Peter N. Oliver

Pierre Varignon and the Parallelogram Theorem

Students of geometry are delighted to discover that a parallelogram is formed when the midpoints of the sides of a convex quadrilateral are joined in order. This intriguing figure, which is easily defined, is known to geometers as the *Varignon parallelogram* because Pierre Varignon (1654–1722), an important French academician of the era of Newton and Leibniz, furnished the first rigorous proof that a parallelogram is formed. Varignon's proof was published in 1731 in *Elemens de Mathematique,* a posthumously published volume of mathematical concepts compiled from Varignon's notes for secondary-level instruction.

The title given to the Varignon collection appreciatively echoes the title of Euclid's *Elements*. Varignon was instrumental in establishing the central place of mathematics, grounded in the thirteen volumes of Euclid, in the schools of western Europe. In addition, he became an influential proponent of the innovations in physics—principally mechanics—and calculus that were being developed by his contemporaries.

Varignon was instrumental in establishing the central place of mathematics in schools of western Europe

These technical interests may have originated in his youth, because he grew up in a family of contracting masons in Caen on the Normandy coast. Reflecting on this heritage, one biographer noted, "Varignon stated that his entire patrimony consisted of his family's technical knowledge; it proved, however, to be of considerable importance for his later career" (Costabel 1976, p. 584).

Varignon did not initially pursue a career in the mathematical sciences. At a relatively late age, he undertook a Jesuit program of study to prepare for the priesthood. He earned his master of arts degree in September 1682; in March 1683, he was assigned to be a priest in the Saint Ouen parish in Caen. As an ecclesiastic, he received more advanced instruction at the University of Caen. When Varignon by chance encountered Euclid's *Elements*, his career diverged toward that of a mathematician. In the Jesuit scholastic tradition, he devoted the remainder of his life to teaching and research in the mathematical sciences.

Varignon began his scientific research while in Caen. In 1686 he relocated to Paris. The following year, he published an article on tackle blocks for pulleys and a book on the composition of forces. Also in 1686, Varignon was appointed to the newly created professorship of mathematics at the Collège de Mazarin in Paris, where he taught and resided for the remaining thirty-six years of his life.

In 1688, Varignon was named geometer to the French Académie Royal des Sciences. Starting with the founding of the Academia Secretorum Naturae in Naples around 1650, scientific associations—usually designated as *academies* or *societies*—were formed throughout Europe by scholars interested in experimental science. The scientific associations were often supported by the patronage of reigning monarchs. Science and mathematics had not been important components of the traditional scholastic training that had come down from the Middle Ages.

In 1704, Varignon was appointed to the chair of Greek and Latin philosophy at the Collège Royal, a position that he held concurrently with his position at Collège de Mazarin. Although Varignon was fully occupied as an educator and had little opportunity for research or writing, he made significant contributions to the advances in mathematics and physical science that occurred during his lifetime. His principal contribution came as an instructor, teaching the mathematical heritage of the Classical

This article is the first of two articles about Pierre Varignon and his parallelogram theorem. The second article, to appear in a future issue of the *Mathematics Teacher,* discusses the "Consequences of the Varignon Parallelogram Theorem."

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world and Arabic civilization, as well as the new techniques of calculus and physics. He was one of the first scholars in France to apply the calculus of Leibniz to problems of Newtonian dynamics. At the time of his death, he was planning an introductory course in calculus for presentation at the Collège Royal in 1722–1723. A posthumous volume of Varignon's materials, *Eclaircissements sur l'analyse des infiniment petits,* published in 1725, contained his notes for teaching the first textbook on differential calculus to appear in print, *Analyse des infiniment petits,* written by the Marquis de l'Hospital (1661–1704) and published in Paris in 1696 (Costabel 1976, p. 586).

