Computing the Determinant Through the Looking Glass

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1 Introduction

The question of how to solve systems of linear equations has been in existence for several thousand years. Ancient Babylonian tablets contain examples of problems that can be described with linear equations. One of the earliest examples of using matrices for solving systems of linear equations can be found in the book *Nine Chapters on the Mathematical Art* [3] written between 200 and 100 BC during the Han Dynasty in China. The problem states:

There are three types of corn, of which three bundles of the first, two of the second, and one of the third make 39 measures. Two of the first, three of the second and one of the third make 34 measures. And one of the first, two of the second and three of the third make 26 measures. How many measures of corn are contained in one bundle of each type?

Exercise 1.1 Set up a system of linear equations based on this problem from the *Nine Chapters on the Mathematical Art*.

The text then shows three columns set up on a counting board (a tool for mathematical calculation) in the following manner:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \\ 26 & 34 & 39 \end{array}$$

Exercise 1.2 Write the equations you found in Exercise 1.1 as an augmented matrix. How does your matrix compare with the numbers on the counting board?

Exercise 1.3 Use row operations to get your augmented matrix from Exercise 1.2 into row-echelon form and solve the system of equations.

What is remarkable about the *Nine Chapters on the Mathematical Art* text is that the author gives instructions to multiply and add columns in such a way that the numbers on the counting board are reduced to:

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0	0	3
0	5	2
36	1	1
99	24	39

Exercise 1.4 Notice that the methods used in the *Nine Chapters* are very similar to our modern approach to solving a system of equations. Verify that the solution that you obtained in Exercise 1.3 is the same as the solution obtained when solving the equations given by the columns on the counting board shown below Exercise 1.3.

One of the earliest European examples (1559) of solving a system of simultaneous linear equations can be found in the *Logistica (Arithmetic)* of the French writer Jean Borrel (1492-1572) (all Borrel translations taken from Stedall [8]). His problem states:

To find three numbers of which the first with the third of the rest makes 14. The second with a quarter of the rest makes 8. Likewise the third with the fifth part of the rest makes 8.

Exercise 1.5 Translate Borrel's language into a system of simultaneous linear equations using the variables A, B and C.

Note that in the text below Borrel is using the symbol [for an equals sign. He continues the solution to the problem in the following way:

To find three numbers of which the first with the third of the rest makes 14. The second with a quarter of the rest makes 8. Likewise the third with the fifth part of the rest makes 8.

Put the first to be 1A, the second 1B, the third 1C. Therefore it will be that 1A, $\frac{1}{3}$B, $\frac{1}{3}$C [14. Likewise, 1B, $\frac{1}{4}$A, $\frac{1}{4}$C [8. And also 1C, $\frac{1}{5}$A, $\frac{1}{5}$B [8. Moreover, having made a second equation from these, you will have the first, second, and third, as I have put here.

3A	1B	1C	[42	1ST
1A	4B	1C	[32	2ND
1A	1B	5C	[40	3RD

From these three equations others are made, by multiplication, or by adding to each other, until by subtracting the smaller from the greater there remains a quantity of only one symbol, which is done in this way. Multiply the second equation by 3, it makes 3A, 12B, 3C [96. Take away the first, there remains 11B, 2C [54.

3A	12B	3C	[96
3A	1B	1C	[42
	11B	2C	[54

Again multiply the third equation by 3, it makes 3A, 3B, 15C [120. Take away the first, there remains 2B, 14C [78.

3A	3B	15C	[120
3A	1B	1C	[42
	2B	14C	[78

Multiply by 11, it makes 22B, 154C [858. Likewise multiply 11B, 2C [54 by 2, it makes 22B, 4C [108. Take that from 22B, 154C [858, there remains 150C [750]. Divide by 150, there comes 5, which is the number of all of C.

22B	154C	[858
22B	4C	[108
	150C	[750

Since now you will have 1C worth 5, from the equation which is 2B, 14C [78, take 14C, that is 70, it leaves a remainder 8, which is worth 2B, therefore 4 is the second number B. Moreover, so that you have the first from the equation where the number of the total is 40, subtract 5C, and 1B, that is, 29 and it leaves a remainder of 11, which is the first number A. Therefore the three numbers are 11, 4, 5, which were to be found.

