A Genetic Context for Understanding the Trigonometric Functions: Babylonian Astronomy and Sexagesimal Numeration

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Trigonometry is concerned with the measurements of angles about a central point (or of arcs of circles centered at that point) and quantities, geometrical and otherwise, which depend on the sizes of such angles (or the lengths of the corresponding arcs). It is one of those subjects that has become a standard part of the toolbox of every scientist and applied mathematician. Today an introduction to trigonometry is normally part of the mathematical preparation for the study of calculus and other forms of mathematical analysis, as the trigonometric functions make common appearances in applications of mathematics to the sciences, wherever the mathematical description of cyclical phenomena is needed. This project is one of a series of curricular units that tell some of the story of where and how the central ideas of this subject first emerged, in an attempt to provide context for the study of this important mathematical theory. Readers who work through the entire collection of units will encounter six milestones in the history of the development of trigonometry. In this unit, we examine the first of these six episodes: the emergence of sexagesimal numeration in ancient Babylonian culture, developed in the service of a nascent science of astronomy.

1 Sexagesimal Numeration

Our story in this unit starts *way, way* back, during the second millennium BCE, when Assyrian scholars in Old Babylonia[1] produced the oldest known records of celestial phenomena, on a collection of clay tablets known as the *Enuma Anu Enlil.* These tablets contain long lists of astrological omens referring to lunar and solar eclipses, or the locations of the planets within certain constellations at certain times. They were compiled over many centuries during the second and first millennia BCE in Babylonia and *they herald the beginning of science as the systematic study of the natural world.* Ironically, these oldest records more properly pertain to *astrology* than to mainstream modern *astronomy.* Astrology is a subject now discredited in Western science as providing untestable (and therefore, unscientific) explanations for the movements of heavenly bodies; it bases explanations of

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[1]Babylonia is the name given to the valley between the Tigris and Euphrates rivers (present day southern Iraq and southwestern Iran), centered on the ancient city of Babylon, which was an important political and cultural center of the region during this era. A number of human cultures flourished there, notable among them that of the Sumerians in the third and second millennia BCE—the period of Old Babylonia; and the Assyrians, whose era of most powerful political control in the area dates from the tenth through sixth centuries BCE. These were followed by the ascendancy of a New Babylonian civilization. This last culture, also known as Chaldean, is familiar to Biblical scholars, as it flourished in the sixth and fifth centuries BCE, its most famous ruler being Nebuchadnezzar, who conquered the Hebrew kingdom in Jerusalem in 587 BCE and razed Solomon's Temple.

the motions of the stars and planets on the actions of supernatural forces or divine entities with whom these bodies are associated, and it discerns from these motions why certain people enjoyed fortune while others were ill-fated. It also identifies auspicious times for adherents to engage in future actions. Astronomy, on the other hand, explains how the heavens move by means of natural physical forces and, by and large, ignores the impacts they make on human actions and social concerns.

The *Enuma Anu Enlil* mark the earliest documents in a long Babylonian tradition of keeping track of the motions of the stars and planets (North, 1994). From these records, they recognized the following general features of the motions of heavenly bodies:

- that the shape of the sky was that of a huge sphere, the *celestial sphere*, which rotated about the Earth roughly once a day, carrying the fixed lights of stars along with it;
- that the stars were arranged in patterns called *constellations*, named for fanciful animals, objects, and mythological figures, and which helped astronomers to gain their bearings and locate other stars and planets;
- that the movement of the Sun around the Earth produced a daily cycle close to but different from that of the celestial sphere; and that its motion across these background stars produced a yearly cycle that governed the seasons;
- that the motion of the Moon across the sky produced a monthly cycle, and that its path was close to but different from that of the Sun;
- that because of this, the Moon came between the Sun and the Earth once in a great while to produce a dramatic *solar eclipse*; and that somewhat more frequently, the Earth blocked the light of the Sun on the face of the Moon, causing a *lunar eclipse*; and that these events occurred with a complicated but identifiable cyclical frequency;
- that the Sun, Moon, Mercury, Venus, Mars, Jupiter, and Saturn—all the *planets*[2] visible to the naked eye—moved within a narrow band around the sky; the constellations found in this band were collectively called the *zodiac*.[3]

