# Gabriel Lamé's Counting of Triangulations

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## 1 Introduction

In a 1751 letter to Christian Goldbach (1690–1764), Leonhard Euler (1707–1783) discusses the problem of counting the number of triangulations of a convex polygon. Euler, one of the most prolific mathematicians of all times, and Goldbach, who was a Professor of Mathematics and historian at St. Petersburg and later served as a tutor for Tsar Peter II, carried out extensive correspondence, mostly on mathematical matters. In his letter, Euler provides a "guessed" method for computing the number of triangulations of a polygon that has $n$ sides but does not provide a proof of his method. The method, if correct, leads to a "formula" for calculating the number of triangulations of an $n$-sided polygon which can be used to quickly calculate this number [3, p. 339–350] [4]. Later, Euler communicated this problem to the Hungarian mathematician Jan Andrej Segner (1704–1777). Segner, who spent most of his professional career in Germany (under the German name Johann Andreas von Segner), was the first Professor of Mathematics at the University of Göttingen, becoming the chair in 1735. Segner "solved" the problem by providing a proven correct method for computing the number of triangulations of a convex $n$-sided polygon using the number of triangulations for polygons with fewer than $n$ sides [8]. However, this method did not establish the validity (or invalidity) of Euler's guessed method. Segner communicated his result to Euler in 1756 and in his communication he also calculated the number of triangulations for the $n$-sided polygons for $n = 1, 2, 3, \ldots, 20$ [8]. Interestingly enough, he made simple arithmetical errors in calculating the number of triangulations for polygons with 15 and 20 sides. Euler corrected these mistakes and also calculated the number of triangulations for polygons with up to 25 sides. It happens that with the corrections, Euler's guessed method provides the correct number of triangulations of polygons with up to 25 sides.

Was Euler's guessed method correct? It appeared so, but there was no proof. The problem was posed as an open challenge to mathematicians by Joseph Liouville (1809–1882) in the late 1830s. He received solutions or purported solutions to the problem by many mathematicians (including one by Belgian mathematician Eugène Charles Catalan (1814–1894) which was correct but not so elegant), some of which were later published in the Liouville journal, one of the primary journals of mathematics at that time and for many decades. The most elegant of these solutions was communicated to him in a letter by Gabriel Lamé (1795–1870) in 1838. Lamé's equation (3), appearing near the end of §2 of this project, was likely Euler's guessed formula, while Lamé's equation (1) appearing at the beginning of §2 offers a recursion relation for these numbers which is quite difficult to solve. The reader is asked to appreciate Lamé's clever reduction of equation (1) to equation (3).

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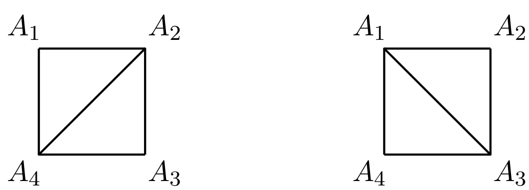
The French mathematician, engineer and physicist Gabriel Lamé was educated at the prestigious Ecole Polytéchnique and later at the Ecole des Mines [5, p. 601–602]. From 1832 to 1844 he served as the chair of physics at the Ecole Polytéchnique, and in 1843 joined the Paris Academy of Sciences in the geometry section. He contributed to the fields of differential geometry, number theory, thermodynamics and applied mathematics. Among his publications are textbooks in physics and papers on heat transfer, where he introduced the rather useful technique of curvilinear coordinates. In 1851 he was appointed Professor of Mathematical Physics and Probability at the University of Paris, and resigned eleven years later after becoming deaf. Gauss considered Lamé the foremost French mathematician of his day [5, p. 601–602].

The triangulation problem can be stated as follows. Given a convex $n$-sided polygon, divide it into triangles by drawing *non-intersecting* diagonals connecting some of the vertices of the polygon. Euler calculated the number, $P_n$, of distinct triangulations of a convex $n$-gon for the first few values of $n$, and conjectured a formula for $P_n$ based on an empirical study of the ratios $P_{n+1}/P_n$ [3, p. 339-350] [4]. Lamé was one of the first to provide the details for a combinatorial proof of Euler's conjectured result for $P_{n+1}/P_n$, a proof which the reader will study in its original (translated) version in this project.

