Race to Refraction: The Repeated Discovery of Snell's Law

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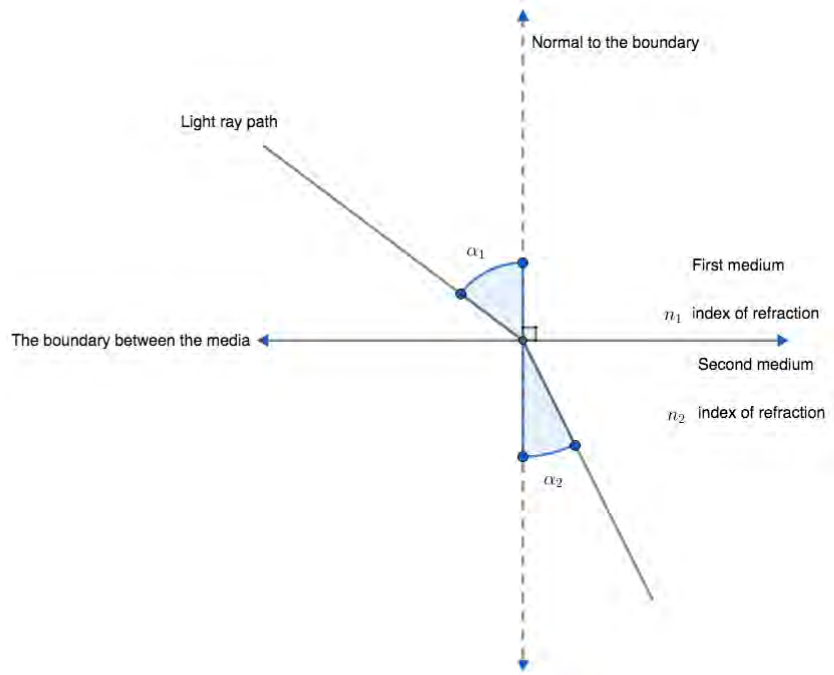
From the year 160 to the 1700's, multiple proofs and experiments were produced to demonstrate the same law of refraction. Considering the boundary between two different materials, such as air and water, light will pass through the first material along a given path, and then bend, known as refraction, once it reaches this boundary. The direction the light will bend depends on the relation between the two media. Although many scholars took different approaches, the foundation of each approach was to show a relationship between the sine of the two angles of refraction and the two media through which the light is passing. Many works and notes on the subject are lost, damaged, or unable to be translated, but the pieces that remain give significant insight as to how the law of refraction can be shown. Over time, the clarity of what is now known as Snell's Law has improved, which can be credited to the work of several mathematicians throughout history.

Today, Snell's Law is written as $\frac{n_1}{n_2} = \frac{\sin \alpha_2}{\sin \alpha_1}$, where α_1, α_2 are the angles of incidence measured from the normal to the boundary between the media, and n_1, n_2 are the indices of refraction of the two media [2].



The indices of refraction will depend on the medium through which light is refracting, and are defined as the ratio of the speed of light in a vacuum, c, to the speed of light, v_i, through the i^{th} medium, meaning each index $n_i = \frac{c}{v_i}$. It is known from experiment that light will bend towards the normal when light enters a denser medium, and will bend away from the normal when light leaves a denser medium [2].

Ptolemy

Ptolemy, living in Alexandria from 100 A.D to around 175 A.D produced ground breaking proofs and experiments in optics around the year 160. Very little is known of Ptolemy's life or the potential influences on his work, though Euclid's *Elements* are likely a direct influence for Ptolemy's use of geometry [7]. It is clear, however, that Ptolemy's work served as a foundation for further research and became a point of criticism for future mathematicians. All comments in square brackets below are mine. Comments in angle brackets are those of the translator. All additional sketches are mine.

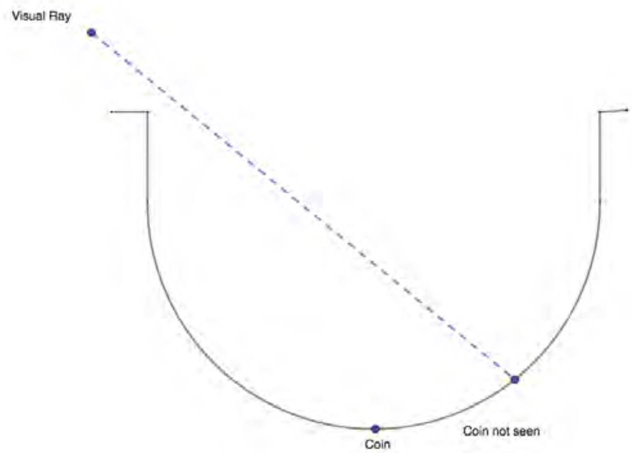
In The Fifth Book of Ptolemy's Optics, Ptolemy states "it has been claimed that this breaking [refraction] does not take place at equal angles; however, the angles <of incidence and refraction> do bear a certain consistent quantitative relation to one another with respect to the normals" [7].

A series of experiments were performed by Ptolemy to obtain the data in Tables V.1-3. Ptolemy describes the set-up of the experiments as follows:

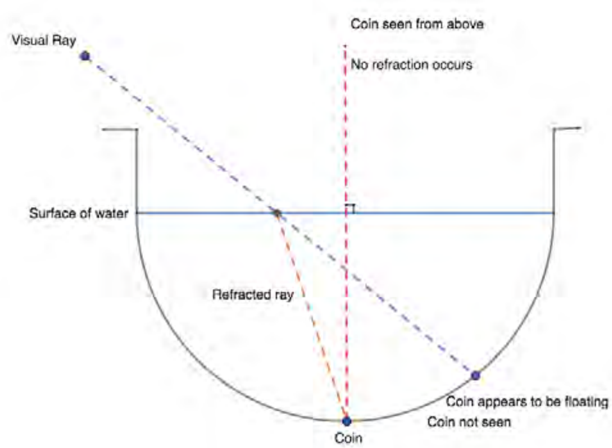
"That this is clear and indubitable we can understand on its own terms by means of a coin that is placed in a vessel called a baptistir.

For, if the eye remains fixed so that the visual ray passing over the lip of the vessel passes above the coin [so that the coin is not seen],

[Figure of baptistir as described by Ptolemy:]



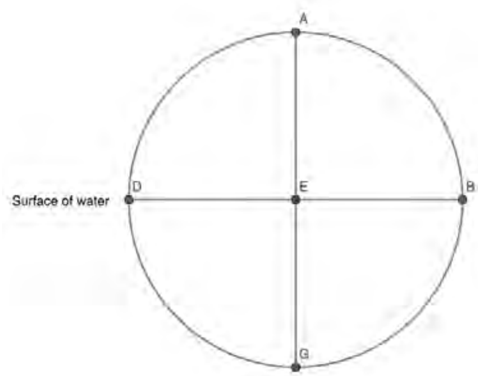
and if water is then poured slowly into the vessel until the ray that passes over the edge of the vessel is refracted toward the interior to fall on the coin, then objects that were invisible before are seen along a straight line extended from the eye to a point higher than the true point <at which the coin lies>.



