Poles, Parking Lots, & Mount Piton:

Classroom Activities that Combine Astronomy, History, and Mathematics

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How big is the earth? How far away is the moon? How big is the moon? How tall are the mountains on the moon? These wonderfully naïve questions have been asked for centuries by children and astronomers alike. A great bonus for mathematics and science teachers is that with basic concepts from the high school curriculum and data that students can collect themselves, these questions can be answered to a high degree of accuracy.

BACKGROUND

Eratosthenes (276–194 BC), born in present-day Libya and educated in Athens, became head librarian at Alexandria in 240 BC. Also around that time, using basic geometry and right angle trigonometry, he made a surprisingly accurate measurement of the circumference of the earth, the equivalent of 39,300 km—only 1.8 percent below today's accepted value of 40,027 km at the equator (Dunham 1990; O'Connor and Robertson 1999).

Eratosthenes knew that the sun is so far away that its rays arriving at Earth are nearly parallel (**fig. 1**). This realization allowed him to use the Euclidean theorem that when parallel lines are cut by a transversal, alternate interior angles are equal. He also knew that a pole in the city of Syene, located on the Tropic of Cancer in southern Egypt about 800 km south of Alexandria, casts no shadow on the day of the summer solstice. Legend has it that he hired professional pacers to step off an accurate distance from Alexandria to Syene (Sagan 1980, pp. 14–17). By knowing angle A and distance B in **figure 1**, Eratosthenes was able to show that the circumference of the earth was about fifty times greater than distance B.

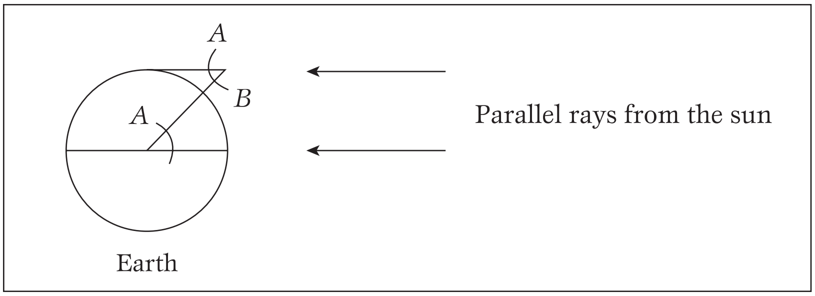


Fig. 1 If angle A and distance B are known, the circumference of the earth can be determined.

OVERVIEW OF OUR PROJECT

In our project, we modified Eratosthenes's method (**figs. 1** and **2**) and extended it to determine other lunar measures, including the altitude of Mount Piton (pronounced pee-´tahn).

We began by measuring the shadow cast by a parking lot light pole of known height at local noon on the day of the autumnal equinox, usually about September 21 (see **fig. 3**). (The vernal equinox, usually about March 21, would work equally well.) Eratosthenes used the summer solstice day; we modified his procedure in this way because school is often not in session on either summer or winter solstices. For further details on how to conduct this measurement with your students, consult William Dunham's *Journey through Genius* or the American Physical Society Web site, or contact us through www.frontieracademy.net.

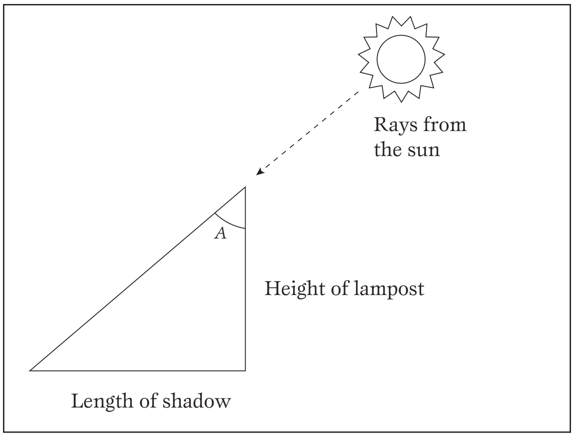


Fig. 2 $\angle A = \tan^{-1}\left(\dfrac{\text{length of shadow}}{\text{height of lamppost}}\right)$

The success of this initial experiment motivated us to extend the inquiry further and reproduce other ancient measurements:

- the distance to the moon by Ptolemy's parallax technique
- the diameter of the moon by means of circular ratios
- the altitude of Mount Piton, a spike-shaped peak on the moon, by using similar triangles

Fig. 3 Students prepare to measure lamppost shadows on the morning of the autumnal equinox.