Although Euler does not state his motivation for studying the triangulation problem, it may have roots in surveying, where a given region to be surveyed is divided into triangles, with the three vertices of the triangle serving as reference points. A modern use of triangulation is the Global Positioning System, where readings from three satellites are used to determine the position of a point on earth. For this project, however, we will consider polygons $\mathcal{P}$ in the plane with the property that if $A$ and $B$ are points in $\mathcal{P}$, then the line segment connecting $A$ and $B$ is also contained in $\mathcal{P}$. This latter property is expressed by stating that $\mathcal{P}$ is convex. As a further simplification, we will often use regular polygons, which have all sides congruent and all angles congruent. Lamé, not Euler, uses the subscripted notation $P_n$ to denote the number of triangulations of a convex polygon with $n$ sides.

**Exercise 1.1.** Explain why $P_3 = 1$.

To determine $P_4$, consider the following triangulations of square $A_1A_2A_3A_4$.




**Exercise 1.2.** Can you find any other ways to triangulate square $A_1A_2A_3A_4$ with non-intersecting diagonals? What is the value of $P_4$?

**Exercise 1.3.** Let $A_1A_2A_3A_4A_5$ be a regular pentagon and compute $P_5$, the number of distinct triangulations of the pentagon. Can you identify how the number of triangulations of a convex quadrilateral (or square) enters into the calculation of the triangulations of the pentagon? Consider triangulations of the pentagon which contain the triangle $A_1A_2A_3$ first, then triangle $A_1A_2A_4$, then triangle $A_1A_2A_5$. Write an equation for $P_5$ in terms of $P_4$ and $P_3$.

**Exercise 1.4.** Let $A_1A_2 \ldots A_6$ be a regular hexagon. Devise a strategy for computing $P_6$ by using the previous results for a pentagon, quadrilateral, and a triangle. Be sure to explain your approach. What is the value of $P_6$?

**Exercise 1.5.** Find a recursion relation for $P_{n+1}$ in terms of the previous $P_k$'s. Be sure to justify your answer. For what values of $k$ must $P_k$ be known in order to compute $P_{n+1}$? Compare your result with §I from Lamé's letter [7] in the next part.

## 2 Lamé's Letter to Liouville

**Extrait d'une lettre de M. Lamé à M. Liouville sur cette question:** *Un polygone convexe étant donné, de combien de manières peut-on le partager en triangles au moyen de diagonales?*[1]



**Excerpt from a letter of Monsieur Lamé to Monsieur Liouville on the question:** *Given a convex polygon, in how many ways can one partition it into triangles by means of diagonals*?[1]

The formula that you communicated to me yesterday is easily deduced from the comparison of two methods leading to the same goal.

Indeed, with the help of two different methods, one can evaluate the number of decompositions of a polygon into triangles: by consideration of the sides, or of the vertices.

I.

Let $ABCDEF\ldots$ be a convex polygon of $n+1$ sides, and denote by the symbol $P_k$ the total number of decompositions of a polygon of $k$ sides into triangles. An arbitrary side $AB$ of $ABCDEF\ldots$ serves as the base of a triangle, in each of the $P_{n+1}$ decompositions of the polygon, and the triangle will have its vertex at $C$, or $D$, or $F\ldots$; to the triangle $CBA$ there will correspond $P_n$ different decompositions; to $DBA$ another group of decompositions, represented by the product $P_3P_{n-1}$; to $EBA$ the group $P_4P_{n-2}$; to $FBA$, $P_5P_{n-3}$; and so forth, until the triangle $ZAB$, which will belong to a final group $P_n$. Now, all these groups are completely distinct: their sum therefore gives $P_{n+1}$. Thus one has

$$(1)\qquad P_{n+1} = P_n + P_3P_{n-1} + P_4P_{n-2} + P_5P_{n-3} + \cdots + P_{n-3}P_5 + P_{n-2}P_4 + P_{n-1}P_3 + P_n.$$

**Exercise 2.1.** For $n = 4$, interpret equation (1) above to reflect the calculation of $P_5$ completed in Exercise (1.3). Note that the term $P_3P_3$ occurs only once in $P_5$.