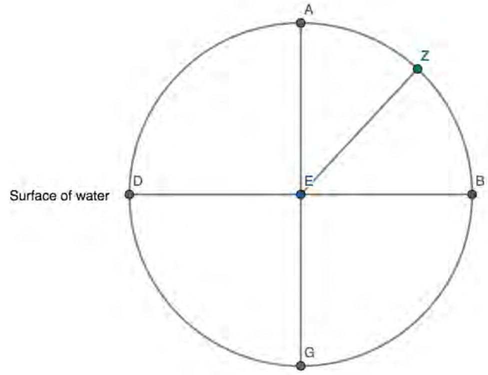
And it will be supposed not that the ray is refracted toward those lower objects but, rather, that the objects themselves are floating and are raised up to <meet> the ray. For this reason, such objects will be seen along the continuation of the <incident> visual ray, as well as [from above] along the normal dropped <from the visible object> to the water's surface-all according to the principles we have previously established."

Ptolemy was able to come to several conclusions from his experiments, which compare different materials for bending light and the different angles of refraction. Again, this is for Ptolemy's Optics, Book V.

<EXPERIMENT V.1>
Let circle *ABGD* <in figure V.2> be described on that [vertical] plaque about centerpoint *E*, and let the two diameters *AEG* and *BED* intersect one another at right angles. Let each of the <resulting> quadrants be divided into 90 equal increments.

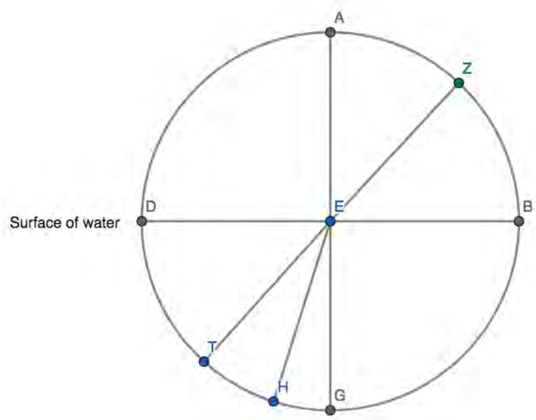


At the centerpoint [*E*] let a small marker of some color or other be attached, and let the plaque be stood upright in the small vessel <discussed in the previous experiment>. Then let a suitable amount of water that is clear enough to be seen through be poured into that vessel, and let the graduated plaque be placed erect at right angles to the surface of the water. Let all of semicircle *BGD* of the plaque, but nothing beyond that, lie under water, so that diameter *AEG* is normal to the water's surface. From point *A*, let a given arc *AZ* be marked off on either of the two quadrants that lie above the water. Furthermore, let a small, colored marker be placed at *Z*.



Now, if we line up both markers at *Z* and *E* along a line of sight from either eye so that they appear to coincide, and if we then move a small, thin peg along the opposite arc *GD* under water until the end of the peg, which lies upon that opposite arc, appears to lie directly in line with the two previous markers, and if we mark off the portion of the arc *GH* that lies between *G* and the point [*T*] at which the object would appear unrefracted, the resulting arc [*GH*] will always turn out to be smaller than *AZ* [arc *GH* < arc *GT* = arc *AZ*]. Moreover, if we join lines *ZE* and *EH*, angle *AEZ* > angle *GEH*, which cannot be the case unless there is refraction -that is, unless ray *ZE* is refracted toward *H* according to the excess of one of the opposite angles over the other.

[Figure V.2]



Furthermore, if we place our line of sight along normal *AE*, we will find the image directly opposite along its rectilinear continuation, which will extend to *G*; and this <radial line> undergoes no refraction.

In the case of all the remaining positions, when arc *AZ* is increased, arc *GH* in turn will be increased, and the refraction will be greater [see table below. Ptolemy gives a list of values, but note that he says “When *AZ* is 40 <degrees>, then *GH* will be 29” which differs from the value 28 in the table].

TABLE V.1

AIR TO WATER	
incidence	*refraction*
10	8.0
20	15.5
30	22.5
40	28.0
50	35.0
60	40.5
70	45.5
80	50.0

Ptolemy repeats the experiments for different combinations of air, water, and glass to obtain the tables of measures of angles of incidence and refraction in Tables V.2 and V.3.

TABLE V.2

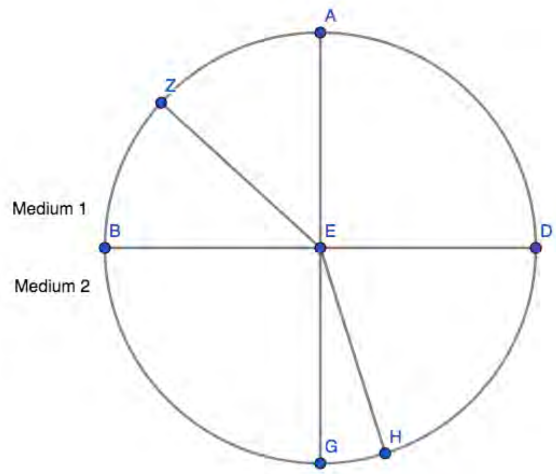
AIR TO GLASS	
incidence	*refraction*
10	7.0
20	13.5
30	19.5
40	25.0
50	30.0
60	34.5
70	38.5
80	42.0

TABLE V.3

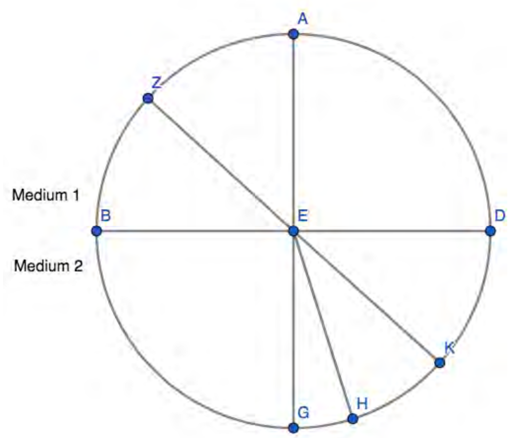
WATER TO GLASS	
incidence	*refraction*
10	9.5
20	18.5
30	27.0
40	35.0
50	42.5
60	49.5
70	56.0
80	62.0

He goes farther to determine a more general relation between the rays, referring to the difference in mediums as rarer and denser (for example air is a rarer medium than water). He concludes in Experiment V.4 that a ray will be refracted either towards the normal when passing from a rarer to denser medium, or away from the normal when viewing from a denser medium into a rarer one.

<EXPERIMENT V.4>
In fact, if we set up the plaque as before and assume that diameter *BD* <in figure V.7> lies on the interface between the two different media, and if we draw normal *AEG* as well as the refracted ray-couple *ZEH* inclined toward the normal, with which it forms angle *GEH*, then the path of refraction remains one and the same.

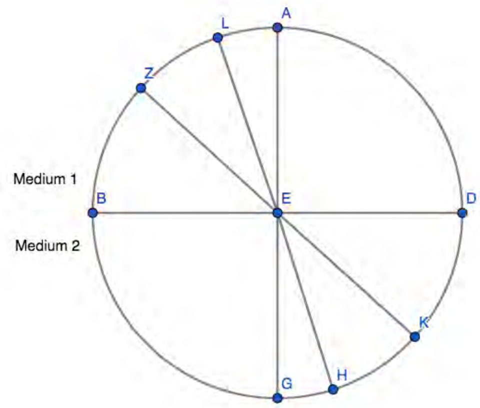


In fact, when the visual ray passes through point *E*, and the eye is stationed at point *Z*, the line <of sight> after refraction-i.e., line *EK* [extended straight from *ZE*] - inclines toward the normal according to its continued passage <along *EH*> while the visible object is seen along the rectilinear continuation <*EK*>.