Extensive astronomical records required the ability to locate objects in the sky and note when and how often they reached these positions. For this purpose, a numeration system was needed to represent quantities encountered when measuring distances between objects, angles of inclination along circles, or times between events. The powerful number system that emerged within Babylonian astronomy to do this evolved from the simultaneous use of at least two different systems of measurement, one with units of 10 (and possible subunits of 5) and one or more other systems with units of 6 (and possibly also larger units of 12). These systems were combined to create what is known as *sexagesimal numeration*, based on units of size 60. Objects were counted in groups of 10 until one reached 60 objects; these would then constitute one larger unit. For numbers larger than 60, the same grouping process would be used with the larger units to produce even larger units. For instance 250 objects would be reckoned as 4 large units of 60 smaller units each, plus an additional

[2]The word *planet* comes from the Greek *planetos* = English *wanderer*. Since both the Sun and the Moon are seen to wander through the sky, both were considered to be planets for many centuries. In contrast, the apparently stationary Earth (like the fixed stars) was not considered a planet.

[3]From the Greek *zodiakos* = English *little animals*.

10 of the smaller size. This could also be done with small fractional parts, since each small unit could be split into 60 smaller parts.

For example, the number we represent as $2\frac{3}{4}$ would be reckoned in this system as 2 units with 45 smaller units, representing three-fourths, or 45 sixtieths, of the original unit.

Sexagesimal numeration shares important features with the decimal system we employ today, wherein we count units from 1 through 9 with single digits, tens using a second digit between 1 and 9 written to the left of the ones digit, hundreds with a third digit of the same type to the left of the tens digit, etc., all the while using 0 as a placeholder when no objects of the corresponding magnitude (units, tens, hundreds, ...) are used. Decimal numeration requires just these ten symbols, the digits 0, 1, 2, ..., 9, to represent any number. This kind of system is called *positional* because the positions of a digit within a specific numeral indicate what size units they represent: ones, tens, hundreds, and so on, with larger powers of 10 off to the left to count larger and larger groupings of units; similarly, tenths, hundredths, thousandths, and even smaller fractions of units (the negative powers of 10) appear to the right of the units digit to measure finer and finer parts of units. In addition, we use a decimal point to separate values of units from their fractional parts. For instance, the number we called $2\frac{3}{4}$ above is written in decimal form as 2.75 because the 2 represents two units, the 7 seven tenths of these units, and the 5 five additional hundredths (that is, tenths of tenths). That is, in decimal form $2.75 = 2 \text{ units} + 7 \text{ tenths} + 5 \text{ hundredths} = 2 + \frac{7}{10} + \frac{5}{100} = 2\frac{3}{4}$.

In the same way, this number would be represented in sexagesimal form as 2 units + 45 sixtieths, requiring only two sexagesimal digits. Note that 45 is a *single sexagesimal digit*, as is any number of units less than 60, in the same way that any number of units less than 10 corresponds to a *single decimal digit*.[4]

Throughout this project we will adopt the following convention for writing numbers in sexagesimal form: such representations will be expressed as a sequence of sexagesimal digits enclosed between brackets [ ], where we use a semicolon (;) to represent the sexagesimal equivalent of a decimal point: it will be used to separate units from sixtieths (in the same way that the decimal point separates units from tenths). Thus,

- $2\frac{3}{4}$ is written in sexagesimal form as $[2; 45]$;
- 78 is written as $[1, 18]$;
- 4.0075 is written as $[4; 0, 27]$; and
- 4800 is written as $[1, 20, 0]$.

However, note that we are using familiar *decimal* forms for the *sexagesimal* digits 0, 1, ..., 59. Thus, the number 78, written above in sexagesimal form as $[1, 18]$ has a two-digit sexagesimal form: remember, 18 is a *one-digit* sexagesimal number!

Task 1 Verify that the numbers 78, 4.0075, and 4800 have the sexagesimal representations identified above. Explain how you know that these representations are correct.

[4]This is the hardest thing to get used to for those new to sexagesimal numeration. Decimal numeration is such an ingrained part of how we understand numbers that it is difficult to recognize that the same number which we write using two or more decimal digits can also be written using fewer sexagesimal digits. But this is quite often the case: because 60 is a larger base than 10, the same number tends to be expressible using fewer sexagesimal digits than decimal digits.

