Explication SAMPLE provided to students

Math 464 WI History of Mathematics R. Delaware

A Transcription and Explication using Modern English and Notation

From: Al-Khwarizmi, c.780 - c.847, born in (what is now) Uzbekistan, from *The Book of Algebra and Almucabola*, as translated in The History of Mathematics, A Reader, ed. John Fauvel and Jeremy Gray, 1987, pp. 230-231.

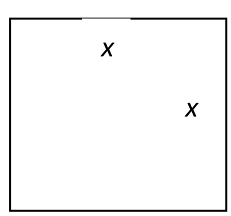
Notes: All comments in [square brackets] and illustrations are mine.

Now, however, it is necessary that we should demonstrate geometrically the truth of the same problems which we have explained in numbers. Therefore our first proposition is this, that a square and 10 roots equal 39 units.

> [With x representing the unknown "root" to be found, and "units" indicating numbers, the proposition, or method, here is "how to solve for x in $x^2 + 10x = 39$". In fact, he will only solve for one positive such "root".]

The proof is that we construct a square of unknown sides, and let this square figure represent the square (second power of the unknown) which together with its root you wish to find. Let the square, then, be *ab*, of which any side represents one root.

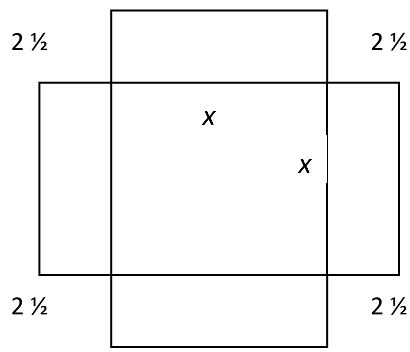
> [We construct a square of side x, abandoning his *ab* notation for the square.]



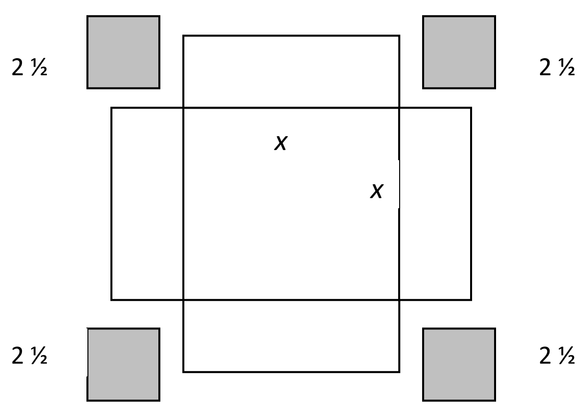
When we multiply any side of this by a number (or numbers) it is evident that that which results from the multiplication will be a number of roots equal to the root of the same number (of the square). Since then ten roots [10*x*] were proposed with the square, we take a fourth part of the number ten [meaning 10/4 = 2 ½] and apply to each side of the square an area [a rectangle] of equidistant sides, of which the length should be the same as the length of the square first described and the breadth 2 ½ , which is a fourth part of 10. Therefore four areas of equidistant sides are applied to the first square, *ab*.

> [meaning, we enlarge the figure from just a square to a square with four rectangles whose total area now represents the left hand side of the equation: $x^2 + 10x$.]

Of each of these the length is the length of one root [*x*] of the square *ab* and also the breadth of each is 2 ½ , as we have just said. These now are the areas *c, d, e, f*. [Again, we abandon his notation.] Therefore it follows from what we have said that there will be four areas [rectangles] having sides of unequal length [lengths 2 ½ and *x*], which also are regarded as unknown.



The size of the areas in each of the four corners, which is found by multiplying 2 ½ by 2 ½ , completes that which is lacking in the larger or whole area. [Literally "completing the square"] Whence it is we complete the drawing of the larger area by the addition of the four products, each 2 ½ by 2 ½ ; the whole of this multiplication gives 25. [Since 4 (2 ½ • 2 ½) = 25.]



And now it is evident that the first square figure, which represents the square of the unknown [x^2], and the four surrounding areas [of the four surrounding rectangles which have total area $10x$] make 39 [because we assume from the beginning that $x^2 + 10x = 39$]. When we add 25 to this, that is, the four smaller squares which indeed are placed at the four angles [corners] of the square *ab*, the drawing of the larger square, called *GH*, [we abandon this notation] is completed (al-jabr).

Whence also the sum total of this [area] is 64 [because 39 + 25 = 64] (al-muqabala), of which 8 is the [positive, square] root, and by this is designated one side of the completed figure [the larger square]. Therefore when we subtract from 8 twice the fourth part of 10, which is placed at the extremities of the larger square *GH*, there will remain but 3. [Because 8 – 2 (2 ½) = 3.] Five being subtracted from 8, 3 necessarily remains, which is equal to one side [x] of the first square *ab*.

[An observation now follows.]

This 3 then expresses one root of the square figure, that is, one [positive] root [x] of the proposed square of the unknown [x^2], and 9 the square itself [because $3^2 = 9$.].

[Now the author expresses part of the procedure above as an algorithm, shortening one step.]

Hence we take half of 10 and multiply this by itself.

[This is NOT the technique illustrated above, where we used instead 4 times 10/4 squared. He explains below why this is equivalent to that technique.]

We then add the whole product of the multiplication to 39, that the drawing of the larger square *GH* may be completed; for the lack of the four [squares at the] corners rendered incomplete the drawing of the whole of this square.

[Assembling all the remarks from above, the algorithm for the solution of $x^2 + 10x = 39$ is thus: Take half of 10 (the coefficient of x), multiply this number by itself (which would be 25), then add it to 39 (yielding a total of 64). The (square) root of 64 is 8. Subtract 5 from 8 (because 5 is twice the fourth part of 10, also interpreted here as half of 10), leaving 3, the desired root x.]

Now it is evident that the fourth part of any number multiplied by itself and then multiplied by four gives the same number as half of the number multiplied by itself. Therefore if half of the root is multiplied by itself, the sum total of this multiplication will wipe out, equal, or cancel the multiplication of the fourth part by itself and then by 4.

> [He is just observing that $4 (a/4) \bullet (a/4) = (a/2) \bullet (a/2)$, for any number *a*, which explains that earlier sentence "Hence we take half of 10 and multiply it by itself."]

Remarks:

Although this appears to be just a solution of a particular equation, in fact it is a nice illustration of a general technique. For any equation of the form $x^2 + bx = c$, where $b, c > 0$, the solution illustrated by the algorithm above is given by

$$x = \sqrt{\frac{b}{2}\frac{b}{2} + c} - \frac{b}{2}$$
$$= -\frac{b}{2} + \sqrt{\frac{b^2}{4} + c}$$
$$= \frac{-b + \sqrt{b^2 + 4c}}{2}$$

which is exactly the form of the positive solution given by the well-known quadratic formula for the solution of the quadratic equation $x^2 + bx - c = 0$.

The negative solution,

$$x = \frac{-b - \sqrt{b^2 + 4c}}{2}$$

which yields $x = -13$ when $b = 10$ and $c = 39$ as above, would not have been recognized as a solution by the author.