We accomplished the whole suite of activities over the course of a few weeks. Our equipment included nothing more than metersticks, a pole of known length, calculators, a sextant, and a photograph of the moon depicting the full length of the shadow of Mount Piton as near as possible to the terminator, the line separating day and night hemispheres of the moon. Sextants capable of making the measurements described here and to the required degree of precision are available from boating supply catalogs for as little as $50.00. If sextants or telescopes with a reticle (a scale etched on the surface of the ocular lens to measure angular size of objects) are not available in the classroom, students may instead use their data from the determination of the earth's circumference together with data tabulated in reference books to determine the moon's distance and diameter. Jay M. Pasachoff's book *Stars and Planets* is an excellent source for this information. The students' first result, the diameter of the earth, can be used through each of the next three activities, giving students a sense of ownership over the entire process.

MEASURING THE DISTANCE TO THE MOON BY PARALLAX

Since $C = \pi d$, knowing the circumference of the earth leads immediately to the determination of its diameter. Knowing Earth's diameter allows us to use Ptolemy's parallax technique to determine the distance to the moon. *Parallax* is the apparent shift in position of an object, with respect to the background, when viewed from two different locations (see **fig. 4**). The result from the previous experiment, the size of the earth, is combined with a new measurement, the parallax of the moon over the course of a night,

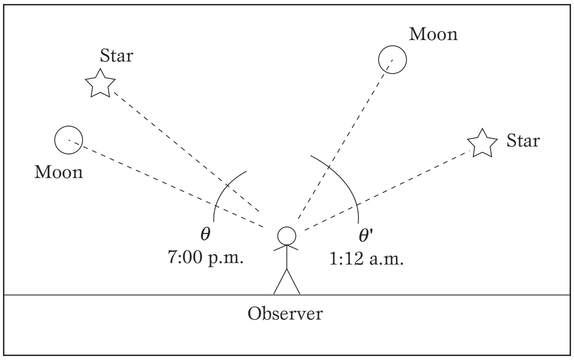
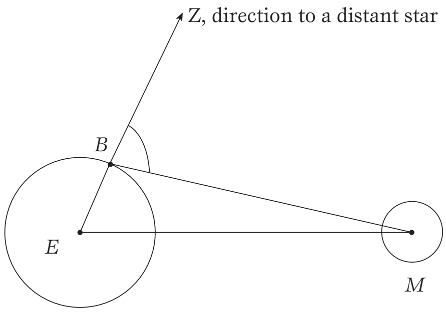


Fig. 4 The moon's apparent angular displacement with respect to a star (parallax) over the course of a night (6.2 hours)

Fig. 5 Students practice with a sextant to discover how measurements of the angular size of the moon and the parallax angle of the moon are made.



During an observation that lasted 6 hours 12 minutes (6.2 hr), the moon's total angular displacement was about 205', or close to 3.4 degrees.

In an almanac we found that the moon requires 27.3 days to orbit the earth. We calculated the fraction of angular displacement due to the moon's own motion as follows.

$$360° \text{ per orbit} \div 27.3 \text{ days/orbit} \approx 13.2°\text{/day}$$
$$13.2°\text{/day} \div 24 \text{ hr/day} \approx 0.55°\text{/hr} \approx 33'\text{/hr}$$

(Note: Be sure to use the orbital period of the moon rather than the full-moon-to-full-moon cycle, which is 29.5 days. The earth advances in orbit during the 27.3 days, and an extra 2.2 days are required for the moon to get back to the point where it can reflect sunlight back to Earth.)

In the 6.2-hr observation, the moon's angular displacement due to its own motion is

$$6.2 \text{ hr} \times 33'\text{/hr} \approx 205'$$

As measured directly by a sextant, the change in the angle between the moon and the reference star over 6.2 hours of observation was 148'. The difference is 205' – 148' = 57' = 0.95°. This number represents our best approximation of the parallax angle of the moon, $\angle EBM$.

$$\frac{EM}{\sin \angle EBM} = \frac{EB}{\sin \angle BME} \rightarrow$$
$$EM = \frac{EB \cdot \sin \angle EBM}{\sin \angle BME}$$
$$= \frac{6{,}350 \text{ km} \cdot \sin 89.5^{\circ}}{\sin 0.95^{\circ}}$$
$$\approx 383{,}000 \text{ km}$$

This result is based upon the assumption that both the moon and the star pass through the observer's zenith. The measurement can be made if this is not the case, but the mathematics will be somewhat more complex.