But if the eye lies at point *H*, and *EZ* lies within the rarer medium delimited by *ABD*, then, after refraction, line *EL* will take an opposite tack from that previously specified, inclining away from normal *AE* <along *EZ*> in such a way that it lies farther out <from the normal> than would the visual ray if it were to continue in a straight line.

[Figure V.7]



Furthermore, when the media and the angles differ from one another by a significant amount, the difference <between the angles of incidence and refraction> increases as the density of either of the media grows.

Indeed, if we assume that [upper semicircle] arc *BAD* lies in the rarer medium and [lower semicircle] arc *BGD* in the denser, and if we take angle *AEZ* as it is represented, then, when the medium within section *BGD* becomes denser than it previously was, the difference between <angle *AEZ* and> angle *GEH* will vary with the difference in density between the two media …

So too, if we suppose that the refraction of another of the visual rays takes place at some other arcal distance <than *AZ*> from normal *AE* e.g., along ray-couple *TEK*-then $AT: AZ > GK: HG$ [meaning $\frac{AT}{AZ} > \frac{GK}{HG}$]. By alternation, $AT: GK > AZ: GH$ [$\frac{AT}{GK} > \frac{AZ}{GH}$]. By separation,

$TZ : AZ > KH: GH.$

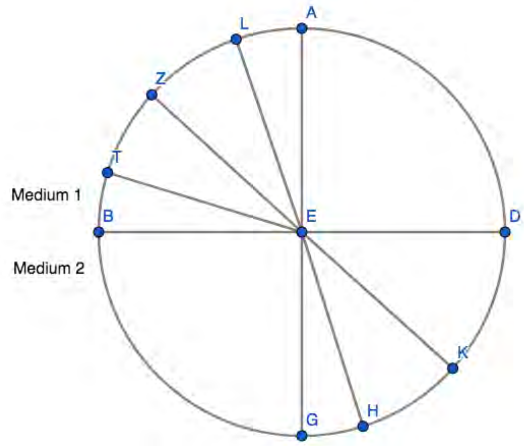
$$[\frac{AT}{AZ} - 1 > \frac{GK}{GH} - 1$$

$$\frac{AT}{AZ} - \frac{AZ}{AZ} > \frac{GK}{GH} - \frac{GH}{GH}$$

$$\frac{AT-AZ}{AZ} > \frac{GK-GH}{GH}$$

$$\frac{TZ}{AZ} > \frac{KH}{GH}]$$

And <by alternation,> $TZ: KH > AZ: GH$ [$\frac{TZ}{KH} > \frac{AZ}{GH}$].



Furthermore, we can determine particular cases on the basis of the refractions as we measured them if we take the resultant numbers and, on their basis, investigate particular measurements of this sort, substituting the numbers so derived for the two arcs *AZ* and *AT*.

Ptolemy's experiments do not yield a universal law, and it is speculated that the inequality he discovered is the best he could have found [7]. So, with the given relationship, as the angle of incidence increases, the ratio increases as well, assuming that angle *AEZ* and *GEH* are constantly proportional.

Ibn Sahl

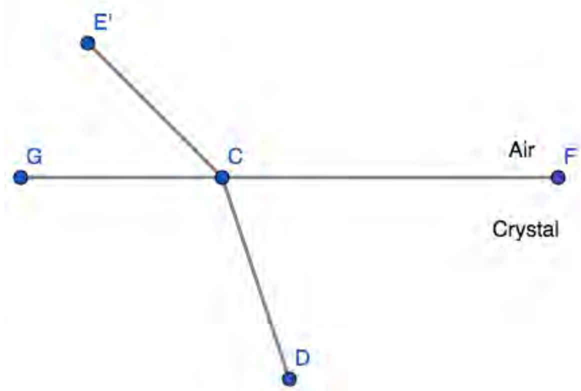
After Ptolemy, the study of optics continued with Ibn Sahl in 984, who knew of Ptolemy's work, but discovered a more universal relationship for refraction using ratios and the sine law [8].

[Ibn Sahl's Demonstration of Refraction]

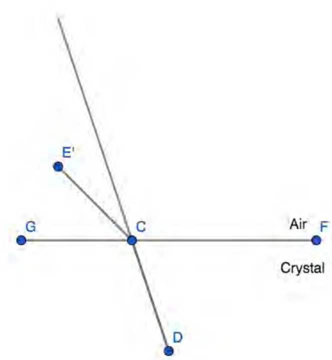
<The hyperbola as a conic section: The law of refraction.
Ibn Sahl first considers refraction on a plane surface. Defining *GF* as the plane surface of a piece of crystal of homogenous transparency, he emphasizes a relation that is the reciprocal of the refractive index n of this crystal in relation to air.>

[Construction]
[Let GF be the surface of a crystal with the air above the crystal.]

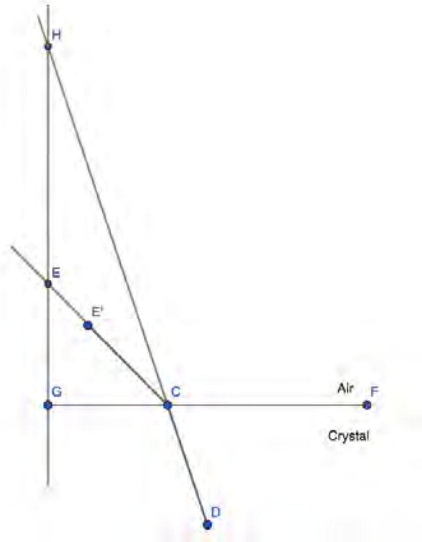
Let *DC* be a light ray in the crystal, which is refracted [at *C*, lying on GF] <Fig. 11; see also Fig. 1> in the air along *CE* [*CE*’].



[Extend line segment *CD* to a ray past point *C*.]



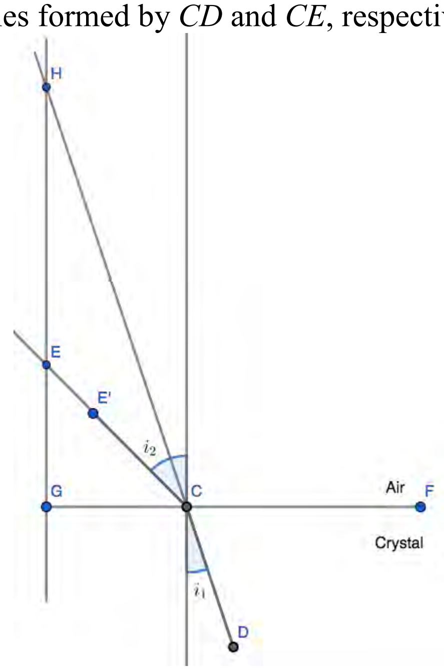
[Construct] the perpendicular [line *GH*] to the plane surface *GF* at *G*[, and] intersect line [ray] *CD* at *H* and the refracted ray at *E*. [Extend line segment *CE*’ so that *E* lies on *GH*.]