[1]See a Memoir of Segner (*Novi Commentarii Acad. Petrop.*, vol. VII, p. 203). The author found equation (1) of M. Lamé; but formula (3) presents a much simpler solution. Formula (3) is no doubt due to Euler. It is pointed out without proof on page 14 of the volume cited above. The equivalence of equations (1) and (3) is not easy to establish. M. Terquem proposed this problem to me, achieving it with the help of some properties of factorials. I then communicated it to various geometers: none of them solved it; M. Lamé has been very successful: I am unaware of whether others before him have obtained such an elegant solution. J. Liouville (This footnote actually appears in the paper.)

**Exercise 2.2.** For $n = 5$, interpret equation (1) above to reflect the calculation of $P_6$ completed in Exercise (1.4).

**Exercise 2.3.** Find an equivalent expression for (1) using summation notation, i.e., $P_{n+1} = \sum \boxed{?}$. If necessary, use the convention $P_2 = 1$.

**Exercise 2.4.** Explain why the triangulations belonging to the groups

$$P_n,\ P_3P_{n-1},\ P_4P_{n-2},\ \ldots\ ,\ P_{n-1}P_3,\ P_n$$

are distinct.

**Exercise 2.5.** Does every triangulation of a convex polygon with $n + 1$ sides occur in one of the groups represented by

$$P_n,\ P_3P_{n-1},\ P_4P_{n-2},\ \ldots\ ,\ P_{n-1}P_3,\ P_n\ ?$$

Why or why not?

**Exercise 2.6.** Use Lamé's recursion relation of §I to compute $P_{10}$. What difficulties do you encounter in this computation?

In §I of his letter, Lamé uses triangle $A_1A_2A_k$ to divide the $(n+1)$-sided polygon $A_1A_2A_3\ \ldots\ A_{n+1}$ into two sub-polygons. The triangulations for the sub-polygons then figure into the recursion relation for $P_{n+1}$. In §II Lamé changes his point of view, and uses instead the diagonal $A_1A_k$ to divide the $n$-gon $A_1A_2A_3\ \ldots\ A_n$ into two sub-polygons. Let's examine the consequences of using a diagonal instead of a triangle to divide the polygon.

**Exercise 2.7.** Consider a regular pentagon $A_1A_2A_3A_4A_5$. Using diagonal $A_1A_3$ to split the pentagon into two figures, how many resulting triangulations of the original pentagon are there? Let $T_1$ denote the set of these triangulations. Using diagonal $A_1A_4$, how many triangulations of the pentagon are there? Let $T_2$ denote the set of these triangulations. Compute the sum of the cardinality (the number of elements) of $T_1$ and $T_2$, and let $S_5$ denote this value. How does $S_5$ compare to $P_5$? Are all elements of $T_1$ and $T_2$ distinct? Is every possible triangulation of the original pentagon an element of $T_1 \cup T_2$? Justify your answers.

**Exercise 2.8.** Consider a regular hexagon $A_1A_2A_3\ \ldots\ A_6$. Let $T_1$ be the set of all triangulations of the hexagon that are formed using the diagonal $A_1A_3$. Let $T_2$ be the set of triangulations of the hexagon using $A_1A_4$, and $T_3$ the set of triangulations using $A_1A_5$. Compute

$$S_6 = |T_1| + |T_2| + |T_3|\,,$$

where $|T_i|$ denotes the cardinality of $T_i$. Do you recognize this sum? How does $S_6$ compare to $P_6$? Are all elements of $T_1$ and $T_2$ distinct? Is every triangulation of the original hexagon an element of $T_1 \cup T_2 \cup T_3$? Justify your answers.

