A Genetic Context for Understanding the Trigonometric Functions
Episode 5: al-Bīrūnī Does Trigonometry in the Shadows

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Trigonometry is concerned with the measurements of angles about a central point (or of arcs of circles centered at that point) and quantities, geometrical and otherwise, which depend on the sizes of such angles (or the lengths of the corresponding arcs). It is one of those subjects that has become a standard part of the toolbox of every scientist and applied mathematician. Today an introduction to trigonometry is normally part of the mathematical preparation for the study of calculus and other forms of mathematical analysis, as the trigonometric functions make common appearances in applications of mathematics to the sciences, wherever the mathematical description of cyclical phenomena is needed. This project is one of a series of curricular units that tell some of the story of where and how the central ideas of this subject first emerged, in an attempt to provide context for the study of this important mathematical theory. Readers who work through the entire collection of units will encounter six milestones in the history of the development of trigonometry. In this unit, we work with passages from an 11th-century text written in Arabic which includes precursors to four modern trigonometric functions: tangent, cotangent, secant, and cosecant.

1 Medieval Islamic Mathematics—why Trigonometry?

Roughly between the years 750 CE and 1400 CE, Islamic scholars, from their early homelands in Arabia, and later from places westward to European Spain and eastward to northern India, fostered attention to the preservation and development of mathematical and astronomical knowledge. While a list of their accomplishments, even only those in mathematics, is impossible to catalog here, there are two famous advances worth noting. First, Islamic scholars adopted the arithmetical techniques of the Hindus, using ten symbols (early versions of the signs 1, 2, 3, 4, 5, 6, 7, 8, 9 and 0) to represent whole numbers in decimal form—what we today call Indo-Arabic numeration. Beginning in about 1000 CE, these methods migrated into Europe and influenced the development of computational methods by Western scholars for centuries to come. Some time before this, however, in about the year 820 CE, a seminal work by Muḥammad ibn Mūsā al-Khwārizmī (c. 780–850 CE) was written. In Arabic it is titled *Al-kitāb al-mukhtaṣar fī ḥisāb al-jabr wa'l-muqābala*, which translated means *The Compendium on Calculation by Completion and Reduction*, and which we remember by the Latinization of one of the words in its Arabic title as *Algebra*.[1] This work outlined general methods for solving equations for unknown quantities and fueled the growth of a whole new main branch of

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[1]For more on al-Khwārizmī's *Algebra*, see the *Convergence* article "Completing the Square: From the Roots of Algebra, A Mini-Primary Source Project for Students of Algebra and Their Teachers," by this author [Otero, 2019].

mathematical activity [Berggren, 2003, Katz, 1998, Suzuki, 2009]. Al-Khwārizmī's text marks the beginning of the second major mathematical advance due to medieval Islamic scholars: namely, the development of methods for a highly general sort of problem solving—algebra—which provided a powerful tool for making further mathematical developments.

Islamic mathematicians were interested in more than arithmetic and algebra, however. Muslim practice requires devotion to *ṣalāt*, or daily ritual prayer, at prescribed times of day. For this and other reasons, Islamic scholars were very interested in accurate timekeeping. Thus, the study of shadow clocks became an important concern, and this helped to further developments in trigonometry in this era. The shadow clock was the simplest of tools, requiring little more than a straight stick planted firmly and vertically into flat ground. In any part of the world where weather conditions permitted the Sun to regularly shine brightly, the stick would cast a shadow on the ground. (See the diagram in the source text on the next page.) As long as the user was sufficiently versed in the movements of the Sun as it traveled slowly but inexorably across the daytime sky, one could infer from the height of the stick and the length of the shadow how far the Sun had gone in its course that day, providing an instant and reliable way to tell time.

An important contributor to the mathematics behind this science of *horology* (or timekeeping; from the Greek *hōra*, for *time*) was the Khwarezmian[2] scholar Abū Rayḥān Muḥammad ibn Aḥmad al-Bīrūnī (973–1048). Al-Bīrūnī was a polymath who made a name for himself as an expert in the history, geography, and the religious and scientific legacy of India, where he traveled extensively; his most famous work, dating from 1017, was *al-Hind* (*A History of India*)[3] [al-Bīrūnī, 1017; 2015]. But in this project, we will read an excerpt from a different work by al-Bīrūnī, *Ifrād al-maqāl fī amr al-ẓilāl* (*The Exhaustive Treatise on Shadows*) [al-Bīrūnī, 1021; 1976], compiled in 1021 while he was living in Ghaznī[4] in the court of a Turkish sultan.

2 Gnomons and the Casting of Shadows

We begin by reading a selection from al-Bīrūnī's treatise *The Exhaustive Treatise on Shadows* in which he addressed the mathematical relationships between the lengths of a shadow cast by the Sun and the angles of elevation of the Sun above the horizon.

The Sixth [Chapter]: On the method by which the use of the shadow and the gnomon[5] is arranged.

[. . . W]e say that plane surfaces upon which shadows fall are numerous. They are all planes of local horizons, which will be determined if their latitudes are known. And the shadows . . . are of two kinds, like [any] category containing its [different] types. One of the two of them is

[2] Khwarezm is a region of central Asia—today part of Uzbekistan—which was also home to the aforementioned al-Khwārizmī, the "Father of Algebra."

