cdots 2^2\cdot 1}.$$

[93] Afterward we shall go back to using only the first equation of each table (which will be all the more simple since this way we will have dispensed with writing the left-hand sides the other equations) to determine the other coefficients, in the order $\mu,\lambda,\ldots,\gamma,\beta,\alpha$.

The extreme simplicity of this permitted me to give in to my curiosity, in researching whether the formula of Mr. Bérard, relating to the case of $2n=12$, deserves the reproach of being false that was addressed to him (*Annales*, vol. VII, p. 245).[38] In this hypothesis $n=6$, and the first equations of the successive

[38] Bérard was the principal of the College at Briançon. Kramp stated that Bérard had the wrong numerical coefficients in this case.

tables are

$$\left.\begin{array}{rcl}\alpha+\beta+\gamma+\delta+\epsilon+\zeta+\eta &=& 6,\\ 11\beta+20\gamma+27\delta+32\epsilon+35\zeta+18\eta &=& 216p,\\ 180\gamma+432\delta+672\epsilon+840\zeta+450\eta &=& 216q,\\ 3024\delta+8064\epsilon+12600\zeta+7200\eta &=& 216r,\\ 40320\epsilon+10080\zeta+64800\eta &=& 216s,\\ 30240\zeta+259200\eta &=& 216t,\\ 259200\eta &=& 216u.\end{array}\right\} \qquad (37)$$

We then obtain $p, q, r, \ldots$ by means of[39]

$$\begin{array}{ll} q=25p-36p'=\frac{2\cdot 89}{1\cdot 5\cdot 3}, & q'=25p'-36p''=\frac{2\cdot 67}{3\cdot 5\cdot 7},\\ r=16q-36q'=\frac{2\cdot 7556}{1\cdot 3\cdot 5\cdot 7}, & r'=16q'-36q''=\frac{2\cdot 4788}{3\cdot 5\cdot 7\cdot 9},\\ s=9r-36r'=\frac{2\cdot 439668}{1\cdot 3\cdot 5\cdot 7\cdot 9}, & s'=9r'-36r''=\frac{2\cdot 65772}{3\cdot 5\cdot 7\cdot 9\cdot 11},\\ t=4s-36s'=\frac{2\cdot 16977600}{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}, & t'=4s'-36s''=\frac{2\cdot 43758144}{3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13},\\ u=t-36t'=-\frac{2\cdot 1354584384}{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13}, & \end{array}$$

[94]

$$\begin{array}{ll} q''=25p''-36p'''=\frac{2\cdot 45}{5\cdot 7\cdot 9}, & q'''=25p'''-36p^{IV}=\frac{2\cdot 23}{7\cdot 9\cdot 11},\\ r''=16q''-36q'''=\frac{2\cdot 3780}{5\cdot 7\cdot 9\cdot 11}, & r'''=16q'''-36q^{IV}=\frac{2\cdot 4532}{7\cdot 9\cdot 11\cdot 13},\\ s''=9r''-36r'''=-\frac{2\cdot 373500}{5\cdot 7\cdot 9\cdot 11\cdot 13}, & \end{array}$$

$$q^{IV}=25p^{IV}-36p^{V}=\frac{2\cdot 1}{9\cdot 11\cdot 33}.$$

The value of u, introduced in the last of equations (37) immediately gives

$$\eta=-\frac{1354584384}{81081000}.$$

Here we have taken ω as unity or the 12th part of the interval between the extreme ordinates. If, with Mr. Bérard, we take the entire interval as unity, we must divide our coefficients by 12. Now, after having divided the previous value of η by 12, and dividing the top and bottom of the fraction by 21, to express the fraction in a simpler form, I find[40]

$$\eta=-\frac{87797136}{63063000},$$

which is precisely the expression of the same coefficient, in the formula of Mr. Bérard. The other coefficients $\zeta, \epsilon, \ldots$ obtained by computation with equations

[39]In [Servois 1817], the value of q was given as $\frac{2\cdot 84}{1\cdot 5\cdot 3}$.

[40]This second value of η can be found by dividing the numerator of the previous value by 12, then multiplying top and bottom by $\frac{7}{9}$.

(37), then divided by 12, also coincide with those in the formulas cited, which is thereby fully justified.

It may not be necessary to observe that the method of which this article is concerned clearly applies, of course, in the same way, to the series (22) which also includes the cases of the interval divided into an odd number or an even number of parts. From this it follows that it is not true to say that Mr. Bérard's method is only immediately applicable to an even divisor (*Annales*, vol. VII, p. 245).[41]

VI. [95] Convinced that the formula for the divisor 12, given by Mr. Bérard, is true, must we then pronounce that Mr. Kramp's formula (*Annales*, vol. VI, p. 377), which differs from it is false? The reply based on the principles of Mr. Kramp himself (vol. VII, p. 245) is affirmative. Moreover, the two methods give the same results for the divisors[42] 1, 2, 3, 4, 5, 6, 7, 8. Thus, if it was only that they began to diverge at the divisor 8 that would be very extraordinary. Consequently, my esteemed friend, the Editor of the *Annales*, thinks that

> We can conclude nothing for or against the formulas of Messrs. Kramp and Bérard from the differences they present in the applications (*Ibid*, p. 246, in the first footnote).

It will be quite easy to decide the question, after the reconciliation that we are about to make among these methods and another, which has long been available to analysts. Here it is.