**Exercise 2.9.** Consider now a regular $n$-gon $A_1A_2A_3\ \ldots\ A_{n-1}A_n$. Let $T_i$ be the set of all triangulations of the $n$-gon which are formed using the diagonal $A_1A_{i+2}$ for $i = 1, 2, 3, \ldots, n-3$. Write an algebraic expression for

$$S_n = |T_1| + |T_2| + |T_3| + \cdots + |T_{n-3}|$$

in terms of the $P_k$'s. How does $S_n$ compare to $P_n$? Are the sets $T_1$ and $T_2$ disjoint? Is every triangulation of the original $n$-gon an element of

$$T_1 \cup T_2 \cup T_3 \cup\ \cdots\ \cup T_{n-3}\,?$$

Justify your answer.

Let's now read from §II of Lamé's letter.

## II.

Let $abcde\ldots$ be a polygon of $n$ sides. To each of the $n-3$ diagonals, which end at one of the vertices $a$, there will correspond a group of decompositions, for which this diagonal will serve as the side of two adjacent triangles: to the first diagonal $ac$ corresponds the group $P_3P_{n-1}$; to the second $ad$ corresponds $P_4P_{n-2}$; to the third $ae$, $P_5P_{n-3}$, and so forth until the last $ax$, which will occur in the group $P_3P_{n-1}$. These groups are not totally different, because it is easy to see that some of the partial decompositions, belonging to one of them, is also found in the preceding ones. Moreover they do not include the partial decompositions of $P_n$ in which none of the diagonals ending in $a$ occurs.

**Exercise 2.10.** In the above statement, what groups is Lamé referring to by "[T]hese groups are not totally different"? What notation have we used for "diagonals ending in $a$"?

Lamé's use of diagonals leads to an enumeration of triangulations which is neither one-to-one nor inclusive of all triangulations. His genius, however, was to slightly alter this strategy to first include all triangulations, and then to count how many times a generic triangulation occurs. Combined with the results of §I, this results in a streamlined computation for $P_n$.

**Exercise 2.11.** Returning to pentagon $A_1A_2A_3A_4A_5$, recall that $S_5$ counts with certain repetitions the number of triangulations arising from diagonals $A_1A_3$ and $A_1A_4$. How many triangulations, counting possible repetitions, would occur if diagonals $A_2A_4$ and $A_2A_5$ are used? Denote this number by $S_5^{(2)}$. How many triangulations, counting possible repetitions, would occur if diagonals $A_3A_5$ and $A_3A_1$ are used? Denote this number by $S_5^{(3)}$. Similarly, let $S_5^{(4)}$ denote the number of triangulations with repetition formed by diagonals $A_4A_1$ and $A_4A_2$. Compute $S_5^{(4)}$. What diagonals would be used to define $S_5^{(5)}$? Compute $S_5^{(5)}$ and $\sum_{i=1}^{5} S_5^{(i)}$, where $S_5^{(1)} = S_5$.

**Exercise 2.12.** Let $\mathcal{T}$ be an arbitrary triangulation of the pentagon. Must $\mathcal{T}$ be included in the count

$$S_5^{(1)} + S_5^{(2)} + S_5^{(3)} + S_5^{(4)} + S_5^{(5)}\ ?$$

Justify your answer. How many times does $\mathcal{T}$ occur in the sum $\sum_{i=1}^{5} S_5^{(i)}$? Why? Use this number of duplications to find integers $K$ and $L$ with

$$K \cdot S_5 = L \cdot P_5$$

and justify your answer.

**Exercise 2.13.** For the hexagon, consider numbers $S_6^{(1)}, S_6^{(2)}, \ldots, S_6^{(6)}$ defined similarly. Based on the number of times a generic triangulation is counted in $\sum_{i=1}^{6} S_6^{(i)}$, find integers $K$ and $L$ with

$$K \cdot S_6 = L \cdot P_6\,,$$

and justify your answer.