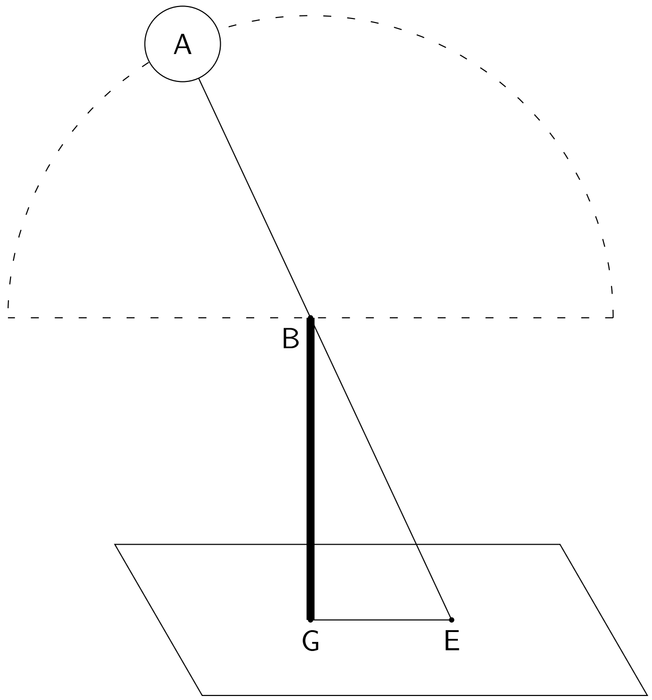
[3] The full title of this text is rather daunting: *Taḥqīq mā li-l-hind min maqūlah maqbūlah fī al-ʻaql aw mardhūlah* (*Verifying All That the Indians Recount, the Reasonable and the Unreasonable*), but in Western histories, it is often abbreviated to simply *al-Hind* (*India*).

[4] Ghaznī is now a town in central Afghanistan.

[5] *Gnomon* is a Greek word that might best be translated 'indicator' (literally, "the one who knows"); it refers to an object whose shadow is used to tell the time of day.

the [*direct*] *shadow*[6] which . . . is bordered by the shadow-caster and that part of the *horizon plane* which is between them. But since light can be perceived on the flat of the earth, a place devoid of light is called shadow, and the shadow-caster is called the *gnomon*, but when it is used, especially in computation, [it is called] the *scale*.

That sort of shadow is always in the plane of a *circle of altitude* through the shadow-caster [and cast] on the part in common between it and the plane of the horizon in the case of the vertical gnomon perpendicular to it . . . .



An example of the *direct shadow* is [the following]: Let A be the body of the sun and BG the gnomon perpendicular to EG, which is parallel to the horizon plane, and ABE is the sun's ray passing through the head of the gnomon BG. So will BGE be the shadow in space. But EG is that which is called the direct shadow such that its base is G and its end E. And EB, the line joining the two ends of the shadow and the gnomon, is the *hypotenuse of the shadow*.

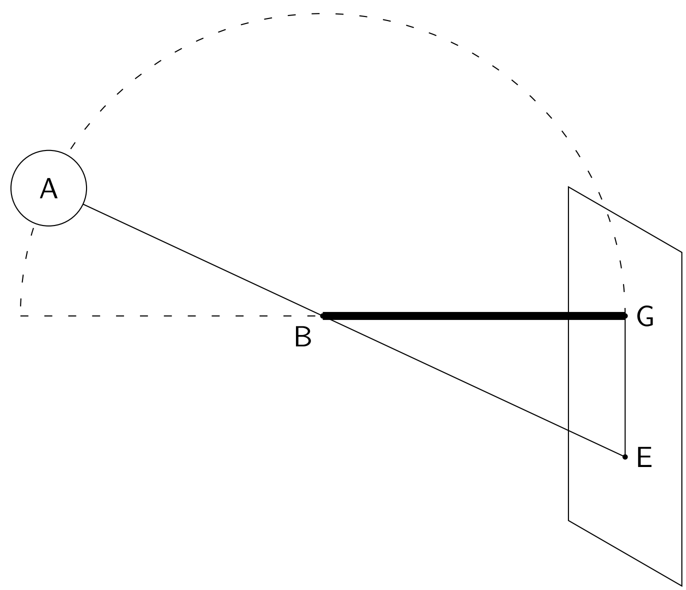
Task 1 Al-Bīrūnī opened this passage by stating that the "planes" on which shadows are cast are "planes of local horizons, which will be determined if their latitudes are known." Why are shadows cast by the Sun different in different places on the Earth, depending on their latitude?

Task 2 What geometric figure does al-Bīrūnī use to model the "shadow in space" cast by the gnomon? Copy the diagram provided with the text, then in your copy label each of these elements: *Sun*, *circle of altitude*, *horizon plane*, *gnomon*, *direct shadow*, *hypotenuse of the shadow*.

[6] Italics have been added by the project author to the source texts to emphasize important technical terms; they are not in the published translation [al-Bīrūnī, 1021; 1976].

We continue our reading of al-Bīrūnī's description of the two kinds of shadows.

However, as for the second type of shadow, it is that whose gnomon is [parallel to] the horizon plane. Then the gnomon is perpendicular to a plane which is itself perpendicular both to the horizon plane and the circle of altitude. And the shadow itself [accordingly] will be along the axis of the horizon. It is called the *reversed* [*shadow*] because its head is under its base.



Task 3 Copy the second diagram provided with the text. Then label these elements of your diagram: *Sun*, *circle of altitude*, *gnomon*, *reversed shadow*. Which points do you think al-Bīrūnī referred to as the *head* and the *base* of the shadow?

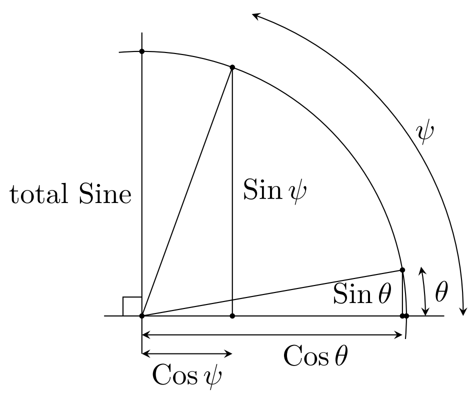
The height of the Sun (also called its *altitude*) along its circular path across the sky, and therefore the length of the shadow it casts at a particular time of day, depends very much on the observer's location—the "planes of local horizons" mentioned by al-Bīrūnī. The further north the observer stands (that is, the higher the location's latitude), the lower the Sun rides across the sky, which in turn means the longer the direct shadows become (and the shorter the corresponding reversed shadows). Regardless of one's latitude, however, there is a fixed relationship between the angle of elevation of the sun and the length of a shadow cast by an object, as measured relative to the length of the object itself. For instance, a stick 1 foot long casts a shadow that is 1 foot long when the sun is at an elevation corresponding to 45 degrees, regardless of where on the earth that object is placed. Of course, *when*—or even if—the sun actually reaches an elevation of 45 degrees at a particular location depends on that location's latitude; for instance, in Cincinnati, Ohio (latitude 39°9′N and longitude 84°28′W),[7] this happens just after 3:00 PM on September 22, 2021. But in other locations further

[7] Not surprisingly, this is where the author lives.

north or south of Cincinnati at different latitudes (but in the same time zone) on that same date, the sun attained that elevation at some other o'clock (or not at all, if that place was too far north).