Let

$$y = v + Au + Bu^2 + Cu^3 + \ldots + Nu^n, \tag{38}$$

be the equation of a *complete* parabolic curve,[43] of order n, passing through the origin of the u's at the vertex of the ordinate v. By requiring it to pass through n other equally spaced ordinates $Ev, E^2v, \ldots, E^nv$ in the interval between the limits $u = 0$ and $u = n$, we will have, for determining the n coefficients $A, B, \ldots, N$, the n equations derived from (38), [96]

$$\left.\begin{array}{rcl} Ev & = & v + A + B + \ldots + N, \\ E^2v & = & v + 2A + 2^2B + \ldots + 2^nN, \\ & & \ldots\ldots\ldots\ldots\ldots, \\ E^nv & = & v + nA + n^2B + \ldots + n^nN. \end{array}\right\} \tag{39}$$

[41]See [Kramp 1816]. In paragraph five, Kramp states that Bérard's method is only applicable to even divisors and suggests that he might have been confused as to the modifications needed to adapt his formulas to the odd case.

[42]The following footnote was given in [Servois 1817]: "There is indeed some difference with respect to the divisor 8, because the common denominator of the coefficients, which are otherwise the same in one and the other, is 28350 in Mr. Bérard's paper and 89600 in Mr. Kramp's. However, it is probable that the difference is due to a typographical error in the latter number, because the first one supports the proof of the hypothesis of the equality of ordinates amongst each other and with unity."

[43]As previously noted, a complete parabolic curve is any polynomial of degree n.

We will then have the area of this curve, between the same limits, by integrating (38) multiplied by du, from $u = 0$ to $u = n$. Additionally, if this area, which I call *Inscribed*, or the *Area inscribed in the parabolic curve*, is taken instead as the area Z of the actual curve, we will have

$$Z = nv + \frac{An^2}{2} + \frac{Bn^3}{3} + \ldots + \frac{Nn^{n+1}}{n+1}. \tag{40}$$

It is clear that we will arrive at the same result by eliminating the coefficients $A, B, C, \ldots$ among the equations (39 and 40).

It is quite obvious that the system of equations (38, 39) can be replaced by this system of equations:

$$y = v + \frac{u}{1}\Delta v + \frac{u}{1} \cdot \frac{u-1}{2}\Delta^2 v + \ldots + \frac{u}{1} \cdot \frac{u-1}{2} \cdots \frac{u-n+1}{n}\Delta^n v, \tag{41}$$

$$\Delta v = Ev - v, \Delta^2 v = E^2 v - 2Ev + v, \ldots, \Delta^n v = E^n v - nE^{n-1}v + \ldots \pm v, \tag{42}$$

because, according to this system, by successively taking $u = 0$, $u = 1$, $u = 2$, $\ldots$, $u = n$ in (41), you will successively find $y = v, y = Ev, y = E^2 v, \ldots y = E^n v$, as it should be. Now, equations (41, 42) being precisely those from which Mr. Kramp eliminated the differences (IV), it is clear that the method of this geometer coincides with that of this article, that is to say, he gives [97] the area of the inscribed parabolic curve of degree n in place of the true area Z.

On the other hand, as demonstrated by Lagrange (*Ecole normale*, vol. IV),[44] the parabolic equation that immediately satisfies conditions (38, 39) is as follows[45]

$$y = \pm\frac{1}{1 \cdot 2 \cdots n}\left\{\begin{array}{l} v \times (u-1)(u-2)(u-3)\ldots(u-n) \\ -nEv \cdot u \times (u-2)(u-3)\ldots(u-n) \\ +\frac{n}{1}\frac{n-1}{2}E^2 v \cdot u(u-1) \times (u-3)\ldots(u-n) \\ \cdots\cdots\cdots\cdots \\ \pm E^n v \cdot u(u-1)(u-2)\ldots(u-n+1) \end{array}\right\} \tag{43}$$

where we must take the upper sign if n is even.

However, by denoting $S^1, S^2, S^3, \ldots$ as the sum of the products 1 by 1, 2 by 2, 3 by 3, $\ldots$, of the terms of the sequence $1, 2, 3, \ldots, n$, and by $S_1^1, S_1^2, S_1^3, \ldots$ these sums of the products when we exclude the term 1; and in general $S_k^1, S_k^2, S_k^3, \ldots$ these same sums of products when we exclude the k^{th} term, it is clear that equation (43) becomes

$$y = \pm\frac{1}{1 \cdot 2 \cdots n}\left\{\begin{array}{l} v\left(u^n - u^{n-1}S^1 + u^{n-2}S^2 - \ldots \pm S^n\right) \\ -\frac{n}{1}Ev\left(u^n - u^{n-1}S_1^1 + u^{n-2}S_1^2 - \ldots \mp S_1^{n-1}\right) \\ +\frac{n}{1}\frac{n-1}{2}E^2 v\left(u^n - u^{n-1}S_2^1 + u^{n-2}S_2^2 - \ldots \mp S_2^{n-1}\right) \\ \cdots\cdots\cdots\cdots \\ \pm E^n v\left(u^n - u^{n-1}S_n^1 + u^{n-2}S_n^2 - \ldots \mp S_n^{n-1}\right) \end{array}\right\} \tag{44}$$

[44] See [Lagrange 1795, p. 277].

[45] This is Lagrange's Interpolating Polynomial for the points $x_k = k$, when $k = 0, 1, 2, \ldots, n$.