Exercise 1.6 Verify that Borrel's solution is correct.

2 Solving Equations Using Determinants

The method that we use today for solving systems of linear equations developed slowly throughout the 17th, 18th and 19th centuries. The first discussion of the connection between the value of the determinant and the solution of a system of linear equations was in a letter that Gottfried Wilhelm von Leibniz (1646 - 1716) sent to Guillaume François Antoine Marquis de L'Hôpital (1661 - 1704) in 1683. In that letter Leibniz uses a computation that resembles a determinant as a basis for his claim that a particular system of linear equations has a solution.

In his 1815 paper *"Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des transpositions opérées entre les variables qu'elles renferment"* (Functions That Only Have Equal and Opposite Values as a Result of Transpositions between their Variables) the French mathematician Augustin Louis Cauchy (1789 – 1857) gives a definition of what we now call determinants. It was Cauchy who attached the word determinant to the computation. He writes (Cauchy translations from Stedall [8]):

I shall now examine in particular a certain kind of alternating symmetric functions which present themselves in a great number of analytic investigations. It is by means of these functions that one expresses the general values of unknowns contained in several equations of the first degree. They appear whenever one needs to form conditional equations, thus as in the general theory of elimination. Messieurs Laplace and Vandermonde have considered them in this respect in the *Memoires of the Academy of [S]ciences* (1772), and Monsieur Bezout has since examined them again from the same point of view in his *Theory of [E]quations*. Monsieur Gauss has made use of them with advantage in his analytic investigations, to discover the general properties of forms of second degree, that is to say, polynomials of second degree with two or several variables; and he has denoted these same functions by the name determinants. I will keep this name which supplies an easy way of stating the results.

Cauchy states that Gauss' use of the word "determinant" is not in the context of linear equations since Gauss allows for second degree terms. However because of some of the similarities, Cauchy adopts the word determinant for his own process.

Recall that the system of equations:

$$\begin{aligned} ax_1 + bx_2 &= u_1 \\ cx_1 + dx_2 &= u_2 \end{aligned}$$

can be written as the matrix equation $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ which we denote as $\mathbf{Ax} = \mathbf{u}$. $\mathbf{Ax} = \mathbf{u}$ has a solution if and only if the matrix $\mathbf{A}$ is invertible and the solution is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{u}$. By the time that Cauchy wrote his 1815 paper it was understood that a matrix is invertible if the determinant of the matrix is non-zero.

In *"Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des transpositions opérées entre les variables qu'elles renferment"* Cauchy uses row and column notation for the positions in a matrix. In particular an $n \times n$ matrix is given by:

$$\begin{bmatrix} a_{1.1}, & a_{1.2}, & a_{1.3}, & \cdots & a_{1.n}, \\ a_{2.1}, & a_{2.2}, & a_{2.3}, & \cdots & a_{2.n}, \\ a_{3.1}, & a_{3.2}, & a_{3.3}, & \cdots & a_{3.n}, \\ etc. & & & \cdots & \\ a_{n.1}, & a_{n.2}, & a_{n.3}, & \cdots & a_{n.n}, \end{bmatrix}$$

where $a_{i,j}$ is the entry in row i and column j. He denotes a determinant by $S(\pm a_{1.1}, a_{2.2}, \ldots.., a_{n.n})$. Cauchy says:

I shall denote in what follows by the name of determinants. If one supposes successively $n = 1, n = 2$, etc..., one will have

$$\begin{aligned} S(\pm a_{1.1}, a_{2.2}) &= a_{1.1}a_{2.2} - a_{2.1}a_{1.2} \\ S(\pm a_{1.1}, a_{2.2}, a_{3.3}) &= a_{1.1}a_{2.2}a_{3.3} + a_{2.1}a_{3.2}a_{1.3} + a_{3.1}a_{1.2}a_{2.3} \\ & \quad -a_{1.1}a_{3.2}a_{2.3} - a_{3.1}a_{2.2}a_{1.3} - a_{2.1}a_{1.2}a_{3.3} \end{aligned}$$

etc.... for the determinants of second, and third order, etc. The quantities affected by different indices being considered in general as unequal, one sees that the determinant of second order contains four different quantities, namely,

$$\begin{matrix} a_{1.1}, & a_{1.2}, \\ a_{2.1}, & a_{2.2}, \end{matrix}$$

and that the determinant of third order contains nine, that is,

$$\begin{matrix} a_{1.1}, & a_{1.2}, & a_{1.3}, \\ a_{2.1}, & a_{2.2}, & a_{2.3}, \\ a_{3.1}, & a_{3.2}, & a_{3.3}, \end{matrix}$$

In general, the determinant of nth order, or

$$S(\pm a_{1.1}, a_{2.2}, \ldots., a_{n.n}),$$

will contain a number n^2 of different quantities, which will be respectively

$$\begin{array}{lllll} a_{1.1}, & a_{1.2}, & a_{1.3}, & \cdots & a_{1.n}, \\ a_{2.1}, & a_{2.2}, & a_{2.3}, & \cdots & a_{2.n}, \\ a_{3.1}, & a_{3.2}, & a_{3.3}, & \cdots & a_{3.n}, \\ etc. & & & \cdots & \\ a_{n.1}, & a_{n.2}, & a_{n.3}, & \cdots & a_{n.n}, \end{array}$$

Exercise 2.1 Use Cauchy's definition to find the determinants of the following:

a. $\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}$

b. $\begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix}$

c. $\begin{bmatrix} -2 & 3 & 0 \\ 1 & 2 & -1 \\ -2 & 3 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 5 & 0 & 4 \\ 1 & 2 & -1 \\ -1 & -2 & 1 \end{bmatrix}$

e. $\begin{bmatrix} 5 & 3 & 4 \\ 1 & 2 & -1 \\ -2 & 3 & 1 \end{bmatrix}$

Exercise 2.2 For each of the matrices in Exercise 2.1 state whether or not it is invertible and explain why.

Exercise 2.3 Try to determine how many terms would be involved in using Cauchy's method for computing the determinant for a 4x4 matrix.

What quickly becomes clear is that the method described by Cauchy involves a large number of multiplications and it would be helpful to have a system for locating all of the necessary combinations. Cauchy does offer a way to calculate determinants in general, but it is quite different from how we calculate them today.

3 Who Was Charles Dodgson?

Charles Lutwidge Dodgson (1832-1898) was born into a clerical family in England. His father Reverend Charles Dodgson was born in 1800 and studied mathematics at the University of Oxford. After graduation Rev. Dodgson became a Mathematics Lecturer and Fellow at Oxford. Because it was expected that university fellows remain single, when Rev. Dodgson married Frances Jane Lutwidge in 1827, he had to give up his position at Oxford. Rev. Dodgson then became a curate at All Saints' Church in Daresbury and it was in the manse at Daresbury that Charles Lutwidge Dodgson and nine of his ten brothers and sisters were born.

Charles L. Dodgson's early education was provided by his parents and from a young age he showed an aptitude for his father's favorite subject, mathematics. As an older child, Charles L. was sent to boarding school but because he had a stammer and was considered of "delicate health," his years in school were socially difficult. However he excelled at his school work, particularly in the areas of mathematics and divinity.

In 1851 Charles L. entered the University of Oxford. By 1852 he had earned degrees in mathematics and classics. Charles L. was soon awarded a Fellowship at Christ Church College in Oxford. With this Oxford University Fellowship came the right to live at Christ Church, and the expectation that he would remain unmarried and be ordained in the Church of England. In 1855 Dodgson became a Mathematics Lecturer at Oxford and in 1861 he took deacon's orders, though he never became an Anglican priest.