Al-Bīrūnī aimed to discover the technical details of this relationship between the altitude of the Sun (that is, the arc from the horizon up to the Sun along the heavenly circle through the Sun and the observer's zenith point) and the length of the shadow that the Sun casts. The gnomon used to create the shadow was used as a scale unit for making this measurement.

To accomplish this goal, al-Bīrūnī (and his contemporaries) made use of the ***Sine* of an arc** in a circle; that is, the side of the right triangle created by dropping the perpendicular from one endpoint of the arc to the radius connecting the center of the circle to the other endpoint of the arc.[8] As the arc increases in size from spanning small angles (like θ in the diagram below) to larger ones (like ψ), the Sine also increases in length and attains its maximum length when the arc is a full quarter of the circle (for angle 90°). This largest possible Sine is called the *total Sine*; note that it is one of the radii of the circle. As we will see in the passages that we will read in the next section of this project, both the Sine of an arc and the total Sine played a role in al-Bīrūnī's study of the relationship between the Sun's altitude and the length of the shadow that it casts. Indeed, by the time of al-Bīrūnī, mathematical astronomers were comfortable using tables of Sine values, called a *zīj* in Arabic, as part of their standard computational toolbox.



Note that the Sine of an arc is one of the legs of a right triangle for which the hypotenuse is a radius of the underlying circle. The other leg of this right triangle is equal to the *Sine* of the *co*mplementary arc, and would eventually be known by geometers as the ***Cosine* of the** (original) **arc**. Where θ is the measure of the angle corresponding to the arc, these quantities are then abbreviated $\operatorname{Sin}\theta$ and $\operatorname{Cos}\theta$, respectively. And when the radius of the circle is of unit length, this is highlighted by using lowercase initial letters: $\sin\theta$ and $\cos\theta$. The capital letter is employed to remind us that the length of the Sine or Cosine of the arc depends on how long the radius of the underlying circle is, even though the degree measure of the angle (or the corresponding arc) does not; in contrast, the lowercase sine or cosine will always be a quantity smaller than 1, the length of the radius of the unit circle.

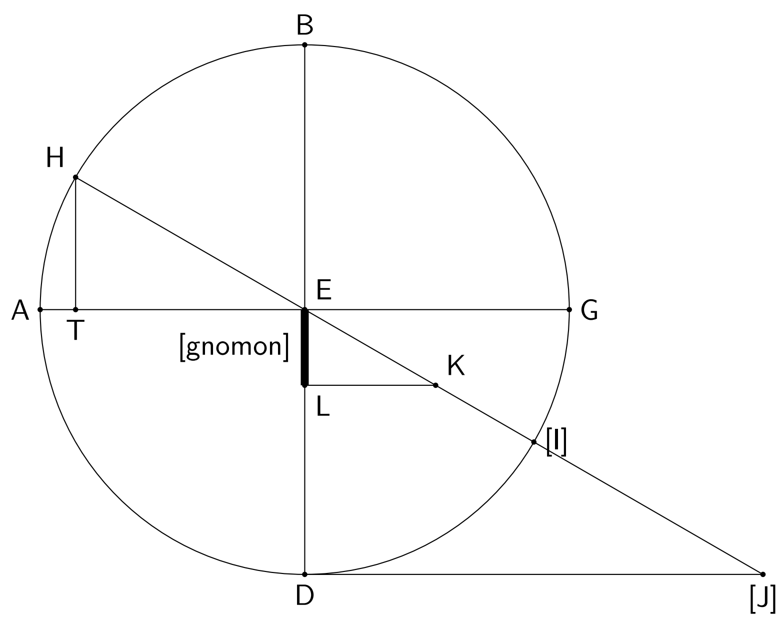
[8]This definition of Sine is equivalent to the one given by the earlier Indian mathematician Varāhamihira (505–587 CE), who composed a treatise that summarized the Babylonian, Egyptian, Greek, Roman and Indian astronomy of his day. This astronomy carried forward an even earlier tradition that employed tables of *chords* in circles associated with the arcs they spanned, rather than the tables of *Sines* of these arcs that featured in Hindu science. For more about this, see the project "Varāhamihira and the Poetry of Sines," in the *Convergence* article "Teaching and Learning the Trigonometric Functions through Their Origins."

3 Move Over, Sine and Cosine!

Here is al-Bīrūnī's statement of the relationship between the altitude of the Sun and the direct shadow, from the Ninth Chapter of his treatise, entitled "On the direct shadow and the altitude, and the extraction of one of the two from the other if [either is] unknown."

The ratio of the gnomon to the hypotenuse of the shadow is as the ratio of the Sine of the altitude to the *total Sine*.

The following diagram was used by al-Bīrūnī to prove this assertion. As this diagram will also help us to decipher what the assertion means, study it carefully as you read through the next passage.



Let ABG be the circle of altitude with center E, representing the gnomon head, and AEG the common part between the plane of the horizon and the plane of this circle, and B [and] D are the two poles of the horizon.[9] We lay off EL equal to the gnomon, and the sun is at point H. So AH will be its altitude and the perpendicular HT is the Sine of this altitude, and HB the complement of its altitude, and ET is equal to its Sine. We extend ray HEK and LK perpendicular to EL. So LK will be the *direct* [*shadow*] of the altitude AH, and KE will be the *hypotenuse of the direct* [*shadow*], and by virtue of the parallelism of the two lines LK [and] TE the angle HET will be the external [one] equal to the angle EKL, and the two angles T [and] L will be right angles. So the triangles EKL [and] HET will be similar, and the ratio of EL, the gnomon, to KE, the hypotenuse of the direct [shadow], will be as the ratio of HT, the Sine of the altitude, to EH, the total Sine.