[98] Multiplying this by du, then integrating between the limits $u = 0$ and $u = n$, and giving the result for the area Z, we finally obtain

$$Z = \pm\frac{1}{1\cdot 2\cdots n}\left\{\begin{array}{l} v\left(\frac{n^{n+1}}{n+1} - \frac{n^n}{n}S^1 + \frac{n^{n-1}}{n-1}S^2 - \ldots \pm nS^n\right) \\ -nEv\left(\frac{n^{n+1}}{n+1} - \frac{n^n}{n}S_1^1 + \frac{n^{n-1}}{n-1}S_1^2 - \ldots \mp \frac{n^2}{2}S_1^{n-1}\right) \\ +\frac{n}{1}\frac{n-1}{2}E^2v\left(\frac{n^{n+1}}{n+1} - \frac{n^n}{n}S_2^1 + \frac{n^{n-1}}{n-1}S_2^2 - \ldots \mp \frac{n^2}{2}S_2^{n-1}\right) \\ \ldots\ldots\ldots\ldots \\ \pm E^nv\left(\frac{n^{n+1}}{n+1} - \frac{n^n}{n}S_n^1 + \frac{n^{n-1}}{n-1}S_n^2 - \ldots \mp \frac{n^2}{2}S_n^{n-1}\right). \end{array}\right\} \tag{45}$$

This (45) is the formula that we promised above (IV), which immediately gives expressions, as functions of the number n, for the coefficients for equidistant coordinates.

If we prolong the right-hand side of (38), up to the power $2n$ of u, which represents the parabolic curve of degree $2n$, we will express that this curve is inscribed in the one whose area has been designated by W (21) by writing the equations, $n+1$ in number,

$$\left.\begin{array}{rl} E^nv + E^{-n}v &= 2v + 2Bn^2 + 2Dn^4 + \ldots, \\ E^{n-1}v + E^{-(n-1)}v &= 2v + 2B(n-1)^2 + 2D(n-1)^4 + \ldots, \\ & \ldots\ldots\ldots\ldots\ldots \\ Ev + E^{-1}v &= 2v + 2B + 2D + \ldots, \\ v &= v. \end{array}\right\} \tag{46}$$

If we then take the two particular areas, one between [99] 0 and $+n$, the other between 0 and $-n$, and we add their absolute values, to have the parabolic area inscribed between $+n$ and $-n$, which area we finally take in place of the area W, we will have

$$W = 2nv + \frac{2n^3}{3}B + \frac{2n^5}{5}D + \ldots. \tag{47}$$

Now, equations (45, 21) and (44, 32) will coincide respectively if we have

$$B = \frac{1}{1\cdot 2}\frac{d^2v}{da^2}, \quad D = \frac{1}{1\cdot 2\cdot 3\cdot 4}\frac{d^4v}{da^4}, \ldots.$$

Therefore, the result of the elimination of $B, D, \ldots$ in (44, 45) will be identical with that of the elimination of the differentials $\frac{1}{1\cdot 2}\frac{d^2v}{da^2}, \frac{1}{1\cdot 2\cdot 3\cdot 4}\frac{d^4v}{da^4}, \ldots$ between equations (21, 32). Now, this final result is that of the method of Mr. Bérard. Thus, it also gives for the approximated area that of the inscribed parabolic curve.

We have the right to conclude, with all rigor, by virtue of the axiom: *Quæsunt eadem*, etc.,[46] that the methods of Messrs. Kramp and Bérard must, for the same divisors, give the same results.

[46] *Quæsunt eadem cum uno tertio sunt eadem inter se*: If two terms agree with the same third, they agree with each other.

The parabolic curve (38) of degree n, inscribed in the given curve, whose area between the limits v and $E^n v$ is Z, has its own proper area Z', between the same limits, expressed in the equidistant ordinates $v, Ev, \ldots$; this is the right-hand side of (45). However, if we immediately treat this parabolic area Z' by the method of Mr. Dobenheim, we would not find a different result, as long as we take a large enough number of otherwise divided areas $T', T'', \ldots$ to eliminate the number $\frac{n}{2}$ of coefficients of the powers of ω that follow T in formula (10), appropriate in this case. Now, it is [100] precisely this result that the method in question gives, in place of the area Z. Thus, the method of Mr. Dobenheim also takes again the area of the inscribed parabolic curve in place of the true area and thus ends the kind of astonishment that it first inspired by presenting, as an approximation for the area of a curve, a linear combination of equidistant ordinates, different from the one that makes up T, or the sum of inscribed trapezoids. This is because nothing keeps the ordinates, which when combined in one way gives the area of the inscribed rectilinear polygon, from giving the area of the inscribed parabola when combined in another way. We also see that these results must concur with those of the other two methods when for n, the divisor of the interval, there is a number $\frac{n}{2}$ of exact divisors $n', n'', \ldots$, and these are used to compose as many auxiliary areas $T', T'', \ldots$. Thus, for example, because $n = 6$ has divisors 1, 2, 3, and 6, we have, besides the area T which corresponds to 1, three other areas T', T'', T''', corresponding to the other divisors 2, 3, and 6, and we conclude that the given method, in this case, gives the same formula as the others (*Compare*: vol. VI, pp. 288 and 376).

VII. The method of approximation (V) may thus be reconciled in the same spirit with that of the preceding article, that is to say, with the method of parabolic curves, and I would have drawn this important conclusion earlier, if I had not been afraid of being criticized for putting forward the proposition: "only whole rational functions eventually give null differences," from which the above is an immediate corollary. We must now try to appreciate the merits of this method of parabolic curves.