Sometime around 1856 Dodgson became interested in photography and purchased a camera. He began taking photos of people, particularly the children of family and friends. Among those he photographed were the children of the writer George Macdonald and the sons of the poet Alfred Lord Tennyson. He also took a number of photographs of the three daughters of Henry George Liddell, the Dean of Christ Church.

Dodgson also enjoyed telling stories to children and in 1862 Dean Liddell's daughter Alice persuaded Dodgson to write down and illustrate some of his stories. A few years later, a friend of the Liddell family convinced Dodgson to publish the stories with their illustrations. Charles Dodgson is better known by his pen name Lewis Carroll and although he was a mathematician, his most famous works are *Alice's Adventures in Wonderland* (1865) and *Through the Looking Glass* (1872).

Dodgson did publish a number of mathematical books during his years at Oxford, but none of them are seen as ground breaking in the field of mathematics. Several of his texts were aimed at helping students to prepare for the Oxford University mathematics graduation exams and these books contain some interesting insights into how to approach specific problems. It is to one of those books, *Elementary Treatise on Determinants with the Applications to Simultaneous Linear Equations and Algebraical Geometry* (1867) [1], that we now turn our attention.

4 Dodgson's Determinant Computation: The "Method of Condensation"

Dodgson had a significant interest in geometry and spent time studying Euclid's *Elements*. He was also interested in algebraic geometry (what we now call analytic geometry), which is the use of algebraic methods to find solutions to geometric problems. For example, solving the system of two linear equations

$$\begin{aligned} 4x + 2y &= 4 \\ 2x + -1y &= 1 \end{aligned}$$

yields the point $(\mathbf{x}, \mathbf{y}) = (\frac{3}{4}, \frac{1}{2})$ where the two lines intersect. Algebraic geometry was one of the topics on the Oxford leaving exams in mathematics and Dodgson's interest in determinants seems to have originated in his interest in solving systems of linear equations.

Dodgson begins *Elementary Treatise on Determinants with the Applications to Simultaneous Linear Equations and Algebraical Geometry* by including a number of well-known definitions related to matrices. Dodgson indicates in his preface that he has drawn on the work of other mathematicians including Isaac Todhunter (1820-1884). In 1861, Todhunter published *An Elementary Treatise on the Theory of Equations* [9] and many of Dodgson's preliminary definitions and methods are similar to those in Todhunter. In the definitions, propositions and corollaries that follow, it is important to note that Dodgson calls a matrix a Block and that his footnotes are essential in clarifying information.

Definition V.

If, in a given Block [matrix], any rows, and as many columns, be selected: the square Block formed of their common Elements is called a Minor of the given Block. Hence any single Element of a Block, being common to one row and one column, is a Minor of it.

Footnote for Definition V:

Thus, in the Block $\begin{Bmatrix} d & b & m & s \\ f & c & g & d \\ e & h & r & l \end{Bmatrix}$, if we select the 2nd and 3rd rows, and the 2nd and 4th columns, we obtain the Minor $\begin{Bmatrix} c & d \\ h & l \end{Bmatrix}$.

Definition VI.

If n be that dimension of a Block which is not greater than the other: its Minors of the nth degree are called its principal Minors; those of the (n -1)th degree its secondary Minors, and so on. Hence a square Block is its own principal Minor.

Footnote for Definition VI:

Thus, in the same Block [in definition V], the Minors $\begin{Bmatrix} d & b & m \\ f & c & g \\ e & h & r \end{Bmatrix}$, $\begin{Bmatrix} d & m & s \\ f & g & d \\ e & r & l \end{Bmatrix}$, &c., are principal Minors; while $\begin{Bmatrix} d & m \\ f & g \end{Bmatrix}$, $\begin{Bmatrix} d & s \\ f & d \end{Bmatrix}$, $\begin{Bmatrix} b & s \\ h & l \end{Bmatrix}$, &c., are secondary Minors.

Definition VII.

If, in a square Block, any rows, and as many columns, be selected: the Minor formed of their common Elements, and the Minor formed of the Elements common to the other rows and columns, are said to be complemental to each other.