[9]B is the zenith point, directly overhead of the observer. D is the observer's *nadir*, and is the point on the celestial sphere opposite the zenith point, below the observer's feet and therefore not in view. The labels of the points I and J have been added to this figure by the project author.

Task 4 Copy the diagram provided with the text. Then label these elements of the diagrams: *Sun*, *circle of altitude*, *gnomon*, *direct shadow*, *hypotenuse of the direct shadow*, *altitude* (of the Sun),[10] *complement of the altitude* (of the Sun),[11] *Sine of the altitude* (of the Sun), *Sine of the complement of the altitude* (of the Sun), *total Sine*.

Task 5 In this passage, al-Bīrūnī argued that "the triangles EKL [and] HLT will be similar." Justify this fact in your own words.

Task 6 In the same way that al-Bīrūnī invoked the similarity of triangles HET and EKL in the diagram above, it is possible to show that both these triangles are similar to triangle EJD. Do this!

Task 7 Explain why al-Bīrūnī's assertion that "the ratio of EL, the gnomon, to KE, the hypotenuse of the direct [shadow], will be as the ratio of HT, the Sine of the altitude, to EH, the total Sine" must be true.

The results of Tasks 5 and 7 make clear that, not only are the ratios EL to KE and HT to EH equal, but both are equal to the ratio of ED to EJ. Because the ratios in these triangles hold greater significance than their absolute sizes, al-Bīrūnī could have taken the triangle of the gnomon and its shadow to be the triangle EJD instead of EKL. Indeed, other astronomers set up their geometry of shadows in just this way, so that the radius of the circle ED was taken to represent the gnomon and the direct shadow corresponded to the line segment DJ tangent to (or touching[12]) the circle.[13] Under this setup, the triangle spans an arc, DI, that is complementary to the arc of the Sun, AH. (Make sure you see why it is that the arc DI is complementary to the arc AH!) Later mathematicians would call the segment DJ the ***Cotangent* of the arc** AH, since DJ is *tangent* to the circle and spans an arc DI which is *co*mplementary to the arc AH. It is abbreviated as $\mathrm{Cot}\,\theta$, where θ is the angle corresponding to the arc AH. If, moreover, the gnomon and radius are both taken to be a unit length, then we call the segment DJ the *cotangent* (with lowercase 'c') and denote it $\cot\theta$.

Task 8 Draw two copies of the previous diagram in which the gnomon EL coincides with the radius of the circle ($L = D$ and $K = J$), but label the diagrams so that in one, the gnomon and radius are of unit length, while in the other, the gnomon and radius have (equal) lengths different from 1 (either larger or smaller than 1). Use similarity of triangles to explain why the *ratio* of the Cotangent of arc AH to its gnomon in the second drawing is equal to the cotangent of the same arc in the first diagram.

If we continue to choose the gnomon to coincide with the radius of the circle ED, then the hypotenuse of the direct shadow for arc AH will be the segment EJ in the figure, cutting through the circle; this hypotenuse of the direct shadow would in later centuries be called the ***Cosecant* of**

[10] Note that this will be an arc of the circle of altitude!

[11] Another arc of the circle of altitude!

[12] The Latin word *tangere* means *to touch*.

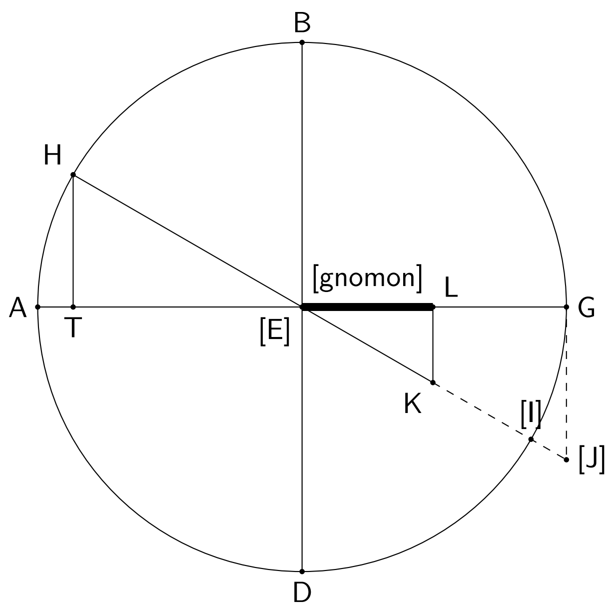
[13] The highly influential Greek mathematician Claudius Ptolemy (fl. ca. 250 CE) also set up the geometry in this way. You can learn more about his work in the project "Ptolemy Finds High Noon in Chords of Circles" from the *Convergence* article "Teaching and Learning the Trigonometric Functions through Their Origins."

the arc (or *cosecant* in a circle of unit radius), abbreviated to Csc θ (or csc θ), since the segment EJ *cuts through*[14] the circle and corresponds to an arc DI which is *co*mplementary to the given arc AH.

$\boxed{\textbf{Task 9}}$ At the end of the preceding passage above, al-Bīrūnī used the similarity of triangles HET and EKL to relate the hypotenuse of the direct shadow to the Sine of the arc. In Task 6, you further showed that those two triangles are similar to triangle EJD. Use the similarity of all three of these triangles to recast al-Bīrūnī's opening assertion that "the ratio of EL, the gnomon, to KE, the hypotenuse of the direct [shadow], will be as the ratio of HT, the Sine of the altitude, to EH, the total Sine" as a formula that relates $\csc\theta$ to $\sin\theta$. Then develop a similar formula that will relate $\cot\theta$ with $\sin\theta$ and $\cos\theta$.

In the Tenth Chapter of *The Exhaustive Treatise on Shadows*, entitled "On the reversed shadow and the altitude, and the extraction of one of the two from the other if [either is] unknown," al-Bīrūnī treated the case of the reversed shadow.