In addition to his pedagogical work, Varignon participated directly in the joint development of calculus and modern empirical physics. Much of his insight into these new fields of inquiry seems to have arisen from the broad, generalizing perspective that he had gained as a classroom teacher. In his teaching and writing, he brought greater coherence to the mechanics component of physics, that is, the science of simple machines, by improving its organization and notation. He is also credited with several specific discoveries relating to the composition of forces and with the Varignon theorem of physics, which establishes a procedure for combining vectors of force. See Rumyantsev (1993, p. 398). In correspondence with Johann Bernoulli (1667–1748), Varignon contributed to the origin of the concept of virtual velocity. At the beginning of the eighteenth century, when traditional mathematicians of the Académie Royal des Sciences, led by Michel Rolle (1652–1719), questioned the validity of calculus, Varignon convinced them of the soundness of the new methods.

As Varignon sought to conciliate this rearguard of traditionalists, he wrote to Leibniz in 1701 for greater elaboration of differentials. In reply, Leibniz encouraged Varignon to start developing mechanical applications for calculus. In 1699, Varignon had published a paper on water clocks, in which he applied differential calculus to the flow of a fluid through an orifice; in 1702, he published a paper that applied calculus to the design of spring-driven clocks. In a 1704 paper, Varignon "continued and extended Jacques [Jakob] Bernoulli's use of polar coordinates, including an elaborate classification of spirals obtained from algebraic curves, such as the parabolas and hyperbolas of Fermat, by interpreting the ordinate as a radius vector and abscissa as a vectorial arc" (Boyer 1968, p. 475).

Late in life, Varignon developed a cordial correspondence with Newton, beginning with a November 1713 letter in which Varignon thanked Newton for a gift copy of the second edition of the *Principia*. After being nominated by Newton, Varignon was elected to the Royal Society in July 1714. For the remainder of Varignon's life, he and Newton sent each other books on scientific developments, and the two aging mathematicians exchanged portraits. Varignon became the directeur, or chief scientific officer, of the Académie Royal des Sciences in 1719. From this position, in the final two years of his life, he oversaw publication of the second, greatly revised, French edition of Newton's *Opticks.*

Varignon's participation in the heady intellectual ferment of his time through his academic pursuits helped place his students and colleagues at the forefront of these developments.

Varignon's insight arose in part from the broad perspective that he gained in the classroom

THE PARALLELOGRAM THEOREM

Varignon's statement and proof of the parallelogram theorem were published nine years after his death in a secondary-level textbook prepared from his notes by former colleagues. The volume, *Elemens de Mathematique de Monsieur Varignon* (Varignon [1731] 1972), is organized in two parts, the first part explaining concepts of arithmetic and elementary algebra and the larger second part covering topics in Euclidean geometry. The two sections have separately numbered pages, with 66 pages on arithmetic and algebra and 155 pages on plane and solid geometry. In addition, to illustrate geometric concepts, the book has an appendix of twenty-two tables of engravings. **Figure 1** shows the title page of the book. Table 8 of *Elemens de Mathematique,* which contains the diagram for the parallelogram theorem, is shown in **figure 2.**

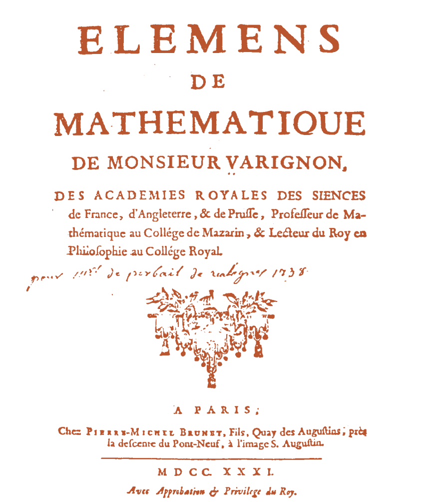
ELEMENS
DE
MATHEMATIQUE
DE MONSIEUR VARIGNON,
DES ACADEMIES ROYALES DES SIENCES
de France, d'Angleterre, & de Prusse, Professeur de Mathématique au Collége de Mazarin, & Lecteur du Roy en Philosophie au Collége Royal.
A PARIS;
Chez PIERRE-MICHEL BRUNET, Fils, Quay des Augustins, près la descente du Pont-Neuf, à l'image S. Augustin.
M DCC. XXXI.
Avec Approbation & Privilege du Roy.