2 Measuring along Circles

Imagine for a moment that you live in ancient Babylon, with no knowledge of modern science and none of the conveniences of modern life—no electric lighting at night, no ability to transport yourself across great distances quickly by car or airplane, no smartphone to connect you to the rest of the world, no standard time from a watch or clock to tell the time of day. Your environment is very likely restricted to a small portion of the earth near where you were born. Your experience of the world includes the reality that when the Sun sets, your life is dominated by darkness, punctuated by occasional firelight—and the beauty of the stars and the Moon!

You are used to the fact that the Sun moves across the background stars of the sky over the course of a year, taking just a bit over 360 ($= 6 \times 60$) days to make one full circular orbit around the sky. You know this because the same constellations of stars are visible in the night sky—at precisely the *opposite* location to the Sun in the heavens—at the same time every year; you know that certain constellations are summer constellations while others are winter constellations. That is, the Sun moves along a circular path against the background stars, called the *ecliptic circle*, about one 360th of its full annual course every day.

These considerations made it convenient for Babylonian astronomers to divide the circular path of the Sun into 360 parts, a measurement system that has endured to the present day. We call these parts of a circle *degrees*. If degrees needed to be further subdivided, one would use units corresponding to sixtieths of a degree, today called *minutes*.[5] If further accuracy was needed, one could use an even smaller unit called the *second*,[6] corresponding to sixtieths of a minute (or one 3600th of a degree, since $3600 = 60 \times 60$). These angular units of 1° (degree), $1'$ (minute), and $1''$ (second), became the standard way to measure along circles for Babylonian astronomers, and almost 4000 years later we use this same system. For instance, an angle measure of 4.0075° is the same as an angle of measure $4^\circ 0' 27''$ (see Task 1).

Task 2 In this task, we start to explore the relationship between angle measure and various lengths within a circle.

(a) Mark a point in the center of a blank sheet of paper and label it O. Then, preferably with a compass, draw a large circle centered at O. Mark a point on the circle at the "3 o'clock" position and label it A. Set your compass to the length OA of the radius of the circle, then, starting at A, mark off points $B, C, D, \ldots$ around the circle so that each of the segments $\overline{AB}, \overline{BC}, \overline{CD}, \ldots$ has the same length OA.[7] You should find that, eventually, the sequence of points $A, B, C, \ldots$ comes back exactly to A. Now draw in the line segments $\overline{AB}, \overline{BC}, \overline{CD}, \ldots$ around the circle. What shape have you now drawn within the circle? Be as specific as you can about describing this figure.

(b) Using a very light pencil line, connect the vertices of your figure through the center O to the points opposite them across the circle (which also happen to be points of the same

[5] From the Latin phrase *partes minutae primae* = English *first small parts*.

[6] From the Latin *partes secundae minutae* = English *second small parts*.

[7] In this project we use the widely adopted convention of referring to line segments using an overscored pair of letters: $\overline{XY}$ is the line segment whose endpoints are denoted X and Y. We use XY (without the overscore) to refer to the numerical length of the segment $\overline{XY}$. Thus again, $\overline{XY}$ is a geometrical object, a line segment, while XY is a number.

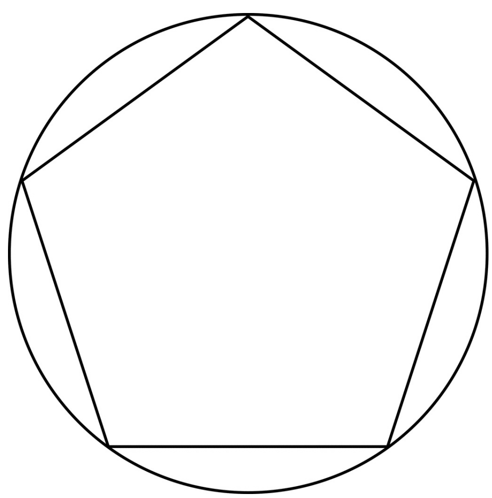
figure). These lines partition the figure into smaller figures. Describe in detail what the shapes of these smaller figures are. How can you tell?

(c) How large, in degrees/minutes/seconds, is the angle spanning the arcs from A to B, B to C, C to D, etc.? Assuming that the radius of the circle has unit length, how long are $\overline{AB}, \overline{BC}, \overline{CD}, \ldots$?