I would not say that it leaves nothing to be desired; I will not even conceal that it is under the weight of a very severe censure, recently pronounced by a judge whom we are not tempted to challenge. [101]

> I consider above all as one of the most defective (methods of approximation), that which supposes that the ordinate of the curve is represented in all of its extent by the formula $y = a + bx + cx^2 + \ldots$, or by an equivalent formula, because, although this curve can pass through a great number of points of a given curve, it does not follow that the two curves are very close to one another. On the contrary, it may happen that the two areas, in spite of all the common points, may be as different from one another as one may wish. (*Exercices de calcul integral*, part III, p. 316).[47]

[47] See [Legendre 1811].

Effectively, between the assigned limits, let the proposed curve be cut in n points by a parabolic curve, in the equation of which (38) you will admit an additional indeterminate coefficient, which will give in (39 and 40), one more term and one more coefficient. Then, determine the $n+1$ coefficients, by means of equations (39 and 40), making Z, in the latter, equal to a given quantity. In this way we will have, between the assigned limits, a parabolic curve of degree $n+1$ which, with n points in common to the given curve, nevertheless has an arbitrary given area, and consequently also an area as different from the area of the proposed curve as we may desire.

However, if between these assigned limits the proposed curve has no singular features, such as multiple branches, infinite branches, conjugate points, cusps, etc., or, if speaking analytically, between these limits, none of the differentials $dFx, d^2Fx, \ldots$ becomes infinite, in a word if the Taylor series may express its ordinates throughout the whole interval — and such is the generally accepted hypothesis — we understand that the more points there are in common between the proposed curve and a parabolic curve of a degree equal to the number of such points, the more the area of this second curve approaches identity with the area of the first one. It is not superfluous to confirm this perception through analytic considerations.

Let's consider a complete parabolic curve of degree n, passing [102] through $n+1$ points of the given curve between the limits 0 and n. It will have as its equation our formula (44), which we will put in the form

$$y = U,$$

U being a whole rational function of u comprising the right-hand side of (44). When u receives any increment α, we will have[48]

$$E^{\alpha}y = U + \frac{\alpha}{1}\frac{dU}{du} + \frac{\alpha^2}{1\cdot 2}\frac{d^2U}{du^2} + \ldots + U'.$$

Now, when we make u equal to one of the numbers of the sequence 1, 2, 3, $\ldots$, n–setting u equal to 4, for example–then U becomes E^4v. Additionally, when one augments u by a unit, $E^{\alpha}y$ becomes Ey, and, in our example, equal to E^5v. The function U' moreover, has only a finite number of terms, because, given that U is a whole rational function of u, its differentials will eventually vanish. I say that the function U' is such that it is null for $\alpha = 0$, and for $\alpha = 1$, it is equal to $E^5v - E^4v$, a quantity which is evidently smaller as the neighboring ordinates approach one another, which is our hypothesis concerning the given curve. Therefore, for any value of α, between 0 and 1, the function U' will be very small, because it is a finite rational function of α. Thus, in the interval between two consecutive ordinates of the given curve, the ordinates on the parabolic curve differ very little from one another and from their limits. Finally, because this is the hypothesis relative to the ordinates of the given curve, the corresponding areas in both curves must also differ by very little.

[48]In [Servois 1817], the following formula was given as $E^{\alpha}y = U + \frac{\alpha}{1}\frac{dU}{du} + \frac{\alpha^2}{1\cdot 2}\frac{d^2U}{du^2} + \ldots U + U'$.

Perhaps I am mistaken, but I dare say that the method of parabolic curves seems to me generally preferable to the direct method (I, II), which consists of taking a certain number of terms of the series (11, 14, etc.) as an approximation, because, without mentioning the difficulties and lengths in which the latter [103] engages, for each particular case, there is trouble which we can apprehend by imagining that we are forced to numerically calculate several successive orders of the differentials, which can often be very complicated. It is entirely impossible when it encounters divergent series, or even series which are very slightly convergent. On the other hand the first method, after a brief examination, necessary to recognize its implementation, brings one to a very close approximation, by very simple calculation, a good part of which is entirely contained in tables.

I take a very simple example–the research on the logarithm of 2–which is the first example that is proposed by Mr. Kramp (*Annales*, vol. VI, p. 288), and we know that the parabolic method applies to it with great ease.

I therefore make $y = Fx = \frac{1}{x}$, from which[49] $Z = \int Fxdx \left\{\frac{x}{a}\right\} = \log\left(\frac{x}{a}\right)$, and to get $Z = \log 2$, I suppose $\omega = 1$, $a = 6$, and $x = 12$. The sequence of the differentials of the function Fx is[50]

$$\frac{dFx}{dx} = -\frac{1}{x^2}, \quad \frac{d^2Fx}{dx^2} = +\frac{2}{x^3}, \quad \frac{d^3Fx}{dx^3} = -\frac{6}{x^4}, \ldots.$$

According to these formulas, the series (19, 20, 22), give, without difficulty,[51]

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots,$$

$$\log 2 = \frac{1}{2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} + \frac{1}{4\cdot 2^4} + \ldots,$$

$$\log 2 = \frac{2}{3}\left(1 + \frac{1}{3\cdot 3^2} + \frac{1}{5\cdot 3^4} + \frac{1}{7\cdot 3^6} + \ldots\right).$$

Of these three series, the first is useless, seeing that it is not sufficiently convergent. The second is not much better. [104] The third may absolutely be used. However, to obtain a result of the same precision, the parabolic procedure of methods (IV, V), aided by the formulas calculated in the *Annales*, etc., seems to me easier. Let us see what series (10) gives. Because

$$T = \frac{1}{2\cdot 6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{2\cdot 12} = 0.694877346,$$

I have

$$\log 2 = 0.694877346 - B_1\frac{2^2-1}{12^2} + B_2\frac{2^4-1}{12^4} - B_3\frac{2^6-1}{12^6} + \ldots.$$

[49]In the notation $\int Fxdx\left\{\frac{x}{a}\right\}$, the numbers in the curly braces represent the limits of integration. In modern days, we would write $\int_a^x F(t)\,dt$.