For the reversed shadow let us repeat the preceding figure and lay off the gnomon along diameter AG.[15]



LK will be the reversed [shadow] of the gnomon EL. And the two triangles HTE (and) KLE will remain similar, so the ratio of LK to KE is as the ratio of HT to HE, and the ratio of HT to TE is as the ratio of LK to LE.

[14]The Latin word *secare* means *to cut*.

[15]The points I and J, and the dashed lines that connect them within the following figure, are not in the original and have been added by the project author to mimic their positions in the earlier diagram from al-Bīrūnī's Ninth Chapter.

Task 10 If al-Bīrūnī had made the gnomon coincide with the radius of the circle of altitude in the last diagram above so that $L = G$ and $K = J$, what segment in the diagram would then represent the gnomon? the reversed shadow? the hypotenuse of the reversed shadow? (You may want to redraw the diagram for yourself to correspond to this situation.)

In later centuries, what al-Bīrūnī called the reversed shadow for a given arc came to be called the ***Tangent* of the arc**, since in the case where the length of the gnomon coincides with the radius EG of the circle of altitude, the reversed shadow segment GJ is tangent to the circle and spans an arc equal to that of the given arc. Similarly, the hypotenuse of the reversed shadow EJ came to be called the ***Secant* of the arc**, since the hypotenuse cuts through the circle and corresponds to an arc equal to that of the given arc. (Once again, make sure you see these relationships in the diagram above!) The notation accompanying these quantities mirrors that described earlier for Cosecant and Secant: if θ is the angle corresponding to the given arc, then $\mathrm{Tan}\,\theta$ and $\mathrm{Sec}\,\theta$ represent the Tangent and Secant of the arc. Likewise, Tangent and Secant have "lowercase" versions, $\tan\theta$ and $\sec\theta$, used when the radius of the circle is assumed to have unit length.

Task 11

(a) Explain why the three triangles HET, KEL and JEG in the last diagram above are mutually similar.

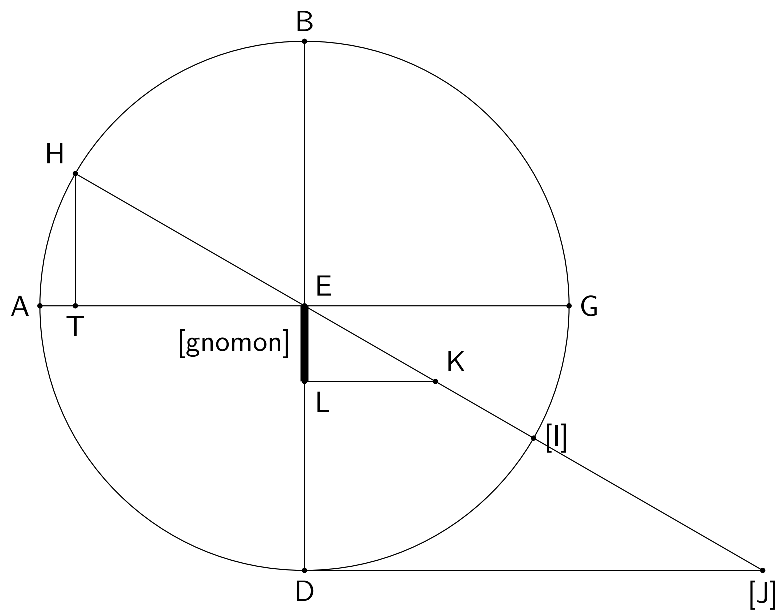
(b) Use similarity of triangles HET and KEL to establish a proportion relating the Tangent LK of arc AH (corresponding to the angle θ) and the gnomon EL to the Sine HT and Cosine ET of arc AH.

(c) Use the similarity of triangles HET and JEG to establish a formula linking $\tan\theta$ to $\sin\theta$ and $\cos\theta$.

4 Making Trigonometry in the Shadows

Al-Bīrūnī was interested in more than just naming the geometric elements of his shadow diagrams. Indeed, he had done all this geometric naming to deal with the fundamental astronomical problem of telling time by the Sun. In this next excerpt we return to the Ninth Chapter of his treatise to see how he employed trigonometry to handle this central problem in the case of a shadow-caster clock that casts a direct shadow. We have also reproduced the corresponding diagram.

In order to simplify the work of making sense of this and subsequent passages, let us suppose that the gnomon we are working with is chosen to be equal to the radius and that both are taken to have unit length. This means that the capital-letter trigonometric quantities (implicitly) referred to here, Cotangent and Cosecant, can be identified with their lowercase counterparts. More practically, however, this also means that a modern calculator can be used to compute these quantities, since scientific calculators have buttons for the sine, cosine and tangent quantities; this removes the need for consulting a *zīj* table, as al-Bīrūnī would have had to do! (More on that in the next section . . .)



If we are given the [direct] shadow at a certain time and we want to find the altitude of the sun for that time, we multiply the shadow by its equal and we take [the square root] of the sum [with the square of the gnomon], and it will be the hypotenuse [of the direct shadow]. Then we divide the product of the gnomon with the total Sine by it [the hypotenuse], and there comes out the Sine of the altitude. We find its corresponding arc in the Sine table and there comes out the altitude of the sun at the time of that shadow. Thus we operate for the Sine of any named arc if it is given.

Task 12 Consider now a particular case of the problem posed in the passage: suppose that "the [direct] shadow at a certain time" is twice the length of the gnomon. We shall follow al-Bīrūnī's directions to determine the value of the arc of the Sun that casts the given shadow.