(d) Again using a light pencil line, draw in the diameters of the circle which are perpendicular to the segments $\overline{AB}, \overline{BC}, \overline{CD}, \ldots$ of the figure. These diameters will strike the circle in another set of points which bisect the arcs between A and B, B and C, C and D, etc. Label these points A' (between A and B), B' (between B and C), C' (between C and D), etc. If we now connect consecutive points around the circle to create segments $\overline{AA'}, \overline{A'B}, \overline{BB'}, \ldots$, what shape do we now produce inside the circle? And how large is the angle spanning these shorter arcs around the circle, from A to A', A' to B, B to B', etc.?

(e) Finally, let P be the point where the diameter having endpoint A' cuts side $\overline{AB}$ of the original figure. What are the angles in the triangle OPA? Use the Pythagorean theorem to determine the lengths of the sides of triangle OPA.

Task 3 Consider the regular pentagon[8] inscribed inside a circle that is pictured below. If we drew in radius lines from the center to the vertices of the pentagon, how large would the angles be between consecutive radii around the figure?

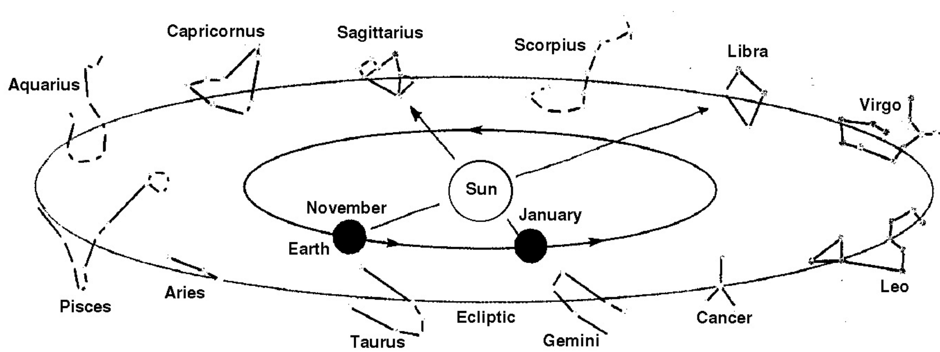


Task 4 If a regular heptagon (7-sided) were inscribed in a circle, and the seven radii drawn in from the center to each of the vertices of the heptagon, how large would the angles be between consecutive radii around the figure? Here, the answer cannot be given exactly in degrees, so round off your answer to the nearest second of arc.

[8] A polygon is called *regular* if all its sides and angles are mutually congruent.

3 The Babylonian Legacy

It takes somewhat more than twelve lunar months for the Sun to make its yearly trek around the heavens along the ecliptic circle and return to the same place relative to the background stars. One-twelfth of the way around this 360-degree circle should be roughly equal to a month, the time it takes for the Moon to go through a full cycle of its phases. This amounts to $360° \div 12 = 30°$, a nice round number in the sexagesimal system (consisting of exactly one half of a unit of sixty parts). Thus the Sun travels along a 30° arc of the ecliptic circle about once a month, and so, each of these 30° arcs was matched to a corresponding constellation along the zodiacal band.[9] This was the mechanism whereby the Babylonians used the sky as a calendar. Since their year began at the spring equinox, the Sun would pass through the first of these constellations, Aries the Ram, during the first month of the year, corresponding to the 0° to 30° band along the ecliptic.[10] During the second month, the Sun would be in Taurus the Bull (30° to 60° along the ecliptic), then Gemini the Twins (60° to 90° along the ecliptic), etc. By the end of the following winter, the Sun would complete its journey through the twelfth zodiacal band, for Pisces the Fish (330° to $360° = 0°$ along the ecliptic), returning to its starting point for the next year's course. (The diagram below sketches how this works, but it is drawn from the modern perspective, showing the Earth orbiting the Sun.)



More importantly to the story of trigonometry, the legacy left by Babylonian astronomers was taken up by Greek astronomers centuries later, thence by Islamic, and later European scholars.[11] Since the science of timekeeping was the business of astronomers who tracked the motions of the Sun and Moon across the sky, its regulation was their responsibility, and the Babylonian tradition of numerical reckoning in sexagesimal numbers became standard for this purpose. This is why the sexagesimal system we use for angle measure also became the system we still use today for measuring

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[9]The zodiacal constellations, in the order through which the Sun passes through them from the beginning of spring, through the summer, fall and winter months, are: Aries, Taurus, Gemini, Cancer, Leo, Virgo, Libra, Scorpio, Sagittarius, Capricorn, Aquarius and Pisces. The reader will recognize that these are also the astrological signs to which modern-day horoscopes are associated. One's horoscope is cast depending on where the Sun was found in the sky on the day one was born.