[50]In [Servois 1817], the following sequence was given as

$$\frac{dFx}{dx} = -\frac{1}{x^2}, \quad \frac{d^2Fx}{dx^2} = +\frac{1}{x^3}, \frac{d^3Fx}{dx^3} = -\frac{1}{x^4}, \ldots.$$

[51]In (19, 20, 22), we have $n = 6$, $v = \frac{1}{6}$, $y = \frac{1}{12}$, and $E^{\frac{1}{2}n}v = F(a + 3\omega) = \frac{1}{9}$.

To assure myself of the convergence of this series, which contains the Bernoulli numbers, I equate the absolute value of two terms of orders n and $n+1$, from which I conclude

$$\frac{B_{n+1}}{B_n} = \frac{\left(2^{2n}-1\right)12^2}{2^{2(n+1)}-1}.$$

Now, Euler, in his *Calcul différentiel*,[52] demonstrated that the ratio of two consecutive Bernoulli numbers converges fairly quickly to the expression $\frac{n^2}{\pi^2}$. We can therefore write

$$\frac{n^2}{\pi^2} = \frac{144\left(2^{2n}-1\right)}{2^{2(n+1)}-1},$$

where we conclude n is approximately equal to 6π, or about 18. That is to say, the series becomes divergent after the first 18 terms. It is therefore *absolutely divergent*, because if you combined the first 18 terms to form a single first term, then you would have an entirely divergent series, which by itself will teach us nothing about the value of $\log 2$, at least in the current state of analysis.

[105] However, one might say, the series in question is of the *semi-convergent* class: those that provide successive approximations, as long as one does not exceed the limit of decreasing terms. I find, to support this proposition, only a feeble induction, whereas a good demonstration is needed. It will be said, *the first term* of a divergent series is, in general, an *approximation*. Whatever this may be, at least it is certain that it is often far away from the exact value, and that nothing in the series can help to judge the degree of approximation. Thus, in our example, it is not from the series, but elsewhere, that I know that the first term 0.694877364 is an approximate value of $\log 2$. We can even calculate, it will be added, the approximation that a *semi-convergent* series can give: we calculate the degree of smallness of the term which is at the beginning of the divergence. So be it, but I do not know that we can demonstrate *a priori* that this is the measure of the approximation that the series infallibly produces: This property itself of giving an approximation whose term is calculable would be a paradox that no induction could admit.

The series[53]

$$1 - \frac{2}{3} - \frac{2}{9} - \frac{4}{27} - \frac{10}{81} - \frac{28}{243} - \frac{28}{243} - \frac{88}{729} - \ldots,$$

is convergent up to the sixth term and is divergent beyond that; it is consequently semi-convergent. Now, this series, multiplied by $\sqrt{3}$, is nothing other than the expansion of $\sqrt{3-4}$ or $\sqrt{-1}$. Thus, as with absolutely divergent series, the semi-convergent series may express imaginary quantities; this never happens for convergent series. From this, it seems to follow that the first two should be united in one and the same single class, as the semi-divergent belong with the convergent.

[52] See [Euler 1755].

[53] In [Servois 1817], the following series was (incorrectly) given as $1 - \frac{2}{3} - \frac{2}{9} - \frac{4}{27} - \frac{10}{81} - \frac{28}{243} - \frac{88}{729} - \ldots$. This is the binomial series for $\left(1-\frac{4}{3}\right)^{\frac{1}{2}}$.

D'Alembert and Condorcet,[54] who engaged themselves so much with series, did not accept these equivocal beings called semi-convergent series. The latter said:

> It is necessary that the sequence given by the [106] method of approximation, must be capable to continue to infinity, without having to stop at any term, and change its *form* or *nature* at that point; and that the more terms we take. . . the less the sum of the sequence differs. . . ; and it must be not only that this holds, but moreover that it has been proved *a priori*. (*Problème des trois corps*, p. 62).[55]

Lagrange expresses himself in perhaps a more positive manner in this regard. After having spoken about the way to evaluate the omitted terms at the end of the Taylor series, he adds:

> By the means of these limits, we have covered the difficulties that may arise from the non-convergence of the series [the value of $(x+i)^m$] . . . if $\frac{i}{x} < 1$ . . . the series will always end up being convergent. However, it will always be divergent at its extremity, if $\frac{i}{x} > 1$, even though it may be convergent in its first terms. Thus, it cannot be used with certainty, however far it may be taken, except by taking into account the limits that we have just given. (*Journal de l'école polytechnique*, book XII,[56] p. 75).[57]

[54]Marie Jean Antoine Nicolas de Caritat Condorcet (1743-1794).

[55]See [Condorcet 1767].

[56]Published in 1804.

[57]The following footnote, inserted by Gergonne, appeared in [Servois 1817]: "While I basically share the opinion of my judicious friend, I nevertheless believe that I must temper it slightly.

First of all, I note with him, that as we may always join as many of the first terms as we may wish of a semi-convergent series into a single term by addition, it follows that the series in this class may always be classified in the class of purely divergent series.

In the second place, I note that such series may always be put into an infinity of different forms. In fact, we may combine their terms two by two, or three by three, four by four, and so on. We may also leave the first term alone, join the *two* following, then the *three* that come after these, the *four* that come next, and so on. We may finally form the terms of this series by any other *regular* combination that we may wish.