<The ratio $\frac{CE}{CH} < 1$, which Ibn Sahl uses throughout his study, is the reciprocal of [the refractive index of the crystal,] n [$\frac{1}{n}$] >

[Construct a line normal to line *GF* from *C*.]

Let i_1 and i_2 be the angles formed by *CD* and *CE*, respectively, with the normal;

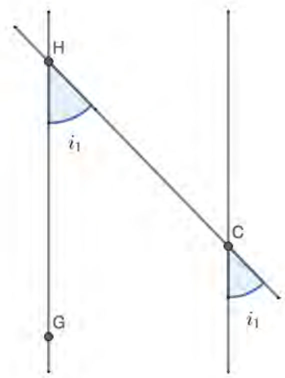


we have

$$\frac{CE}{CH} = \frac{CE}{CG} \cdot \frac{CG}{CH} = \frac{\sin i_1}{\sin i_2} = \frac{1}{n}.$$

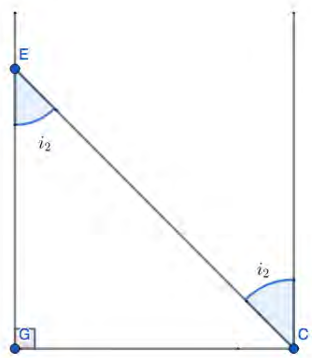
[Observe, $\frac{CG}{CH} = \frac{\text{opposite in } \Delta \text{ CHG}}{\text{hypotenuse } \Delta\text{CHG}} = \sin(\measuredangle\, CHG) = \sin(i_1)$

$\measuredangle\, CHG = i_1$ because CH is a transversal cutting two parallels



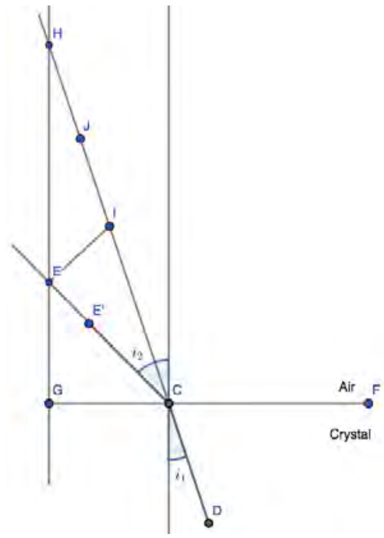
and $\frac{CG}{CE} = \frac{\text{opposite in } \Delta\, CEG}{\text{hypotenuse } \Delta\, CEG} = \sin(\measuredangle\, CEG) = \sin(i_2)$

$\measuredangle\, CEG = i_2$ because both add to $\measuredangle\, ECG$ to give 90°].



[where *n* refers to the index of refraction of the crystal in relation to air]

Let *I* be a point on segment *CH* such that *CI* = *CE*, and let point *J* be the middle [the midpoint] of *IH*. We have $\frac{CI}{CH} = \frac{1}{n}$. Therefore *C*, *I*, *J*, *H* characterize the crystal for any refraction.



[End of proof]

Thomas Harriot

Throughout Thomas Harriot's life (1560 – 1621), he kept manuscripts of new ideas and personal notes on various mathematical topics, but never published them while he was alive. Many still remain unpublished and most are inexplicable to the average reader. An early manuscript, dated July 1601 by Harriot himself, show use of the sine law, which appears to be an attempt to find the relationship of refraction [11]. Harriot's manuscripts show that there were several drafts of his discovery of refraction, where the final draft shows significant alterations marked on the original document. His 1601 statement of refraction is as follows from his final draft:

> The eye and the visible point being in two transparent media, in themselves homogeneous, but between themselves differing in kind, and conjoined at a plane or spherical interface.
>
> Then, given (in position) a single incident ray together with its corresponding refracted ray, to specify, in position, the refracted ray corresponding to any other given incident ray, according to the laws of geometry.
>
> The same being given in numbers (that is, a single angle of incidence and its corresponding angle of refraction), to express, also in numbers, the angle of refraction corresponding to another angle of incidence.

> Another: the line from the eye (on one of those rays) to the visible object (on another) being given in position and magnitude, to find the point of refraction on the common surface at the boundaries of the media. [The final draft adds this clarification to show where the point of refraction is.]
>
> Another: Given two transparent media, in themselves homogeneous, but between themselves differing in kind, and conjoined at a plane or spherical interface; and given also two straight line[s] containing an angle, the complement of which with respect to two right angles does not exceed the greatest angle of refraction, to place geometrically the common surface of the media at the common intersection <of the two lines>, such that the eye, placed on one of the given lines, will see the other, along the same straight line.

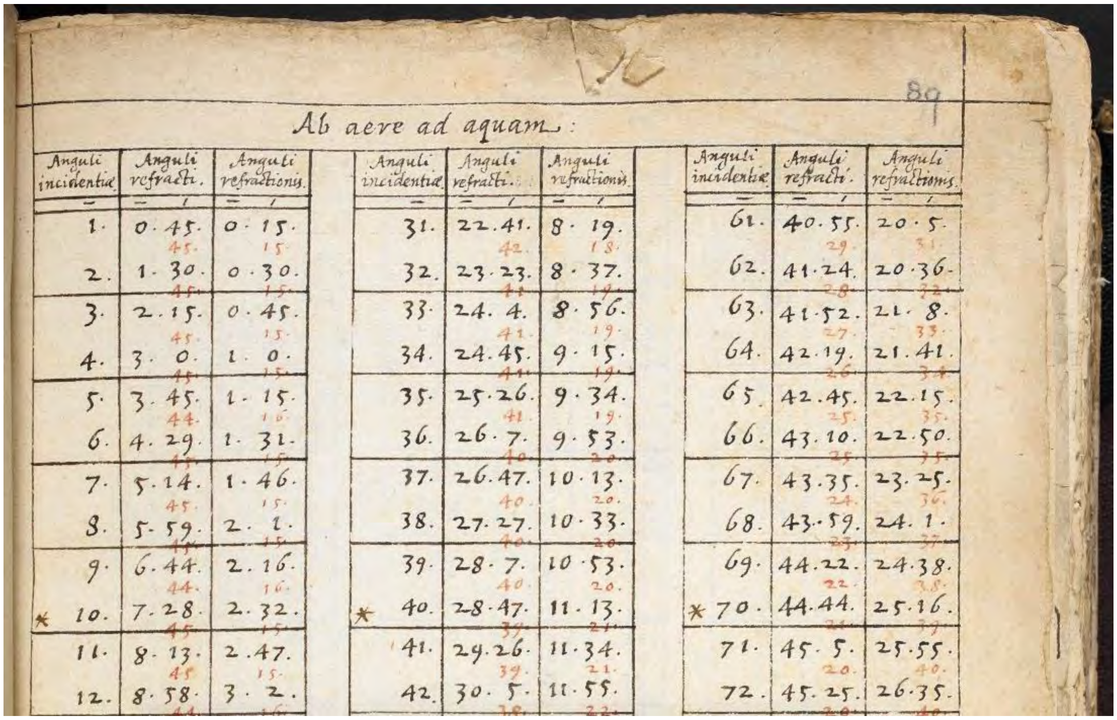
From his first draft, Harriot notices and comments that "the angles of refraction from water to crystal, or vice versa, smaller than those from air to water, when (according to the theory of weakness) the refraction ought to be larger."