[10]Today, because the axis of the earth's rotation performs a slow gyroscopic rotation relative to the plane of its orbit around the Sun (a motion called the *precession of the equinoxes* by astronomers), the spring equinox no longer finds the Sun in the constellation Aries, but rather in Pisces. In fact, this slow movement will soon push the Sun at equinox into the next zodiacal constellation, Aquarius. That is, we are inexorably moving into the Age of Aquarius, as announced in the famous 1967 Broadway musical *Hair!*.

[11]Subsequent episodes in this series of curricular units investigate other milestones from this long tradition.

time: hours of 60 minutes each, and minutes of 60 seconds each. In fact, long after astronomers in ancient Egypt divided the nighttime into 12 hours, and the daytime similarly into 12 hours, Greek astronomers adopted Babylonian sexagesimal methods to similarly subdivide the hour into minutes and seconds. Later scientists, from medieval through modern times, continued these traditions (Richards, 1998). Thus, the Babylonian sexagesimal numeration persists in our modern lives in substantial ways.

References

J. L. Berggren. *Episodes in the Mathematics of Medieval Islam.* Springer, New York, NY, 2003.

Barnabas Hughes, O.F.M. *Regiomontanus: On Triangles.* University of Wisconsin Press, Madison, WI, 1967.

Victor Katz. *A History of Mathematics: an introduction.* Addison Wesley, Reading, MA, second edition, 1998.

Victor Katz, Annette Imhausen, Eleanor Robson, Joseph W. Dauben, Kim Plofker, and J. Lennart Berggren. *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: a Sourcebook.* Princeton University Press, Princeton, NJ, 2007.

E. S. Kennedy. *The Exhaustive Treatise on Shadows, by Abū Rayḥān Muḥammad ibn Aḥmad al-Bīrūnī.* Institute for the History of Arabic Science, Aleppo, Syria, 1976.

Otto Neugebauer and David Pingree. *The Pañcasiddhāntikā of Varāhamihira.* Munksgaard, Copenhagen, Denmark, 1970/1972.

John North. *The Fontana History of Astronomy and Cosmology.* Fontana Press, London, 1994.

Olaf Pedersen. *A Survey of the Almagest, with annotation and new commentary by Alexander Jones.* Springer, New York, NY, 2011.

Kim Plofker. *Mathematics in India.* Princeton University Press, Princeton, NJ, and Oxford, 2008.

E. G. Richards. *Mapping Time: the calendar and its history.* Oxford University Press, Oxford, 1998.

Jeff Suzuki. *Mathematics in Historical Context.* Mathematical Association of America, Washington, DC, 2009.

G. J. Toomer. *Ptolemy's Almagest.* Princeton University Press, Princeton, NJ, 1998. Translated and annotated by G. J. Toomer; with a foreword by Owen Gingerich.

Glen Van Brummelen. *The Mathematics of the Heavens and the Earth: the early history of trigonometry.* Princeton University Press, Princeton, NJ, and Oxford, 2009.

Notes to Instructors

This project is one of a collection of curricular units drawn from a Primary Source Project (PSP) titled *A Genetic Context for Understanding the Trigonometric Functions*. The full PSP is designed to serve students as an introduction to the study of trigonometry by providing a context for the basic ideas contained in the subject and hinting at its long history and ancient pedigree among the mathematical sciences. Each of the six individual units in that PSP looks at one of the following specific aspects of the development of the mathematical science of trigonometry:

- the emergence of sexagesimal numeration in ancient Babylonian culture, developed in the service of a nascent science of astronomy;
- a modern reconstruction (as laid out in (Van Brummelen, 2009)) of a lost table of chords known to have been compiled by the Greek mathematician-astronomer Hipparchus of Rhodes (second century, BCE);
- a brief selection from Claudius Ptolemy's *Almagest* (second century, CE) (Toomer, 1998), in which the author blue(Ptolemy) shows how a table of chords can be used to monitor the motion of the Sun in the daytime sky for the purpose of telling the time of day;
- a few lines of Vedic verse by the Hindu scholar Varāhamihira (sixth century, CE) (Neugebauer and Pingree, 1970/1972), containing the "recipe" for a table of sines as well as some of the methods used for its construction;
- passages from *The Exhaustive Treatise on Shadows* (Kennedy, 1976), written in Arabic in the year 1021 by Abū Rayḥān Muḥammad ibn Aḥmad al-Bīrūnī, which include precursors to the modern trigonometric tangent, cotangent, secant and cosecant;
- excerpts from Regiomontanus' *On Triangles* (1464) (Hughes, 1967), the first systematic work on trigonometry published in the West.