Fig. 1
Title page of *Elemens de Mathematique*

From the History of Science Collections, University of Oklahoma

The theorem about the Varignon parallelogram may be stated as follows:

THEOREM. *The figure formed when the midpoints of the sides of a quadrilateral are joined in order is a parallelogram.*

Varignon's proof conforms to the logical rigor and formality that have long been demanded of geometry students

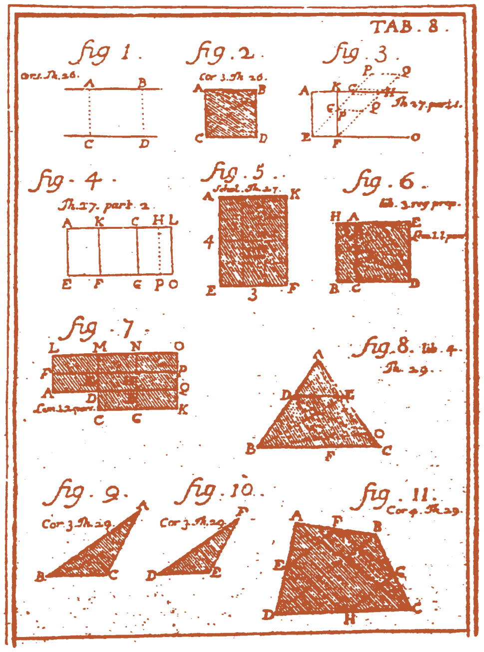

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Fig. 2
Table 8 from
Elemens de Mathematique de Monsieur Varignon;
figure 11 in this table illustrates
Varignon's parallelogram theorem.

As originally published in 1731, the theorem was presented as a corollary to the theorem that states that if a line intersects two sides of a triangle and separates the sides into segments of proportional lengths, then the line is parallel to the third side. **Figure 3** shows the statement and proof of the Varignon parallelogram theorem, as published in *Elemens de Mathematique*.

Varignon's proof may be translated from its original French as follows:

If sides *AB, BC, CD,* and *DA* of a quadrilateral are each divided into two equal parts by points *F, G, H,* and *E,* such that the midpoints are joined by segments *FE, EH, HG,* and *GF,* the quadrilateral figure *FEGH* is a parallelogram; this may be shown by the extension of diagonals *DB* and *AC,* since by hypothesis, *AF* = *FB* and *AE* = *ED, AF* · *FB* :: *AE* · *ED,* and thus (by Part. 2) *EF* is parallel to *DB*. In like manner, by hypothesis, *BG* = *GC* and *DH* = *HC; BG* · *GC* :: *DH* · *HC,* and consequently (by Part. 2) *GH* will also be parallel to line *BD.* Since *EF* and *GH* are parallel to the same third line, they are therefore parallel between themselves.

By the same reasoning, one could prove that the lines *FG* and *EH* are parallel to the diagonal *AC,* and as a consequence, are parallel between themselves. Thus, the quadrilateral *EFGH* is a parallelogram.

COROLLAIRE IV.

T.b. 8. Fig. 11. Si les côtés *AB*, *BC*, *CD*, *DA* d'une figure rectiligne de quatre côtés, ſont diviſés chacun en deux parties égales en *F*, *G*, *H*, *E*, & que les points des diviſions ſoient joints par les lignes droites *FE*, *EH*, *HG*, *GF*, la figure quadrilaterale *FEHG* eſt un parallélograme; car en menant les lignes *DB*, & *AC*, comme par l'hypothéſe, *AF* = *FB* & *AE* = *ED*, *AF*· *FB* : : *AE*· *ED*, & ainſi (*Part.* 2.) *EF* eſt paralléle à *DB*. De même puiſque, par l'hypothéſe, *BG* = *GC*, & *DH* = *HC*; *BG*· *GC* : : *DH*· *HC*, & par conſéquent (*Part.* 2.) *GH* ſera encore paralléle à la ligne *BD*. Donc *EF* & *GH* ſont paralléles à la même troiſiéme ligne, elles ſont donc auſſi paralléles entre elles.