Harriot concludes in a corollary a statement popular and consistent with the arguments proposed by previous mathematicians.

> Corollary: A ray, perpendicularly incident on a denser medium, is contracted [bent towards the normal]; on a rarer medium, is extended [bent away from the normal].

While Harriot does not expound, legibly, on the idea of different media, it is clear that he observed such differences as researchers before him. Harriot recorded his angles of incidence and refraction in a page of tables.

[First entries in the tables of incident angles, refracted angles, and the difference between incident and refractive angles, from air to water]



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Ab aere ad aquam:

Anguli incidentiæ	Anguli refracti.	Anguli refractionis.	Anguli incidentiæ	Anguli refracti.	Anguli refractionis.	Anguli incidentiæ	Anguli refracti.	Anguli refractionis.
°	° ′	° ′	°	° ′	° ′	°	° ′	° ′
1.	0. 45.	0. 15.	31.	22. 41.	8. 19.	61.	40. 55.	20. 5.
2.	1. 30.	0. 30.	32.	23. 23.	8. 37.	62.	41. 24.	20. 36.
3.	2. 15.	0. 45.	33.	24. 4.	8. 56.	63.	41. 52.	21. 8.
4.	3. 0.	1. 0.	34.	24. 45.	9. 15.	64.	42. 19.	21. 41.
5.	3. 45.	1. 15.	35.	25. 26.	9. 34.	65.	42. 45.	22. 15.
6.	4. 29.	1. 31.	36.	26. 7.	9. 53.	66.	43. 10.	22. 50.
7.	5. 14.	1. 46.	37.	26. 47.	10. 13.	67.	43. 35.	23. 25.
8.	5. 59.	2. 1.	38.	27. 27.	10. 33.	68.	43. 59.	24. 1.
9.	6. 44.	2. 16.	39.	28. 7.	10. 53.	69.	44. 22.	24. 38.
* 10.	7. 28.	2. 32.	* 40.	28. 47.	11. 13.	* 70.	44. 44.	25. 16.
11.	8. 13.	2. 47.	41.	29. 26.	11. 34.	71.	45. 5.	25. 55.
12.	8. 58.	3. 2.	42.	30. 5.	11. 55.	72.	45. 25.	26. 35.

Willebrord Snellius

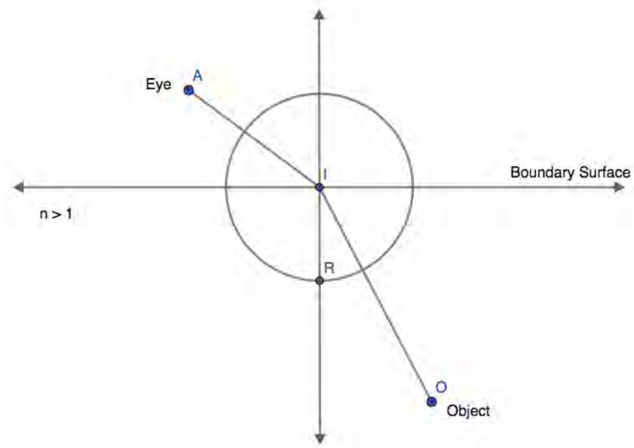
The work of Willebrord Snellius was not published during his lifetime, however the law of refraction is termed Snell's Law, suggesting that Snellius was the one to discover it. It is clear that several mathematicians had a similar idea prior to the 1621 discovery by Snell. The original statement by Snell reads in translation:

> The real radius has to the apparent radius the same proportion in one and the same different medium. The secant of the complementary angle of the inclination in the rarer medium has the same ratio to the secant <of the complementary angle> of the broken <radius> in the denser medium, as the apparent radius has to the true or incident radius [1, 104-105].

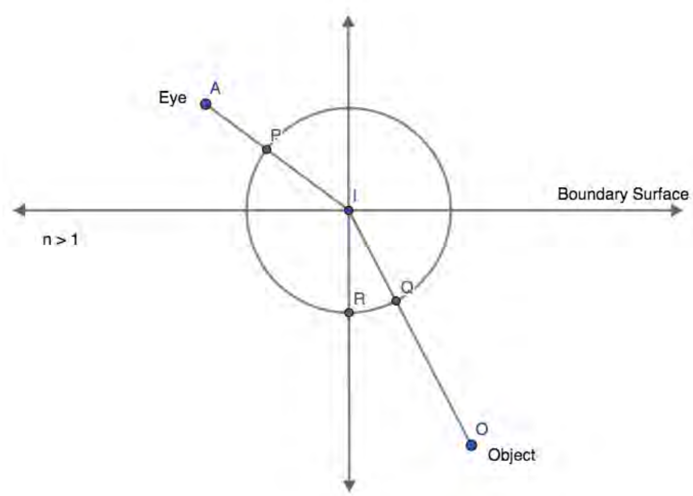
This statement corresponds to a lemma using the construction of circles to determine the now known value for the index of refraction of the denser medium.

> <Reconstruction of the lemma [statement] before prop. 34 of Book I [of Snell] (with denser medium below).>

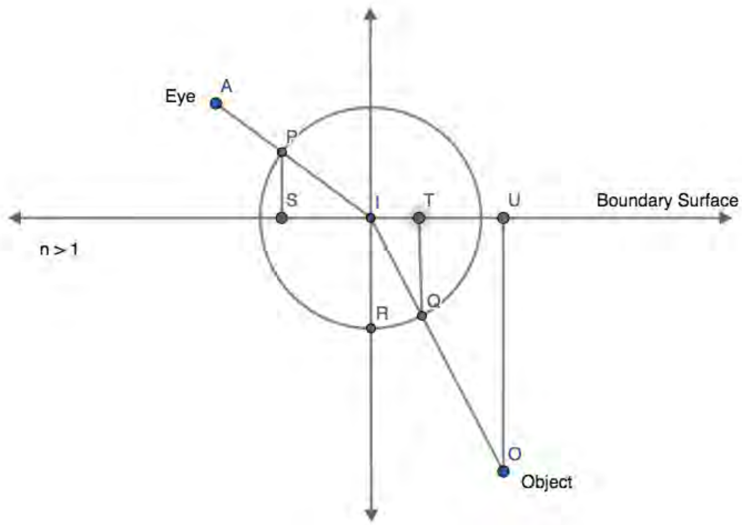
If eye point *A*, incidence point on boundary surface *I* and object point *O* are given, then draw a circle with radius *R* smaller than the distances between the three points around the incidence point.

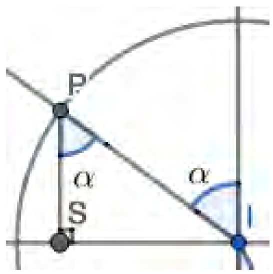


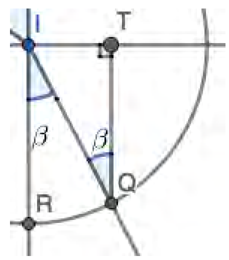
The connecting lines between *A* and *I* or *O* and *I* produce two points of intersection *P* and *Q* with the circle.



From each of these [*P* and *Q*], the [line] perpendicular is dropped onto the boundary surface, and the distances *SI* and *IT* obtained then behave in the same way as the given ratio [$\frac{\sin\alpha}{\sin\beta}$] <in modern notation> [3].