**Exercise 2.14.** For a regular $n$-gon, find integers $K$ and $L$ with

$$K \cdot S_n = L \cdot P_n \,,$$

where $L$ indicates the number of times a fixed triangulation occurs in the count $\sum_{i=1}^{n} S_n^{(i)}$.

Lamé continues:

But if one does the same for each of the other vertices of the polygon, and combines all the sums of the groups of these vertices, by their total sum $n\,(P_3P_{n-1} + P_4P_{n-2} + \cdots + P_{n-2}P_4 + P_{n-1}P_3)$ one will be certain to include all the partial decompositions of $P_n$; each of these is itself repeated therein a certain number of times.

Indeed, if one imagines an arbitrary such decomposition, it contains $n-2$ triangles, having altogether $3n-6$ sides; if one removes from this number the $n$ sides of the polygon, and takes half of the remainder, which is $n-3$, one will have the number of diagonals appearing in the given decomposition. Now, it is clear that this partial decomposition is repeated, in the preceding total sum, as many times as these $n-3$ diagonals have ends, that is $2n-6$ times: since each end is a vertex of the polygon, and in evaluating the groups of this vertex, the diagonal furnished a group including the particular partial decomposition under consideration.

Thus, since each of the partial decompositions of the total group $P_n$ is repeated $2n-6$ times in $n\,(P_3P_{n-1} + P_4P_{n-2} + \cdots + P_{n-2}P_4 + P_{n-1}P_3)$, one obtains $P_n$ upon dividing this sum by $2n-6$. Therefore one has

$$P_n = \frac{n\,(P_3P_{n-1} + P_4P_{n-2} + \cdots + P_{n-2}P_4 + P_{n-1}P_3)}{2n-6}. \tag{2}$$

**Exercise 2.15.** In equation (2) identify terms which play the role of $S_n$, $K$ and $L$.

Lamé's equations (1) and (2) thus represent two strategies for computing $P_n$, although each equation is itself a recursion relation requiring the value of $P_3$, $P_4$, …, $P_{n-1}$ to compute $P_n$. Can these two equations be combined to solve for $P_n$ directly?

**Exercise 2.16.** Find a fraction $N$ with $P_{n+1} = NP_n$. Be sure to carefully justify your work. Use this equation to find a fraction $N'$ with $P_{n+1} = N'P_{n-1}$.

Lamé concludes:

III.

The first formula (1) gives

$$P_3P_{n-1} + P_4P_{n-2} + \cdots + P_{n-2}P_4 + P_{n-1}P_3 = P_{n+1} - 2P_n,$$

and the second (2) gives

$$P_3P_{n-1} + P_4P_{n-2} + \cdots + P_{n-2}P_4 + P_{n-1}P_3 = \frac{2n-6}{n}P_n;$$

so finally

$$P_{n+1} - 2P_n = \frac{2n-6}{n}P_n,$$

or

$$P_{n+1} = \frac{4n-6}{n}P_n. \tag{3}$$

This is what was to be proven.

Paris, 25 August, 1838.

## 3 A Modern Formula

**Exercise 3.1.** Let $P_2 = 1$. Using the simple recursion relation $P_{n+1} = NP_n$, explain why

$$P_3 = \frac{2}{2}\,P_2$$
$$P_4 = \frac{2}{2}\cdot\frac{6}{3}\,P_2$$

Find an integer $M$ with

$$P_5 = \frac{2\cdot 6\cdot M}{2\cdot 3\cdot 4}\,P_2.$$

Letting $M_1 = 2$ and $M_2 = 6$, find integers $M_3$, $M_4$, $M_5$, …, $M_{n-1}$ with

$$P_{n+1} = \frac{M_1\cdot M_2\cdot M_3\cdot M_4\,\cdots\,M_{n-1}}{2\cdot 3\cdot 4\cdot 5\,\cdots\,n}\,P_2.$$