Footnote for Definition VII:

Thus, in the Block $\begin{Bmatrix} b & g & h & r \\ c & l & t & v \\ d & m & f & e \\ a & s & x & q \end{Bmatrix}$, the Minors $\begin{Bmatrix} b & g \\ c & l \end{Bmatrix}$ and $\begin{Bmatrix} f & e \\ x & q \end{Bmatrix}$ are complemental to each other; as also are the Minors $\begin{Bmatrix} c & v \\ a & q \end{Bmatrix}$ and $\begin{Bmatrix} g & h \\ m & f \end{Bmatrix}$. Thus, again, the single Element f and the Minor $\begin{Bmatrix} b & g & r \\ c & l & v \\ a & s & q \end{Bmatrix}$ are complemental to each other.

Exercise 3.1 For the matrix $\begin{bmatrix} -2 & 3 & 0 \\ 1 & 2 & -1 \\ -2 & 3 & 1 \end{bmatrix}$ identify two different minors and their complements.

Next Dodgson provides a definition of a determinant that involves minors. Again, he clarifies what he means in the footnotes. Proposition I defines a determinant. Note that his notation $2\backslash 3$ indicates the element in the 2nd row and 3rd column, what we would write as $a_{2,3}$.

Proposition I.

The determinantal coefficient of any Element of a square Block is the Determinant of its complemental Minor, affected with $+$ or $-$ according as the numerals which constitute its symbol are similar or dissimilar.

Here the word "similar" is used to mean that both terms are even or both terms are odd. Dissimilar means that one term is even and one is odd. For example $2\backslash 4$ and $3\backslash 3$ are similar and $3\backslash 4$ is dissimilar. The footnote below is essential to understanding the proposition.

Footnote for Proposition I:

Thus the Determinant of the Block $\begin{Bmatrix} a & b \\ c & d \end{Bmatrix}$ is $(ad - bc)$; and that of the Block $\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{Bmatrix}$ is $(aek - ahf - bdk + bgf + cdh - cge)$. Here the Determinantal coefficient of e is $(ak - cg)$, i.e. $\begin{vmatrix} a & c \\ g & k \end{vmatrix}$; and as e corresponds to the symbol $2\backslash 2$, the numerals of which are similar, the sign of this Determinant ought to be $+$, and so we find it. Again, the Determinantal coefficient of f is $(-ah + bg)$, i.e. $-\begin{vmatrix} a & b \\ g & h \end{vmatrix}$; and as f corresponds to the symbol $2\backslash 3$, the numerals of which are dissimilar, the sign of this Determinant ought to be –, and so we find it.

The corollary to Proposition I gives an explanation of the computation.

Corollary 1 to Proposition I.

If, in a square Block, any row, or column, be selected: the Determinant of the Block may be resolved into terms, each consisting of one of the Elements of that row, or column, multiplied by the Determinant of its complemental Minor.

Footnote for Corollary 1:

This gives us a simple method for computing the value of a Determinant arithmetically. Thus,

$$\begin{vmatrix} 3 & 1 & 2 & 4 \\ 4 & 5 & 2 & 3 \\ 3 & 1 & 3 & 2 \\ 4 & 2 & 1 & 3 \end{vmatrix} = 3\begin{vmatrix} 5 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 4 & 2 & 3 \\ 3 & 3 & 2 \\ 4 & 1 & 3 \end{vmatrix} + 2\begin{vmatrix} 4 & 5 & 3 \\ 3 & 1 & 2 \\ 4 & 2 & 3 \end{vmatrix} - 4\begin{vmatrix} 4 & 5 & 2 \\ 3 & 1 & 3 \\ 4 & 2 & 1 \end{vmatrix}$$

$$= 3\times \left\{ 5\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} \right\} - \&c. = 3 \times \{35 + 2 - 15\} - \&c.$$

$$= 3 \times 22 - \&c. = 66 - \&c.$$

Exercise 3.2 Use Dodgson's definition to find the determinants via minors of the matrices (note b and c appear as part of his footnote for the Corollary to Proposition I):

a. $\begin{bmatrix} -2 & 3 & 0 \\ 1 & 2 & -1 \\ -2 & 3 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 4 & 2 & 3 \\ 3 & 3 & 2 \\ 4 & 1 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 4 & 5 & 3 \\ 3 & 1 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

Exercise 3.3 Now finish his computations for the 4x4 matrix in his footnotes for Proposition I, Corollary 1, and get a numerical value for that determinant.