Now, if all the terms of the series are not of the same sign, or are not always so starting at a certain term, we may imagine that among the new series that we may have deduced, it may well be that we find some that are convergent, and even that we may prove that they must remain so after a certain term. Now, given that the latter may have an assignable sum, those that we have deduced from them must equally be so.

Among the examples that we may produce in support of these reflections, one of the simplest is undoubtedly that of the divergent series

$$\frac{0}{1} - \frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \dots. \tag{A}$$

In combining its terms two by two, it becomes

$$-\frac{1}{1\cdot 2} - \frac{1}{3\cdot 4} - \frac{1}{5\cdot 6} - \frac{1}{7\cdot 8} - \frac{1}{9\cdot 10} - \frac{1}{11\cdot 12} - \dots, \tag{B}$$

a perpetually convergent series, whose rule is manifest, and which may consequently be used with all certainty of conscience, as a means of approximation. Now, the first is the expansion

VIII. [107] Until some fortunate discovery has taught us to turn convergent those series that are very little so or not at all, that is to make use of divergent series, the parabolic method remains the resource of the calculating mathematician, and it is [108] consequently this method that we must endeavor to perfect. The parabolic area incontestably approaches closer to the true area if, in addition to the number n of common points, the two curves always have, at these points, more or less intimate contacts. Now, it is always possible to satisfy this new condition when we have the equation of the proposed curve. Indeed, in differentiating equation (38), which is therefore no longer terminated at the term Nu^n, we find

$$\begin{aligned} \frac{dy}{dx} &= A + 2Bu + 3Cu^2 + \dots \\ \frac{d^2y}{dx^2} &= 2B + 2\cdot 3Cu + 3\cdot 4Du^2 + \dots \\ \frac{d^3y}{dx^3} &= 2\cdot 3C + 2\cdot 3\cdot 4Du + 3\cdot 4\cdot 5Eu^3 + \dots . \end{aligned}$$

These are as many formulas that give the differential coefficients at the vertices of each of the ordinates $v, Ev, E^2v, \dots$, by successively making $u = 0, u = 1, u = 2, \dots$. In order to clarify these ideas, suppose we wish that the parabolic curve is to have, at the points in common with the proposed curve, contacts of the first-order, or common tangents. To abbreviate, we use the letters $\alpha, \beta, \dots, \mu, \nu$, alone or marked with accents to represent the $n+1$ equidistant ordinates and their successive differential coefficients, respectively. Noting that $\alpha = v$ and that $\alpha' = A$, we have the equations [109]

$$\left.\begin{array}{rcl} \beta &=& \alpha + \alpha' + B + C + \dots \\ \gamma &=& \alpha + 2\alpha' + 2^2B + 2^3C + \dots \\ && \dots\dots\dots\dots\dots \\ \nu &=& \alpha + n\alpha' + n^2B + n^3C + \dots . \end{array}\right\} \qquad (48)$$

of $\frac{1}{2} - \log 2$, because we have

$$\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + \dots,$$

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{7} + \dots,$$

which, in fact gives,

$$\frac{1}{2} - \log 2 = \frac{0}{1} - \frac{1}{2} + \frac{2}{3} - \frac{1}{4} + \frac{4}{5} - \frac{5}{6} + \dots .$$

From this it follows that the divergent series (A) may, like the convergent series (B), be used an an approximation to $\frac{1}{2} - \log 2$.

However, along with Mr. Servois, I also reject, as an instrument of approximation, any series that are divergent, or even semi-convergent, the terms of which are all of the same sign, or become of the same sign starting from any one of its terms, as well as every divergent or semi-convergent series, always having terms that are both positive and negative, but of which we cannot prove that, by some transformation, it can be reduced to a truly convergent series, whether immediately or from any of its terms. J.D.G."

$$\left.\begin{array}{rcl}\beta' & = & \alpha' + 2B + 3C + \ldots \\ \gamma' & = & \alpha' + 2\cdot 2B + 3\cdot 2^2 C + \ldots \\ & & \ldots\ldots\ldots\ldots\ldots \\ \nu' & = & \alpha' + 2nB + 3n^2 C + \ldots.\end{array}\right\} \tag{49}$$

The first ones (48), are the same as (39), expressing the equality of the n points. The latter ones (49) express the equality of the n tangents. To find the area Z we must combine them with the equation

$$Z = n\alpha + \frac{1}{2}n^2\alpha' + \frac{1}{3}n^3 B + \frac{1}{4}n^4 C + \ldots. \tag{50}$$

The equations (48, 49), separately n in number, together $2n$ in number, will determine the $2n$ coefficients $B, C, \ldots$, that is to say, the $2n$ coefficients following A, so that the last term of (38) will be of order u^{2n+1}. Thus, we may always make a parabolic curve of order $2n+1$ pass through n points of the given curve with common tangents at these points. If we wished the parabolic curve had both first and second order contacts at the same time, that is to say, common tangents and radii of curvature, we must join the following equations to the equations (48, 49), also n in number,

$$\begin{array}{llll}\beta'' & = \alpha'' + 2\cdot 3C & +3\cdot 4D & + \ldots \\ \gamma'' & = \alpha'' + 2\cdot 3\cdot 2C & +3\cdot 4\cdot 2^2 D & + \ldots \\ & \ldots\ldots\ldots\ldots\ldots & & \\ \nu'' & = \alpha'' + 2\cdot 3\cdot nC & +3\cdot 4\cdot n^2 D & + \ldots.\end{array}$$

By means of $3n$ equations, we will determine $3n$ coefficients [110] $C, D, \ldots$, such that (38) will rise to order $3n+2$, and so on. We see in general that we may always determine a parabolic curve which, at the n points of intersection, has the m contacts of successive orders all at the same time, and that this curve will be of the order $mn + m - 1$.