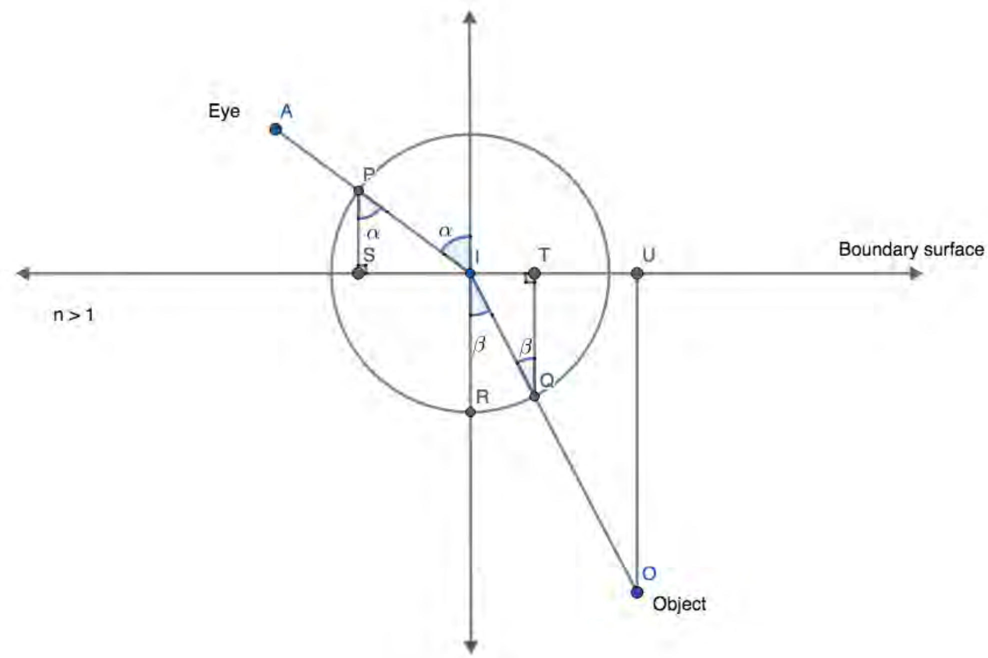






[$PI = IQ$ = radius of circle.

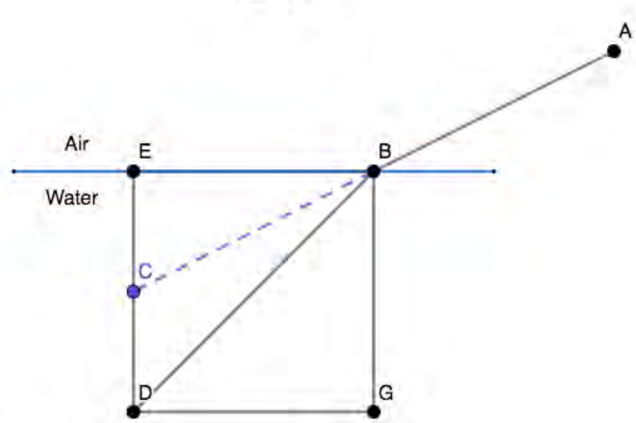
So, $\frac{\sin\alpha}{\sin\beta} = \frac{\left(\frac{SI}{PI}\right)}{\left(\frac{IT}{IQ}\right)} = \frac{SI}{IT}$.]



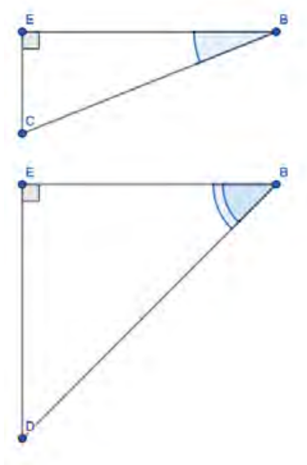
[End of lemma statement]

Snell defined angle of incidence and angle of refraction with a series of relationships corresponding to geometric drawings. Figure 8 shows the ray *AB* in air and the observed ray *BD* in water. A modern interpolation by scholars of sine tables shows $n = \frac{4}{3}$, which is the accepted index for water, even though Snell did not denote specific indices of refraction, like those used today [3].

Figure 8



Observe the triangles *EBC* and *EBD*



Snell used the relationship

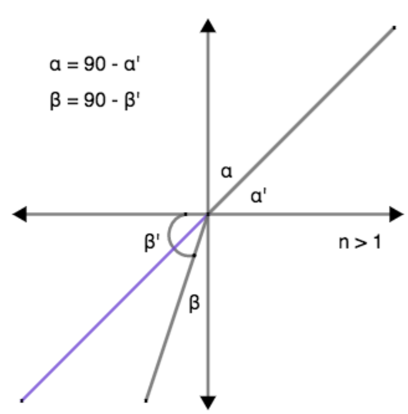
$$\frac{\text{length of apparent ray}}{\text{length of true ray}} = \frac{CB}{DB} = \frac{\frac{CB}{EB}}{\frac{DB}{EB}} = \frac{\sec(\text{angle } EBC)}{\sec(\text{angle } EBD)}$$ to determine a general relationship

for the angles of the apparent and true ray.

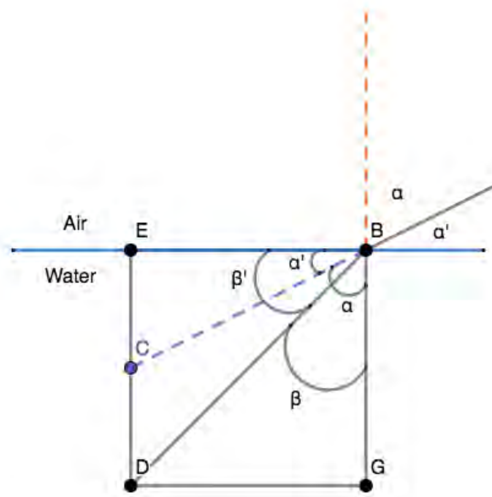
Modern Demonstration from Snell's Given Figures and Ratios:

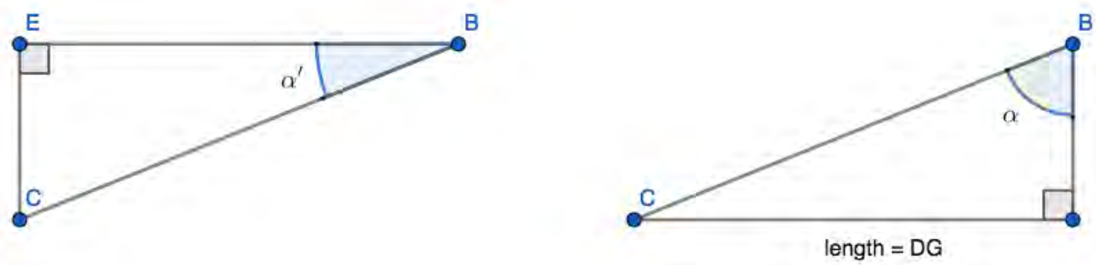
Snell gives the following designations for angles $\alpha, \alpha', \beta, \beta'$ where

$\alpha = 90^{\circ} - \alpha'$ and $\beta = 90^{\circ} - \beta'$.