SOURCE: ABELL 1975, PP. 381-82; CHAISSON AND MCMILLAN 2004, PP. 10-13

Fig. 6 Data from Abdell, *Exploration of the Universe*

by measuring the angular displacement between the moon and a fixed star over the span of several hours.

Principles

When the moon is observed throughout the night, its position relative to the background stars changes. The angular displacement is about one Moon diameter per hour (**fig. 4**).

The amount of the moon's angular displacement is attributed to two factors: (1) the moon's own motion as it orbits around the earth and (2) the parallax caused by seeing the moon from two different locations in space. For example, the full moon observed at sunrise is seen from a position one Earth diameter (about 12,900 km) farther east than where it was seen the previous night at sunset.

Technique

Though students could certainly perform the necessary measurements, doing so requires that they observe the angle between the moon and a star over the course of several hours on a clear night. In the most simplified scenario, parallax is measured between the moon and a star as they pass through the observer's zenith. (Interested readers may want to explore the mathematics of the more common situation in which the moon and star do not pass directly overhead.) The equipment required includes either a sextant or a homemade astrolabe. Our students practiced making this measurement outdoors during the day when the moon is visible (see **fig. 5**). The required data for our example assumes the parallax angle is measured over the course of six or more hours as the moon passes through the zenith.

Because taking nighttime measurements of the moon and stars is not generally practical in the school-day setting, we instead provide students with data from Abell (1975, p. 389; see **fig. 6**) and conduct a discussion of the required nightlong measurements and subsequent calculations in the classroom. The Starpath School of Navigation (Burch 2003; see also www.starpath.com) will serve as an excellent resource if teachers want to give advanced students

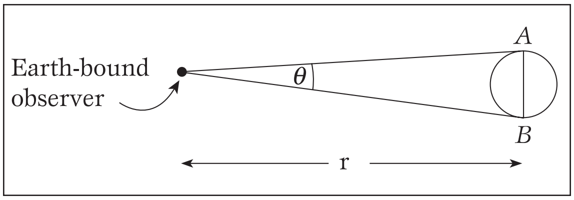


Fig. 7 The angular size of the moon represents a fraction of its complete orbit. Thus, the diameter *AB* of the moon may be found.

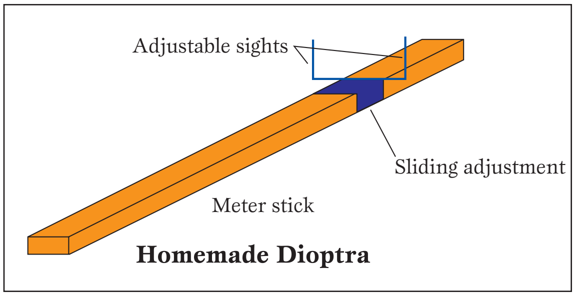


Fig. 8 Schematic of a dioptra, a device for measuring the angular size of the moon

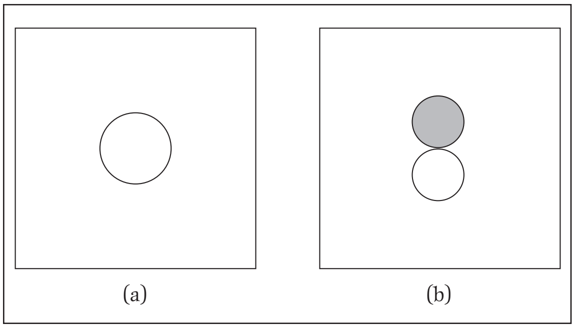


Fig. 9 With a properly aligned sextant, the scale should read 0° 0.0', and both the direct and reflected images of the moon should coincide, as in (a). Next, the micrometer drum is rotated until the direct and reflected images are tangent, as in (b). The difference between the scale readings in (b) and (a) represents the angular size of the moon.

the opportunity to stay up late and make the measurements using a sextant. These students may then use their own data in the calculations that follow, which is in many ways a much more satisfying experience.

We encourage mathematics teachers to collaborate with science department faculty at their school to give this problem to ambitious students as an additional challenge. We welcome suggestions from readers about how to approach this problem. Hirshfeld (2004) provides an excellent discussion of Aristarchus's technique for determining lunar distances, which teachers can also try with their students.