On peut par la même raiſon prouver que les lignes *FG* & *EH* ſont paralléles à la ligne droite *AC*, & par conſéquent paralléles entre elles. Donc le quadrilatére *EFGH* eſt un parallélograme.

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Fig. 3
Varignon's proof of the parallelogram theorem,
as given in *Elemens de Mathematique*

The notation of the expressions "*AF* · *FB* :: *AE* · *ED*" and "*BG* · *GC* :: *DH* · *HC*" represents proportionality between the lengths of the segments. The notation could also be used to show proportional numbers or analogous relationships. The first proportion would be read "*AF* is to *FB* as *AE* is to *ED*." Such proportions are now commonly represented as two equivalent fractions.

At two places in the proof, Varignon refers to "Part. 2" as a reason that lines are parallel. In Varignon's arrangement of the Euclidean geometric propositions, part. 2 of theorem 29 is the converse of the theorem that states that a line parallel to one side of a triangle cuts the other two sides proportionally, that is, Euclid, VI.2. Translated from Varignon's text, the converse may be stated as "the line that cuts a triangle proportionally is parallel to its base" (Varignon [1731] 1972, p. 60).

Varignon's proof conforms to the logical rigor and formality that have long been demanded of geometry students. He begins with the given assertion that points *E, F, G,* and *H,* as shown in **figure 4,** are midpoints of the sides of a general quadrilateral, *ABCD*. He next draws diagonal *DB*. Examining $\triangle ADB$, one of the two triangles formed by the diagonal, he shows that the two equal divisions of segment *AB* are proportional to the two equal divisions of segment *AD.* The equal segments of each side would have a one-to-one ratio. From this proportionality, Varignon deduces on the basis of Euclid, VI.2, that $\overline{EF}$ is parallel to $\overline{DB}$. Turning his attention to $\triangle CBD$, he employs the same logical sequence of reasons to show that $\overline{GH}$ is also parallel to $\overline{DB}$. Since $\overline{EF}$ and $\overline{GH}$ are both parallel to $\overline{DB}$, it is proved that they are parallel to each other, since by theorem (Euclid, I.30), two distinct lines that are parallel to the same line are parallel to each other.

In the second paragraph of the proof, Varignon points out that the same line of reasoning would show that $\overline{FG}$ and $\overline{EH}$, both being parallel to diagonal *AC,* are parallel to each other. Having demonstrated that the opposite sides of quadrilateral *EFGH* are parallel, he concludes that the figure

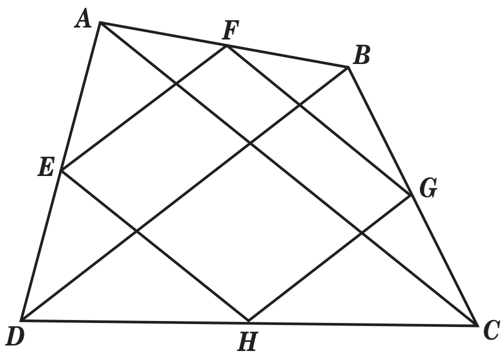

Fig. 4
Quadrilateral *ABCD*

meets the definition of a parallelogram (Varignon [1731] 1972, pp. 62–63).

When the Varignon parallelogram appears as a practice exercise in present-day geometry textbooks, students are sometimes asked to show that its opposite sides are congruent, each side having half the length of the diagonal to which it is parallel. However interesting these pairs of congruences may be, they are not essential for establishing that the figure is a parallelogram.

The parallelogram theorem gives evidence of the lucidity and craftsmanship of the gifted teacher for whom it is named.

A future article in the *Mathematics Teacher* will examine some implications of Varignon's theorem and suggest exercises for exploratory inquiry and deduction.

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Geometric Approaches to Quadratic Equations from Other Times and Places

(Continued from page 313)

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