So, figure 8 becomes:





So, $\frac{CB}{DB} = \frac{CB}{EB} \cdot \frac{EB}{DB} = \frac{1}{\cos\alpha'} \cdot \frac{\cos\beta'}{1} = \frac{\sec\alpha\prime}{\sec\beta\prime}$ as claimed by Snell, where angle $EBC = \alpha'$, and angle $EBD = \beta'$.

Observe, that by the cofunction theorem in right triangles,

$$\sin\alpha = \cos\alpha'.$$

Similarly, $\sin\beta = \cos\beta'$.

Then, $\frac{1}{\cos\alpha'} \cdot \frac{\cos\beta'}{1} = \frac{\sin\beta}{\sin\alpha} = n.$

However, Snell mistakenly concludes that

$$\frac{\sec(\text{angle } EBC)}{\sec(\text{angle } EBD)} = \frac{\sec\alpha\prime}{\sec\beta\prime} = \frac{1}{\cos\alpha\prime} \cdot \frac{\cos\beta'}{1} = \frac{\sin\alpha}{\sin\beta} = n.$$

Snell is showing that $\cos\alpha' = \sin\beta$, however, it has just been shown that $\sin\alpha = \cos\alpha'$.

From the lemma, it must be that $\frac{\sin\beta}{\sin\alpha} = \frac{1}{n}$ since $\frac{\sin\alpha}{\sin\beta} = n$, meaning that the index of air is 1 and Snell omitted the rearrangement of the fraction. If not, Snell is incorrect in his simplification as stated.

The Influences of Descartes, Fermat, and Huygens

Following Snell, Descartes began to research sine law of refraction and first assumed that light acts as a projectile, with his specific example of a tennis ball. Descartes assumes that light is being refracted into a medium 1.5 times as dense as air, and using geometry, concludes that a quotient of sines is equivalent to 1.5, which he specifies is the index of refraction [9, 393-395]. Claims have been made that Descartes obtained information from Snell, and Snell's law is often referred to as the Snell-Descartes Law, even though no accusations against Descartes have been confirmed.

Pierre de Fermat and Christiaan Huygens were both quick to criticize Descartes's work, arguing that light does not travel instantaneously as Descartes suggested. Fermat's principle of least time was shown as light traveling along the incident and refracted rays will take more time than the hypotenuse of the triangle made by the incident and refracted ray. Geometrically, it is clear that the sum of the lengths of two sides of a triangle is longer, and thus would take more time to travel along the longer path, than the length of the third side, as shown in Euclid. This proposition was also favored by Huygens, who was skeptical of Descartes' work, criticizing that:

> "For it has always seemed to me, that even Mr. Des Cartes, whose aim has been to treat all the subjects of natural philosophy intelligibly, and who assuredly has succeeded in this much better than anyone before him, has said nothing that is not full of difficulties, or even inconceivable, in dealing with Light and its properties." [9, p. 397]

Such discrepancies led Huygens to prove for himself the laws of refraction.

Isaac Newton

Around 1664, Newton began to investigate optics, and specifically the law of refraction. Shortly after, Newton began to read the works of Robert Hooke and Descartes. Hooke had recently published an experiment using liquor, and specific investigations for color where he concluded

> "*By this I have also found that look what* proportion *the* Sine *of the Angle of* one Inclination *has to the* Sine *of the Angle of* Refraction*, correspondent to it, the same* proportion *have all the* Sines *of other Inclinations to the* Sines *of their appropriate Refractions"* [5].

It is apparent that the inclusion of color by Newton in the following axioms follows from research initiated by Hooke. Following the publication of Hooke's experiment, Newton published an essay on refraction where he specifically credits Descartes's work, which also included color. Newton ultimately negates Descartes, but was undoubtedly influenced by his work [10]. The critical nature of Newton's evaluation of Descartes and Hooke led to the discovery of his axioms [4].

[Newton's Axioms]

AX. I.

The Angles of Reflexion and Refraction, lie in one and the same Plane with the Angle of Incidence.

AX. V. [Really, a theorem]

[Claim]

The Sine of Incidence is either accurately or very nearly in a given Ratio to the Sine of Refraction.

[Restatement of Claim]

Whence if that Proportion be known in any one Inclination of the incident Ray, 'tis known in all the Inclinations, and thereby the Refraction in all cases of Incidence on the same refracting Body may be determined.

Thus if the Refraction be made out of Air into Water, the Sine of Incidence of the red Light is to the Sine of its Refraction as 4 to 3. If out of Air into Glass, the Sines are as 17 to 11.

In Light of other Colours the Sines have other Proportions: but the difference is so little that it need seldom be considered.

[Original Fig. 1]

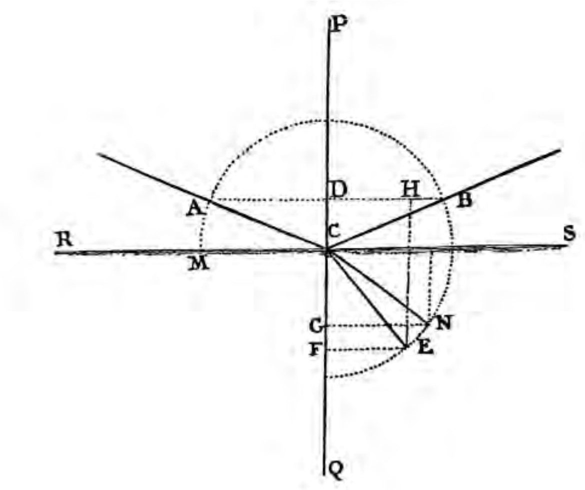
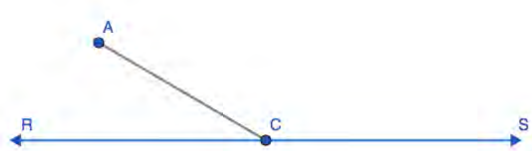


Fig. 1

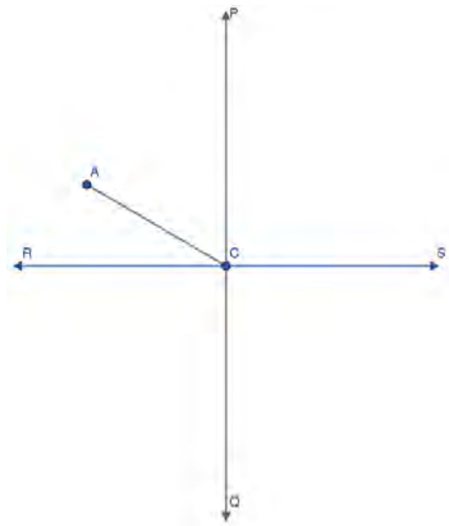
[Construction]

Suppose therefore, that *RS* <in *Fig.* 1.> represents the Surface of stagnating Water,

and that *C* is the point of Incidence in which any Ray coming in the Air from *A* in the Line *AC* is reflected or refracted, and I would know whither this Ray shall go after Reflexion or Refraction:



I erect upon the Surface of the Water from the point of Incidence the Perpendicular *CP* and produce it downwards to *Q*,