(a) Let θ represent the angle for "the altitude of the Sun for that time." Further assume that "the gnomon" equals the radius of the underlying circle, which we take to be 1 unit long so that all the trigonometric quantities mentioned in the passage will have lowercase labels (as in the first diagram in Task 8). Which of the quantities $\sin\theta, \cos\theta, \tan\theta, \sec\theta, \cot\theta, \csc\theta$, or 1 corresponds with each of these items mentioned by al-Bīrūnī in the passage above:

(i) "the [direct] shadow"

(ii) "the gnomon"

(iii) "the hypotenuse [of the direct shadow]"

(iv) "the total Sine"

(v) "the Sine of the altitude"

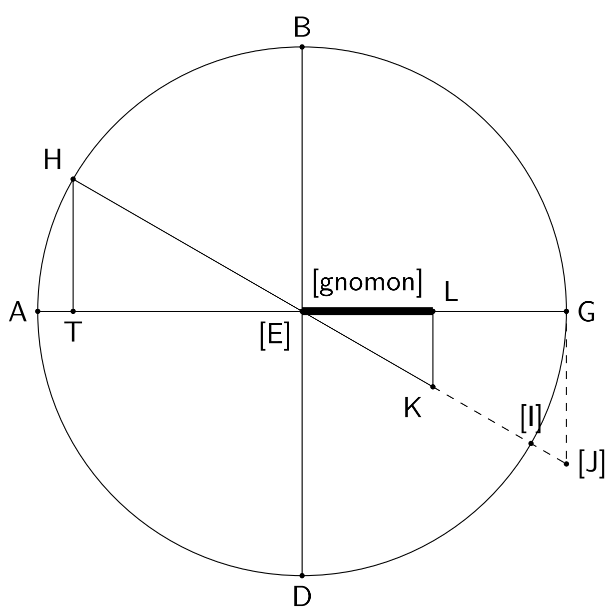
(b) Al-Bīrūnī directed his reader to "multiply the shadow by its equal" and "take [the square root] of the sum [with the square of the gnomon]" to obtain "the hypotenuse [of the direct shadow]." Rewrite this procedure in the form of a trigonometric equation using the representations found in part (a).

(c) Continuing, al-Bīrūnī proceeded to "divide the product of the gnomon with the total Sine" by the value of this "hypotenuse," resulting in "the Sine of the altitude." In a similar way, rewrite this procedure in the form of a trigonometric equation.

(d) If, as we assumed at the outset, "the [direct] shadow at a certain time" is twice the length of the gnomon, then what, according to the equations derived here, is the value of the "the Sine of the altitude?" Use the SIN^{-1} button on your calculator[16] to determine the value of the arc θ and find how high (in degrees) the Sun is off the horizon.

Task 13 Now suppose that "the [direct] shadow at a certain time" is precisely equal to the length of the gnomon. It shouldn't be too difficult to guess how high the Sun must be in the sky at that moment by thinking about the triangle that corresponds to this situation, but work through al-Bīrūnī's instructions nonetheless to confirm your guess.

Al-Bīrūnī applied similar trigonometric machinery in his Tenth Chapter to deal with the case of a wall-mounted shadow-caster. (Again, we reproduce the associated diagram.)



If we are given that the reversed [shadow] is known and we need its altitude, we would take the root of the sum of the squares of the reversed [shadow] and its gnomon so that there

[16]The notation SIN^{-1} denotes what is called the *inverse sine*, or *arcsine*, and is used to identify the arc or angle whose sine has a given value. Thus, if $a = \sin\theta$ (and θ is within 90° of 0°), then $\theta = \sin^{-1} a$. There are similar inverses which are attached to the other trigonometric quantities; for instance, $\cos^{-1} b$, the *inverse cosine* or *arccosine* of b, is the measure of the arc or angle whose cosine equals b. Most scientific calculators will include SIN^{-1}, COS^{-1} and TAN^{-1} buttons to compute these quantities.

would result the hypotenuse [of the reversed shadow]. Then we multiply the reversed [shadow] by the total Sine and we divide the result by the hypotenuse [of the reversed shadow], and there will result the Sine of the altitude.

Task 14 (a) Explain why al-Bīrūnī was correct to state that "the root of the sum of the squares of the reversed [shadow] and its gnomon" produced "the hypotenuse [of the reversed shadow]." Refer to the segments within the figure above. What fundamental geometric result did he invoke here?

(b) In the final sentence in the passage above, al-Bīrūnī asserted that if "we multiply the reversed [shadow] by the total Sine and we divide the result by the hypotenuse [of the reversed shadow], ... there will result the Sine of the altitude." Recast this statement in terms of the trigonometric quantities $\sec\theta, \tan\theta$ and $\sin\theta$.

(c) Provide an argument based on similar triangles for the truth of the statement in part (b).

Task 15 Use the geometry of the diagrams from al-Bīrūnī's Ninth and Tenth Chapters to explain why the following general trigonometric formulas are true:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

5 A Table of His Own

In *The Exhaustive Treatise on Shadows*, al-Bīrūnī also provided a table—or using his jargon, a *zīj*—that recorded the lengths of both direct and reversed shadows (cotangents and tangents, respectively) for arcs between 1° and 90°; we only display the beginning of the table here, for arcs up to 15°. As you examine the table, note that the lengths that appear in the final column of al-Bīrūnī's table are in sixtieths of the gnomon length. Indeed, since al-Bīrūnī was working within the astronomical tradition dating back 2000 years before him to Babylonian times, he, like all other early astronomers, computed using a hybrid decimal-sexagesimal numeration.[17] So, for instance, the tangent of an arc corresponding to an angle of 82° is given as [426; 15], or $426 + \frac{15}{60} = 426.25$ sixtieths of a gnomon unit. Division of this number by 60 shows that it corresponds to a bit more than 7 times the gnomon unit. You can easily check with a scientific calculator that tan(82°) is indeed a bit larger than 7.

[17] The ancient Babylonians used a base-60, or *sexagesimal*, numeration system, entirely similar to our familiar Indo-Arabic numerals, except that instead of employing ten symbols for the numbers 0 to 9 and concatenating digits to the right to represent decreasing powers of 10, they employed 59 digits to represent the numbers 1 to 59 and concatenated digits to represent decreasing powers of 60. For more about this, see the classroom unit "Babylonian Astronomy and Sexagesimal Numeration" from the first episode of the *Convergence* article "Teaching and Learning the Trigonometric Functions Through Their Origins."

The Twelfth [Chapter]: On tables containing shadows, exclusive of their computation, and how to obtain them.

It is customary among authors of *zījes* to put the values of the shadows corresponding to their arcs in tables arranged part by part, and this arrangement is befitting it, and this is how we put them.