Results

The result for the distance to the moon compares favorably with accepted values, which because of the eccentricity of the moon's orbit vary between 406,610 km at apogee and 356,334 km at perigee, with a mean distance of 384,404 km (Abell 1975, p. 155; Pasachoff 2000, p. 535).

THE SIZE OF THE MOON

Principle

The goal of the next problem is to find the diameter of the moon. Buoyed by their success in determining the moon's distance from Earth, students can tackle this problem by the relatively straightforward technique of circular ratios. The critical data in this experiment is the angle subtended by the moon. A necessary prerequisite is the Earth-Moon distance, which was determined previously.

The basic principle is that the moon sweeps out an approximately circular orbit around the earth. In its path, the moon occupies a fraction of this orbit. Thus, the angular size of the moon, as measured from Earth, represents a fraction of the essentially circular orbit (**fig. 7**). So the following ratio can be used to determine *AB,* the diameter of the moon:

$$\frac{AB}{2\pi r} = \frac{\theta}{360^{\circ}}$$

Technique

The angular size of the moon may be measured with a sextant or even the reticle on a telescope. Ptolemy measured the angular size of the moon to a great degree of accuracy using the dioptra, obtaining a value of 31 1/3 ft. (Ptolemy 1952, pp. 143–88). In his *Commentary* on book 5 of Ptolemy's *The Almagest,* Pappus describes the dioptra (see **fig. 8**). Some students may want to build this device themselves, although this step is not necessary: Our students obtained similar results using a sextant. They are shown using a sextant in exactly this manner in **figure 5**. **Figure 9** describes how the sextant can be used to obtain the angular size of the moon.

Results

Since students may not have access to such instruments or be skilled in their use, our example uses data taken from Abell (1975, pp. 154–55). He gives a mean value of 31.083 feet for the angular size of the moon. This value will vary because of the elliptical shape of the moon's orbit around the earth. Substituting appropriate values and using 31 ft. ≈ 0.52° and $r = 383{,}000$ km, our previous result, yields a value of 3,480 km, within a close margin of error to the accepted value of 3,476 km (Abell 1975, p. 154).

THE ALTITUDE OF MOUNT PITON

Principle

The grand finale is to determine the height of the moon's Mount Piton. This can be accomplished using a photograph of the moon (**fig. 10**), which is available at nominal cost from Lick Observatory at the University of California, www.ucolick.org/

publications/. The underlying mathematical principle involves similar triangles. In order to establish a set of similar triangles, the photograph must show the moon's phase at first or third quarter, when the Earth-Moon-Sun system forms a right triangle. Galileo is credited with having first developed this technique (Galileo, trans. Van Helden, 1989, pp. 42–53), which involves some clever imagination to obtain the required similar triangles (Holton et al. 1975, unit 2, "Motion in the Heavens," pp. 8–14).

Figure 11a depicts a mountain near the moon's terminator as it appears from an earthbound observer's line of sight. **Figure 11b** depicts the same situation, except the observer's viewpoint has been rotated 90° with respect to the moon's terminator. Now the pair of similar triangles becomes apparent (see **fig. 11**) (Abell 1975, p. 301). It must be noted that length AB as measured on the photograph is actually a curve that follows the surface of the moon. Nevertheless, because the distance is short compared with the radius of the moon, only a negligible error is introduced by assuming this length is the same as AB in the right triangle ABM.

Fig. 10 The moon at first quarter. Mount Piton is circled near the bottom of the photograph.