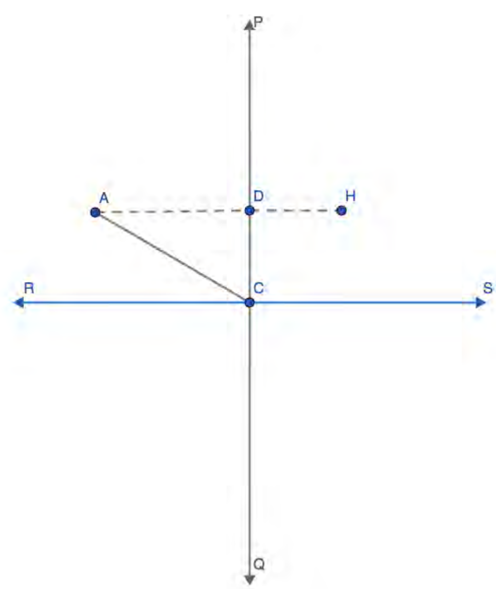


and conclude by the first Axiom, that the Ray after Reflexion and Refraction, shall be found somewhere in the Plane of the Angle of Incidence *ACP* produced, I let fall therefore upon the Perpendicular *CP* the Sine of Incidence *AD* [assuming *AC* = 1]… [information on reflection is omitted]

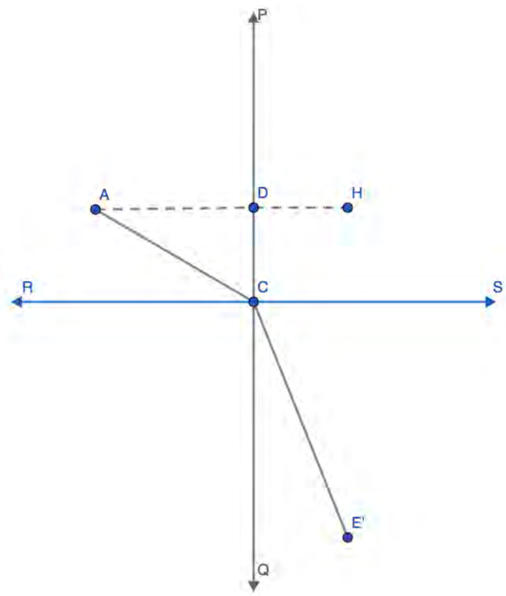
[For refraction]

…But if the refracted Ray be desired, I produce *AD* [from *A*, parallel to *RS*, where *D* is the point *AD* intersects *PQ*,] to *H,* so that *DH* may be to *AD* as the Sine of Refraction to the Sine of Incidence, that is, (if the Light be red) as 3 to 4

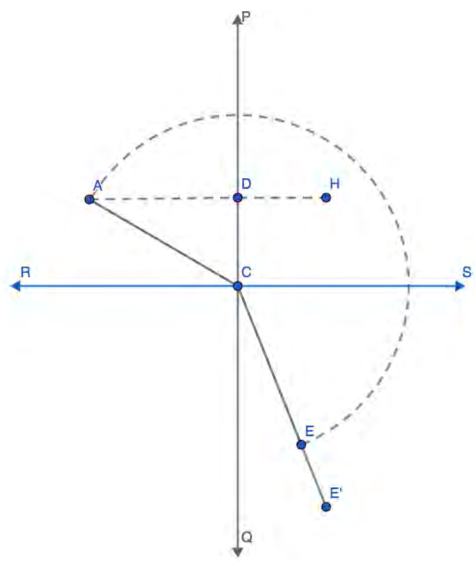
$$[\frac{DH}{AD} = \frac{Sine\ of\ Refraction}{Sine\ of\ Incidence} = \frac{3}{4}];$$



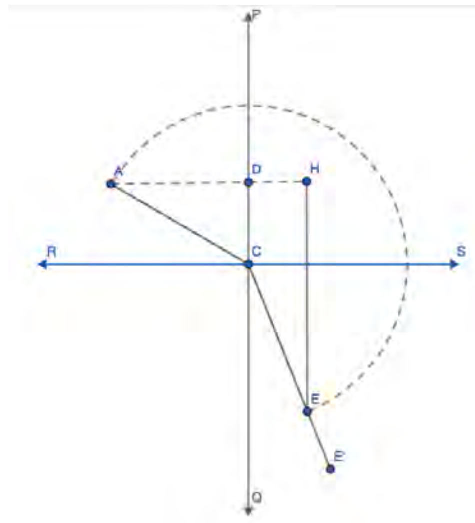
[construct the refracted ray *CE*']



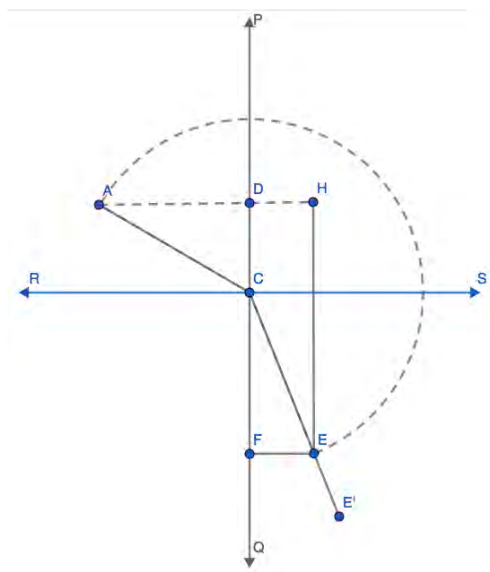
and about the Center *C* and in the Plane *ACP* with the Radius *CA* describing a Circle *ABE* [where *E* is the point that *CE*' intersects the circle, and B is unspecified yet],



I draw a parallel to the Perpendicular *CPQ*, the Line *HE* cutting the Circumference in *E*, and joining *CE*, this Line *CE* shall be the Line of the refracted Ray.



For if *EF* be let fall perpendicularly on the Line *PQ* [where *F* is the point at which *EF* intersects *PQ*],



this Line *EF* shall be the Sine of Refraction of the Ray *CE*, the Angle of Refraction being *ECQ*; and this Sine *EF* is equal to *DH* [*EF* = *DH*], and consequently in Proportion to the Sine of Incidence *AD* as 3 to 4.

[Meaning, where the first equality is from above, $\frac{3}{4} = \frac{DH}{AD} = \frac{EF}{AD} = \frac{\sin(\measuredangle ECQ)}{\sin(\measuredangle ACD)}$.]

[End of proof]

Comparison of Values from Air to Water

Angle of Incidence	Angle of Refraction - Ptolemy	Value for n - Ptolemy	Angle of Refraction - Harriot	Value for n - Harriot	Modern Angle of Refraction (Using n = 1.33)
10	8	1.25	7.28	1.37	7.48
30	22.5	1.31	22	1.33	22.02
60	40.5	1.33	40.26	1.34	40.51

Comparison of the Indices of Refraction

Index of Refraction Stated	Water	Glass	Air
Snell	$\frac{4}{3}$	n/a	1
Newton	$\frac{4}{3}$	$\frac{17}{11}$	1
Modern	$\frac{4}{3}$	$\frac{3}{2}$	1

Overall, each proof or experiment over time shows that there is a relationship between the angle of incidence, the angle of refraction, and the ratio of the two different mediums through which light in passing. Since many works were not published and readily available, it is possible that the sine law of refraction, specifically known as Snell’s Law, can be honestly attributed to several mathematicians. It has been shown that several results on the law have been produced, both with and without influence of previous research, and ultimately conclude the same result.

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