Suppose, to give an example, that having divided the interval between the limits in $n = 3$ equal parts, we wish to make a parabolic curve pass through the vertices of the four ordinates α, β, γ, and δ, and in addition to these points the two curves have common tangents. I take the first three equations of (48, 49) as far as the coefficient G, inclusive. By means of these, I determine the six coefficients B, C, D, E, F, G. I substitute into (50) and I finally find

$$Z = \frac{465\,(\alpha+\delta) + 1215\,(\beta+\gamma) + 57\,(\alpha'-\delta') - 81\,(\beta'-\gamma')}{1120}. \tag{51}$$

Let us apply this formula to the logarithm of 2. Because the interval is divided into three units, we must make $a = 3$ and[58] $x = 6$, to have $Z = \log 2$. Given this, we have

$$\alpha = \frac{1}{3}, \quad \beta = \frac{1}{4}, \quad \gamma = \frac{1}{5}, \quad \delta = \frac{1}{6},$$

[58] In [Servois 1817], this was given as $n = 6$.

$$\alpha' = -\frac{1}{9}, \quad \beta' = -\frac{1}{16}, \quad \gamma' = -\frac{1}{25}, \quad \delta' = -\frac{1}{36},$$

from which

$$\alpha + \delta = \frac{1}{2}, \quad \beta + \gamma = \frac{9}{20}, \quad \alpha' - \delta' = -\frac{1}{12}, \quad \beta' - \gamma' = -\frac{9}{400},$$

values which, when substituted in (51), give

$$\log 2 = 0.693145\ldots,$$

the exact expression to the *fifth* decimal place, inclusive.

Formula (51) is easily verified in another way, by making [111]

$$\alpha = \beta = \gamma = \delta = 1 \quad \text{and} \quad \alpha' = \beta' = \gamma' = \delta' = 0.$$

This is of course a line parallel to the x-axis at a distance of 1. Thus, one must have $Z = 3$. Another verification is to make

$$\alpha = 0, \beta = 1, \gamma = 2, \delta = 3 \quad \text{and} \quad \alpha' = \beta' = \gamma' = \delta' = 1.$$

This is the case of a straight line passing through the origin, and inclined to the x-axis by half a right angle. Thus, we must have $Z = \frac{3^2}{2}$.

It should be noted that the form in which equation (51) is presented is not accidental, and, we shall see why, in general, by combining the differential coefficients of the same order two by two in the final expression of the area, following the summary account of the procedure that I recommended be followed by those who wish to construct tables, according to the ideas set out in this article. This procedure is entirely similar to that which we have applied to the series (21), and consists in eliminating from this series the differential coefficients $\frac{d^2v}{da^2}, \frac{d^4v}{da^4}, \ldots,$ not only by means of equations (32), but by means of those equations joined with the entirely similar equations that exist among the successive differential coefficients of the ordinates $v, Ev, \ldots$. Thus, for formulas of the different systems of equations to be used, we have, using the simple letters $\alpha, \beta, \ldots$, in place of the differential quotients [112][59]

$$\begin{aligned}
E^n v + E^{-n} v &= 2v + 2n^2\alpha + 2n^4\beta + \ldots \\
\frac{dE^n v}{da} - \frac{dE^{-n} v}{da} &= \ldots 2 \cdot 2n^2\alpha + 4 \cdot 2n^3\beta + \ldots \\
\frac{d^2E^n v}{da^2} - \frac{d^2E^{-n} v}{da^2} &= \ldots 2 \cdot 2n\alpha + 3 \cdot 4 \cdot 2n^2\beta + \ldots \\
\frac{d^3E^n v}{da^3} - \frac{d^3E^{-n} v}{da^3} &= \ldots 2 \cdot 3 \cdot 4 \cdot 2n\beta + \ldots \\
\frac{d^4E^n v}{da^4} - \frac{d^4E^{-n} v}{da^4} &= \ldots 1 \cdot 2 \cdot 3 \cdot 4 \cdot 2\beta + \ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots &
\end{aligned}$$

[59] In [Servois 1817], the third equation was given as $\frac{d^2E^n v}{da^2} - \frac{d^2E^{-n} v}{da^2} = \ldots 2 \cdot 2\alpha + 3 \cdot 4 \cdot 2n^2\beta + \ldots$. The ellipses following the equal sign presumably indicate a constant term of 0.

The first of these formulas gives the $n+1$ equations (32): each of the following ones gives as much. Multiplying the first system by $A, B, C, \ldots$, the second by $A', B', C', \ldots$, the third by $A'', B'', C'', \ldots$, and so on; and then adding them all, and denoting by V the sum of the left-hand sides, we shall set $W = V$, and we have, going up to the contact of order m, the number $(m+1)(n+1)$ of equations among as many coefficients, which once having been determined give us

$$W = \left\{ \begin{array}{l} A\left(E^n v + E^{-n} v\right) + B\left(E^{n-1} v + E^{-(n-1)} v\right) + \ldots + Nv \\ +A'\left(\frac{dE^n v - dE^{-n} v}{da}\right) + B'\left(\frac{dE^{n-1} v + dE^{-(n-1)} v}{da}\right) + \ldots + N' v \frac{dv}{da} \\ +A''\left(\frac{d^2E^n v - d^2E^{-n} v}{da^2}\right) + B''\left(\frac{d^2E^{n-1} v + d^2E^{-(n-1)} v}{da^2}\right) + \ldots + N'' v \frac{d^2v}{da^2} \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array} \right\} \tag{52}$$

That is to say, in the final expression of the area W, the ordinates equally distant from the extremities, as well as their successive differential coefficients, are gathered together under the same numerical coefficient, but separated by the $+$ sign, for the differentials $d^0, d^2, d^4, \ldots$ of even ranks, and by the $-$ sign, for the differentials $d^1, d^3, d^5, \ldots$ of odd ranks.