This collection of units is not meant to substitute for a full course in trigonometry, as many standard topics are not treated within here. Rather, it is the author's intent to show students that trigonometry is a subject worthy of study by virtue of the compelling importance of the problems it was invented to address in basic astronomy in the ancient world. Each unit may be incorporated, either individually or in various combinations, into a standard course in College Algebra with Trigonometry, a stand-alone Trigonometry course, or a Precalculus course. These lessons have also been used in courses on the history of mathematics and as part of a capstone experience for pre-service secondary mathematics teachers.

Student Prerequisites

Nearly nothing in the way of special prerequisites is required of students for this particular unit beyond what a typical high school student knows about basic plane geometry.

Suggestions for Classroom Implementation

The central features of most of the units in the full PSP described above are the primary source texts and the sequence of tasks that accompany them. The particular unit presented here is unusual in that it does not include any primary source text. Instructors who wish to expose students to a primary source in conjunction with implementation of this unit are encouraged to use the Primary Source Project *Babylonian Numeration* by Dominic Klyve, which is available for free download at `https://blogs.ursinus.edu/triumphs/`.

This particular unit is meant to be completed in one 50-minute (or 75-minute) classroom period, plus time in advance for students to do some initial reading and time afterwards for them to write up their solutions to the tasks. It should be emphasized that student written work should be far more explicit and detailed in its production than the oral communication in which they engage in the classroom, communication that is often accompanied by the recording of rather telegraphic notes.

Sample Implementation Schedule (based on one 50- or 75-minute class period)

Students should have read through the entire unit and written up their calculations for Task 1 before class begins. The first 5–10 minutes can be devoted to checking the mechanics of converting decimal numbers into sexagesimals. Instructors may find that many handheld calculators have a tool that converts radian measures to degrees/minutes/seconds, a useful tool for conversion of decimal numbers into sexagesimal form. But one should also be aware that it may be too tempting for students to employ as a means of avoiding the conceptual work of understanding the procedure. Some instructors may want to give classroom time to review the six bulleted items in Section 1, and illustrate these basic facts about the heavens by presenting dynamic images of the sky with "planetarium" software that shows the transits of the sun, moon, planets and stars above an artificial horizon in speeded-up time. Many such software packages supply overlaid images of the constellations and allow the user to set display parameters to suit local geographic coordinates as well.[12]

Approximately 30 minutes should be set aside to allow students to work together, in groups of 3–5, on Task 2. This will likely comprise the greater part of the classroom period. Students will need to have on hand blank paper, pencils for drawing, straightedges (rulers) and compasses. The latter can be challenging to use effectively, so it may be best to have at least two students per group working to construct the diagrams required for the Task to allow their drawings to be compared and improved upon. Still, artistic perfection is not the ultimate goal here, so you might ask them to be somewhat forgiving of their drawings.

The most significant mathematical challenges of this project are (1) the decimal to sexagesimal conversion of Task 1; and (2) Task 2(e). Allow sufficient time for students to make progress on both of these.

Tasks 3 and 4 can be assigned for homework. Indeed, the work of all four Tasks should be written up as a final assignment.

LaTeX code of this unit is available from the author by request to facilitate preparation of 'in-class task sheets' based on tasks included in the project. The project itself can also be modified by instructors as desired to better suit their goals for the course.

[12] I have enjoyed success using a rather user-friendly freeware package called *Stellarium* (`stellarium.org`). It can be downloaded to run on an individual computer and has an online version as well.

Recommendations for Further Reading

Instructors who want to learn more about the history of trigonometry are recommended to consult Glen van Brummelen's masterful *The Mathematics of the Heavens and the Earth: The early history of trigonometry* (Van Brummelen, 2009), from which much of this work took inspiration.

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