We have $n! = 2\cdot 3\cdot 4\cdot 5\,\cdots\,n$. Factor a 2 from each $M_i$ to write

$$P_{n+1} = 2^x\,\frac{(M_1/2)(M_2/2)\,\cdots\,(M_{n-1}/2)}{n!}\,P_2.$$

What is the value of $x$? Do you recognize the product

$$\prod_{i=1}^{n-1}(M_i/2) = (M_1/2)(M_2/2)(M_3/2)\,\cdots\,(M_{n-1}/2)\,?$$

What is missing from $\prod_{i=1}^{n-1}(M_i/2)$ to form a factorial? Include the missing terms in both the numerator and denominator to write

$$P_{n+1} = \frac{(2n-2)!}{D_1\,n!}\,P_2\,,$$

for some integer $D_1$. Find an integer $D_2$ with

$$P_{n+1} = \frac{1}{D_2}\binom{2n-2}{n-1} P_2.$$

Justify your answer via a direct argument.

**Exercise 3.2.** Use the above equation for $P_{n+1}$ to compute $P_{10}$ and compare this to Exercise (2.6).

## References


[1] Barnett J., Bezhanishvili G., Leung H., Lodder J., Pengelley D., Ranjan D., *Teaching Discrete Mathematics via Primary Historical Sources*, `http://www.math.nmsu.edu/hist_projects/`.

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[3] Euler, L., *Leonhard Euler und Christian Goldbach, Briefwechsel 1729–1764*, Juskevic, A. P., Winter, E. (eds.), Akademie Verlag, Berlin, 1965.

[4] Euler, L., *Novi Commentarii Academiae Scientarium Imperialis Petropolitanque* **7** (1758–59), p. 9–28.

[5] Gillispie, C. C., Holmes, F. L. (eds.) *Dictionary of Scientific Biography*, Vol. VII, Scribner, New York, 1970.

[6] Koshy, T. *Catalan Numbers with Applications* Oxford University Press, New York, 2009.

[7] Lamé, G., "Un polygone convexe étant donné, de combien de manières peut-on le partager en triangles au moyen de diagonales?," *Journal de Mathématiques Pures et Appliquées*, **3** (1838), p. 505–507.

[8] Segner, A. "Enumeratio Modorum Quibus Figurae Planae Rectilinae per Diagonales Dividuntur in Triangula", *Novi Commentarii Academiae Scientarium Imperialis Petropolitanque* **7** (1758-59), p. 203–209.

## Notes to the Instructor

This project contains original source material from Gabriel Lamé's 1838 publication "Given a convex polygon, in how many ways can one partition it into triangles by means of diagonals?" [7]. The paper, written as a letter to Joseph Liouville, develops a clever and highly original method for counting the number of triangulations of a convex polygon, yielding what today is called the sequence of Catalan numbers. Catalan's own derivation of these numbers, however, is somewhat difficult to follow. Lamé's method relies on an averaging argument over certain symmetries of a (regular) polygon. The project offers engaging material for an upper-divisional course in combinatorics or discrete mathematics. Section two of the project carefully leads the reader through Lamé's argument with several student exercises. This section closes with Lamé's simple recursion relation for the number of triangulations of a convex polygon. The third section offers a formulation of these numbers in terms of binomial coefficients. A prerequisite for the project is an introductory course in discrete mathematics covering binomial coefficients and the concept of a one-to-one correspondence. If the project is covered in its entirety, allow about three weeks.

A version of this curricular module for a computer science course has been written by Desh Ranjan [1, 2] in which the running times for Lamé's equations (1) and (3) are compared. Both the programming version and this mathematical version share the same introduction, co-authored by Desh Ranjan and Jerry Lodder. The Catalan numbers occur quite naturally in other enumeration problems, such as counting the number of rooted, binary planar trees. Every triangulation of a convex polygon corresponds to such a binary tree and vice versa. Although this correspondence is not developed in the project, it could be discussed in class. For a study of other applications of these numbers, see the text by Koshy [6].