IX. [113] Although the method of the preceding article expresses the area as a function of equidistant ordinates and their successive differential coefficients, it must not be confused with the method of the same features given by Euler in his *Calcul intègral* (vol. I, sect. I, chap. VII).[60] The latter clearly amounts to dividing the total area into a number n of partial areas, having their bases a on the x-axis, and taking the sum of these areas, evaluated separately, by the series of Bernoulli. I shall not insist on proving that it is always practicable, and that it may be sometimes quite advantageous to evaluate these partial areas using the methods that we have just studied. I will equally abstain from making any comparison between the results of the method of parabolic curves and those of the methods which represent the order of the curve by functions of the abscissa, whether by a finite rational fraction or a recurrent series derived from it, or by a finite sequence of sines or cosines and their multiples at the abscissa, or a finite series of exponentials, etc. I close with the following two general observations:

1. By Taylor's Theorem, we are permitted, in general, to assume:

$$Fx = Fa + \alpha\,(x-a) + \beta\,(x-a)^2 + \gamma\,(x-a)^3 + \ldots. \tag{53}$$

If we know a certain number of the values $Fx', Fx'', \ldots$ of Fx, which correspond to $x', x'', \ldots$, or else a certain number of values of $V, V', V'', \ldots$, which must satisfy the equations

$$\left. \begin{array}{rcl} V &=& A + BFa + \alpha C + \beta D + \ldots, \\ V' &=& A' + B'Fa + \alpha C' + \beta D' + \ldots, \\ && \ldots\ldots\ldots\ldots\ldots, \end{array} \right\} \tag{54}$$

in which the coefficients $A, B, \ldots$ and $A', B', \ldots$ are also [114] known, then by eliminating a number n of coefficients $\alpha, \beta, \ldots$, between a number $n+1$

[60] *Institionum calculi integralis*, Vol I., E342, 1768.

of equations expressing, according to (52) or (53), an equal number of known values $Fx', Fx'', \ldots$, and $V, V', \ldots$, we finally obtain an equation of the first degree in Fa, from which we immediately obtain an expression of this function in terms of known quantities, an expression which is an approximate value, provided that the particular expansion deduced from (52 and 53) is possible. This is, in general, the spirit of the method that has principally occupied us in this memoir; from this it follows that it is applicable to many other things besides quadratures.

2. Suppose it is *possible* to assume

$$Fn = \phi + \frac{\alpha}{n} + \frac{\beta}{n^2} + \frac{\gamma}{n^3} + \ldots, \tag{55}$$

ϕ being what Fn becomes, when n is infinite. If we know the values of Fn corresponding to the values $n = 1, n = 2, n = 3, \ldots$, we have

$$\left.\begin{array}{rcl} F1 &=& \phi + \frac{\alpha}{1} + \frac{\beta}{1} + \frac{\gamma}{1} + \ldots, \\ F2 &=& \phi + \frac{\alpha}{2} + \frac{\beta}{2^2} + \frac{\gamma}{2^3} + \ldots, \\ F3 &=& \phi + \frac{\alpha}{3} + \frac{\beta}{3^2} + \frac{\gamma}{3^3} + \ldots, \\ && \ldots\ldots\ldots\ldots\ldots \end{array}\right\}$$

Among these, assumed to be $n + 1$ in number, we eliminate a number n of coefficients $\alpha, \beta, \gamma, \ldots$, and we will have ϕ by an equation of the first degree that can be used to express it in terms of $F1, F2, F3, \ldots$ and the different powers of $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$, by approximation, if the form (54) and equations that are derived from it are possible.

[115] In (54), does Fn denote, for example, the perimeter or area of a regular polygon of n sides inscribed in or circumscribed about a circle? The term ϕ will be the perimeter or the area of that with an infinity of sides, or the circle itself, and $F1, F2, F3, \ldots$ will be the polygons of $1, 2, 3, \ldots$ sides. Does Fn represent the sum of the n first terms of an infinite series? Then, ϕ will be the infinite sum as long as $F1, F2, F3, \ldots$ are the sums of the same series carried out to the 1^{st} 2^{nd}, 3^{rd}, ... terms. Is Fn the ordinate interpolated in a curve by means of n other given ordinates? Then, ϕ will be the one that was interpolated by means of an infinite number of given ordinates, that is to say, the incontestable ordinate,[61] as long as $F1, F2, F3, \ldots$ are the interpolations deduced from the $1, 2, 3, \ldots$ assigned ordinates, and so on.

I have just referred back to, in substance, a very beautiful idea that the Editor of the *Annales* published among several others of the same type in the *réflexions* that he made following Mr. Kramp's first memoir (*Annales*, vol. VI, p. 303ff). I hasten to take this opportunity to recommend this small work of my worthy friend to the attention of all geometers.

[61]This was given as *l'ordonée rigoureuse*, literally, the "rigorous ordinate."

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