[for the] direct shadow	[for the] reversed shadow	parts [sixtieths]
1	89	[3437; 22]
2	88	[1718; 11]
3	87	[1144; 52]
4	86	[858; 2]
5	85	[685; 43]
6	84	[570; 52]
7	83	[488; 40]
8	82	[426; 15]
9	81	[378; 49]
10	80	[340; 17]
11	79	[308; 40]
12	78	[282; 17]
13	77	[259; 54]
14	76	[240; 39]
15	75	[223; 55]
...	...	...
90	0	[0; 0]

Task 16 Make three independent observations about what you notice about the structure of this table or about any regularities you observe in the values presented here.

Task 17 Verify that al-Bīrūnī's values for $\tan(75°), \tan(80°)$, and $\tan(85°)$ are quite close to the more accurate values you will obtain from a scientific calculator.

Task 18

(a) Suppose that the elevation of the Sun is 15° above the horizon; according to al-Bīrūnī's table, how long should the (direct) shadow cast by a 1 meter long (vertical) gnomon be? [Task 15 will be useful here!]

(b) How long should the shadow cast by a wall-mounted 0.5 meter long gnomon be if the Sun is 85° above the horizon?

6 On Beyond *Zījes*

Modern trigonometry has today become the study of the six functions of a real variable: the sine and cosine, tangent and cotangent, secant and cosecant functions. Trigonometry investigates the algebraic and graphical behavior of these functions and the interrelations they have with each other. But it is instructive to be aware that this fundamental group of mathematical functions, vital today for any scientific study of periodic behavior, had its genesis in the humble problem of telling the time of day, the most prosaic of periodic phenomena that humans engage with on a daily basis. In an era when accurate time is crucial for the maintenance of our society, we also find ourselves managing time in ways that are nearly divorced from connections to the heavens.[18]

The words tangent and cotangent, secant and cosecant, were not the terms al-Bīrūnī used for the elements of the triangles that formed his geometric models of shadows cast by the Sun. Those terms were coined hundreds of years later, in Europe during the Renaissance. In our next and final episode of this series, we shall see how trigonometry finally came down to earth in 15th-century Europe, and how mathematicians began to use it in matters having nothing to do with astronomy.

References

Abū Rayḥān Muḥammad ibn Aḥmad al-Bīrūnī. *Alberuni's India: An account of the religion, philosophy, literature, geography, chronology, astronomy, customs, laws and astrology of India about AD 1030.* Edited, with translation, notes and indices, by Edward C. Sachau, Munshiram Manoharlal Publishers Pvt Ltd, New Delhi, New Delhi, 1017; 2015.

Abū Rayḥān Muḥammad ibn Aḥmad al-Bīrūnī. *The Exhaustive Treatise on Shadows.* Edited and translated by E. S. Kennedy, Institute for the History of Arabic Science, Aleppo, Syria, 1021; 1976.

J. L. Berggren. *Episodes in the Mathematics of Medieval Islam.* Springer, New York, NY, 2003.

Barnabas Hughes, O.F.M. *Regiomontanus: On Triangles.* University of Wisconsin Press, Madison, WI, 1967.

Victor Katz. *A History of Mathematics: An Introduction.* Addison Wesley, Reading, MA, second edition, 1998.

Victor Katz, Annette Imhausen, Eleanor Robson, Joseph W. Dauben, Kim Plofker, and J. Lennart Berggren. *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: a Sourcebook.* Princeton University Press, Princeton, NJ, 2007.

Otto Neugebauer and David Pingree. *The Pañcasiddhāntikā of Varāhamihira.* Munksgaard, Copenhagen, Denmark, 1970/1972.

John North. *The Fontana History of Astronomy and Cosmology.* Fontana Press, London, 1994.

Daniel E. Otero. Completing the Square: From the Roots of Algebra, A Mini-Primary Source Project for Students of Algebra and Their Teachers. *Convergence*, September 2019. URL https://www.maa.org/press/periodicals/convergence.

[18]When was the last time you checked your sundial to make sure you weren't late for class?

Olaf Pedersen. *A Survey of the Almagest, with annotation and new commentary by Alexander Jones.* Springer, New York, NY, 2011.

Kim Plofker. *Mathematics in India.* Princeton University Press, Princeton, NJ, and Oxford, 2008.

E. G. Richards. *Mapping Time: The Calendar and Its History.* Oxford University Press, Oxford, 1998.

Jeff Suzuki. *Mathematics in Historical Context.* Mathematical Association of America, Washington, DC, 2009.

G. J. Toomer. *Ptolemy's Almagest.* Translated and annotated by G. J. Toomer; with a foreword by Owen Gingerich. Princeton University Press, Princeton, NJ, 1998.

Glen Van Brummelen. *The Mathematics of the Heavens and the Earth: The Early History of Trigonometry.* Princeton University Press, Princeton, NJ, and Oxford, 2009.

Notes to Instructors

This project is the fifth in a collection of six curricular units drawn from a Primary Source Project (PSP) titled *A Genetic Context for Understanding the Trigonometric Functions*. The full project is designed to serve students as an introduction to the study of trigonometry by providing a context for the basic ideas contained in the subject and hinting at its long history and ancient pedigree among the mathematical sciences. Each of the individual units in that PSP looks at one of the following specific aspects of the development of the mathematical science of trigonometry:

- the emergence of sexagesimal numeration in ancient Babylonian culture, developed in the service of a nascent science of astronomy;
- a modern reconstruction (as laid out in [Van Brummelen, 2009]) of a lost table of chords known to have been compiled by the Greek mathematician-astronomer Hipparchus of Rhodes (second century, BCE);
- a brief selection from Claudius Ptolemy's *Almagest* (second century, CE) [Toomer, 1998], in which the author (Ptolemy) showed how a table of chords can be used to monitor the motion of the Sun in the daytime sky for the purpose of telling the time of day;
- a few lines of Vedic verse by the Hindu scholar Varāhamihira (sixth century, CE) [Neugebauer and Pingree, 1970/1972], containing the "recipe" for a table of sines as well as some of the methods used for its construction;
- passages from *The Exhaustive Treatise on Shadows* [al-Bīrūnī, 1021; 1976], written in Arabic in the year 1021 by Abū Rayḥān Muḥammad ibn Aḥmad al-Bīrūnī, which include precursors to the modern trigonometric tangent, cotangent, secant, and cosecant;
- excerpts from Regiomontanus' *On Triangles* (1464) [Hughes, 1967], the first systematic work on trigonometry published in the West.