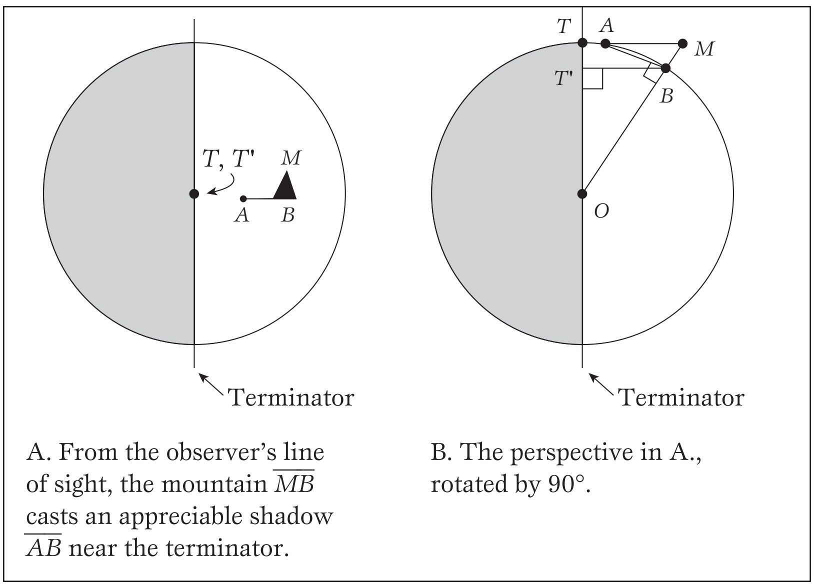


Fig. 11 $\triangle ABM$ and $\triangle BT'O$ are similar, which allows height of $\overline{BM}$ to be found.

Technique

Students may measure lengths on a photograph (**fig. 10**) of the moon to obtain the ratios for the similar triangles described above. Mount Piton is a prominent peak, near the terminator, about one-fourth the distance from the lower edge or limb. We have found that taking measurements is easier if using a transparency of this photograph projected onto a screen. In the previous activity, students determined the moon's diameter, $2 \cdot OB = 3480$ km. The diameter of the moon on the photograph is about 177 mm. Thus, the scaling factor between photograph and actual distances on the moon is 1 mm ≈ 20 km. These values depend on the size of the image used.

Results

Using a photograph like the one in **figure 10**, students might obtain the following values: $BT' = 6.0$ mm, $AB = 2.0$ mm, and $OB = 88.5$ mm. By substituting these values they obtain

$$BM = \frac{(2.0 \text{ mm})(6.0 \text{ mm})}{88.5 \text{ mm}} \approx 0.14 \text{ mm}.$$

Multiplying by the scaling factor,

$$0.14 \text{ mm}\left(\frac{20 \text{ km}}{1 \text{ mm}}\right) \approx 2.8 \text{ km}$$

for the actual altitude of Mount Piton. The accepted value, using more sophisticated techniques, is 2.3 km (Holton et al. 1975, unit 2, p. 14).

We celebrate such findings with our students. Although this calculation yields a result that differs from the accepted value by about 22 percent, it conveys the idea that a measurement-based value for the altitude of mountains on the moon can be obtained. In addition, a discussion of the sources of error is a rich activity. In determining the altitude of Mount Piton, students had to estimate the location of the terminator on the photograph. The resolution of the picture influenced this decision. Measuring the diameter of the moon, and especially the distance of the base of the mountain to the terminator and the length of the mountain's shadow, involved estimation that naturally brings error with it. Finally, the technique itself works best if the mountain of interest is very near the terminator.

CONCLUSION

In this age of Global Positioning System devices and other black box technologies, students often take for granted information printed in books, such as the altitude of mountains on the moon, which they can discover themselves. We have found the preceding series of activities appealing from several other perspectives:

1. These four activities allow students to reflect upon the accomplishments of the ancients, who were capable of sophisticated astronomy.
2. The measurements and calculations involved provide a meaningful context for high school geometry and trigonometry.
3. The lives of famous astronomers such as Eratosthenes, Aristarchus, Ptolemy, and Galileo and the historical contexts in which they lived can add greatly to classroom discussions held while these experiments are performed.
4. These activities might also be used to complement those described by Kurt J. Rosenkrantz in the September 2004 issue of the *Mathematics Teacher.*
5. These activities demonstrate the transitive, mathematical logic behind much of scientific knowledge. The first measurement, together with an assumption of the earth's sphericity, served as the basis for the determination of the moon's distance. In turn, the moon's distance was a prerequisite to determining its absolute size. Finally, the diameter of the moon was necessary in calculating the altitude of Mount Piton.

The authors of the astronomy text we use at Frontier Academy add another powerful perspective to the value of these activities when they summarize the use of techniques such as parallax. "Although the observations are straightforward and the geometrical reasoning elementary, simple measurements such as these form the basis for almost every statement made in this book about the size and scale of the universe" (Chaisson and McMillan 2004, p. 13).

ACKNOWLEDGMENTS

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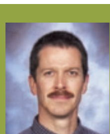

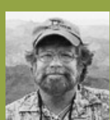

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