This collection of units is not meant to substitute for a full course in trigonometry, as many standard topics are not treated here. Rather, it is the author's intent to show students that trigonometry is a subject worthy of study by virtue of the compelling importance of the problems it was invented to address in basic astronomy in the ancient world. Each unit may be incorporated, either individually or in various combinations, into a standard course in College Algebra with Trigonometry, a stand-alone Trigonometry course, or a Precalculus course. These lessons have also been used in courses on the history of mathematics and as part of a capstone experience for pre-service secondary mathematics teachers.

In this unit, students are introduced to the genesis of the trigonometric quantities tangent, cotangent, secant and cosecant in the context of timekeeping by the Sun in an 11th-century work of al-Bīrūnī (973–1048) called *The Exhaustive Treatise on Shadows* [al-Bīrūnī, 1021; 1976]. Using simple geometric diagrams, al-Bīrūnī models shadows cast by a straight gnomon as the Sun travels in its circular path across the sky around the tip of the gnomon, creating a shadow in the shape of a right triangle. If the gnomon is vertical in the ground, al-Bīrūnī calls it a *direct shadow*, whereas if the gnomon is horizontally mounted on a vertical wall, he calls it a *reversed shadow*. When the gnomon is vertical, the line in the model corresponding to the edge of the shadow on the ground is the genesis of the cotangent of the arc of the elevation of the Sun; similarly, when the gnomon is horizontal, the line corresponding to the edge of the shadow on the wall becomes the tangent of the same arc. The corresponding hypotenuses of these right triangles form the cosecant and secant, respectively, of the arc. Students will contend with al-Bīrūnī's choice not to set the radius of the underlying circle equal to the length of the gnomon, which helps them to better appreciate that doing so simplifies matters considerably and leads to the modern conventions concerning these trigonometric quantities.

Examples of how al-Bīrūnī may have used this trigonometry to deal with the practical use of a shadow-casting clock are included throughout so that the mathematics is never too far from the astronomical context in which it arose.

Student Prerequisites

Very little mathematical machinery is needed to work this PSP. But it would help for students to have been exposed to sexagesimal numeration (to handle the arithmetic of angle measures in degrees and minutes) and to a reasonably good high school geometry course, especially with regard to the geometry of the circle and the measure of arcs and angles. A simple introduction to sexagesimal numeration is provided by the first episode in this series, "Babylonian Astronomy and Sexagesimal Numeration," and a presentation of the geometry of the circle and the measure of arcs and angles that serve as the foundation of trigonometry will be obtained by working the second episode, "Hipparchus' Table of Chords." A familiarity with some basic astronomy can also be helpful, and the third episode, "Ptolemy Finds High Noon in Chords of Circles," addresses this. Finally, an introduction to the sine of an arc, which is the launching point for this project, can be found in the fourth episode, "Varāhamihira and the Poetry of Sines." Nonetheless, it should be possible for an interested student to successfully tackle this project with not much more than the brief introduction to the definition of the sine of an arc that is provided at the end of Section 3.

Suggestions for Classroom Implementation

This project can be implemented in two 50-minute class periods. (There is rather too much to do to expect that it can also be accomplished in a single 75-minute period, so instructors who teach twice a week will need to adapt what is described here to a two-75-minute-period schedule.) For a detailed plan for how to structure the periods, one that suggests preparatory reading, activities for the classroom, and a minimal collection of Tasks whose solutions should be written up for homework, see the next subsection below.

The critical experiences in this project occur in Tasks 8 through 11, where students should be making the connections between the segments in the diagrams that describe al-Bīrūnī's direct and reversed shadows and the four new trigonometric quantities (tangent, cotangent, secant, cosecant). A possible hurdle will likely come in the final Task, as the student will have to negotiate hybrid sexagesimal numeration in al-Bīrūnī's *zīj* to complete the necessary computations. Instructors who choose to do so may also omit that particular Task, as well as the entire Section 5 (A Table of His Own) in which it appears. Site testers who have opted to assign this (relatively short) section, however, report that their students have benefited from seeing the level of accuracy that can be obtained without computers or calculators, which in turn often leads to a "wow!" moment that increases their appreciation of mathematics.

Sample Implementation Schedule (based on two 50-minute class periods)

In preparation for the first class period, students should be asked to read through Task 4 of the project and bring to class their responses for Tasks 1-4. Inform students that it is not expected that everything they read will make sense to them immediately, and that classroom discussion should help to clear up any difficulties they encounter. The first class can begin with such discussion, but an effort should be made to move quickly to getting students working in small groups through the remainder of Section 3, focusing on working the exercises presented in Tasks 5-11. The technical issues involved in Tasks 5 and 10 are thorny enough that these should be given highest priority for public consideration during that class period. For homework, ask students to formally write up Tasks 7, 9 and 11.

To prepare for the second class period, have students read the beginning of Section 4, through Task 12, and bring their answers to part (a) of this Task to class. In class, after discussing students' responses to Task 12(a), set students to work in their groups to work through the rest of this and the next section, focusing on completion of Tasks 12 and 14. For homework, assign formal solutions to be submitted for these same two Tasks, along with Task 17 (and for a special challenge, include Task 18). Ask students to read the concluding Section 6 as a follow-up assignment, if time does not permit this to be done during the second class period.

LaTeX code of this entire PSP is available from the author by request to facilitate implementation. The PSP itself may also be modified by instructors as desired to better suit the goals for their course.

Recommendations for Further Reading

Instructors who want to learn more about the history of trigonometry are recommended to consult Glen Van Brummelen's masterful *The Mathematics of the Heavens and the Earth: The early history of trigonometry* [Van Brummelen, 2009], from which much of this work took inspiration.

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