As you can see from Exercise 3.3, even using this systematic method for computing determinants can be complicated because of the number of minors that must be computed to get down to 2x2 matrices whose determinants are easy to compute.

Based on his book *Elementary Treatise on Determinants with the Applications to Simultaneous Linear Equations and Algebraical Geometry*, Dodgson published a short paper "Condensation of Determinants, Being a New and Brief Method for Computing their Arithmetical Values" [2]. This paper was published in the 1866-67 edition of the *Proceedings of the Royal Society,* and it is that paper that we will examine for an explanation of Dodgson's method. We begin with his introduction:

If it be proposed to solve a set of n simultaneous linear equations, not being all homogeneous, involving n unknowns, or to test their compatibility when all are homogeneous, by the method of determinants, in these, as well as in other cases of common occurrence, it is necessary to compute the arithmetical values of one or more determinants - such, for example, as

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 1 & 4 \\ 3 & 5 & -1 \end{bmatrix}$$

Now the only method, so far as I am aware, that has been hitherto employed for such a purpose, is that of multiplying each term of the first row or column by the determinant of its complemental minor, and affecting the products with the signs + and − alternately, the determinants required in the process being, in their turn, broken up in the same manner until determinants are finally arrived at sufficiently small for mental computation.

Dodgson is describing in words Cauchy's system for computing determinants. For the example given above, the calculation would be:

$$\begin{vmatrix} 1 & 3 & -2 \\ 2 & 1 & 4 \\ 3 & -5 & 1 \end{vmatrix} = 1 \times \begin{vmatrix} 1 & 4 \\ 5 & -1 \end{vmatrix} - 3 \times \begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} + -2\times \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = -21 + 42 - 14 = 7$$

Dodgson however recognizes the obvious problem with this method and says:

But such a process, when the block consists of 16, 25, or more terms, is so tedious that the [new] method of elimination is much to be preferred for solving simultaneous equations; so that the [old] method, excepting for equations containing 2 or 3 unknowns, is practically useless.

The new method of computation, which I now proceed to explain, and for which "Condensation" appears to be an appropriate name, will be found, I believe, to be far shorter and simpler than any hitherto employed.

In the following remarks I shall use the word "Block" to denote any number of terms arranged in rows and columns, and "interior of a block" to denote the block which remains when the first and last rows and columns are erased.

The process of "Condensation" is exhibited in the following rules, in which the given block is supposed to consist of n rows and n columns:

(1) Arrange the given block, if necessary, so that no ciphers [0's] occur in its interior. This may be done either by transposing rows or columns, or by adding to certain rows the several terms of other rows multiplied by certain multipliers.

(2) Compute the determinant of every minor consisting of four adjacent terms. These values will constitute a second block, consisting of n-1 rows and n-1 columns.

(3) Condense this second block in the same manner, dividing each term, when found, by the corresponding term in the interior of the first block.

(4) Repeat this process as often as may be necessary (observing that in condensing any block of the series, the rth for example, the terms so found must be divided by the corresponding terms in the interior of the (r-1)th block), until the block is condensed to a single term, which will be the required value.

Dodgson's algorithmic description of his method is not clear. Robin Wilson, in his book *Lewis Carroll in Numberland: His Fantastical Mathematical Logical Life* [10], gives a nice step by step example of Dodgson's method for a 3×3 matrix. Here is that example.

Suppose that we wish to calculate the determinant of the following matrix (the reason for the bolding the central 2 will soon become clear):

$$\begin{vmatrix} 1 & 4 & 2 \\ 1 & \mathbf{2} & 3 \\ 1 & 1 & 1 \end{vmatrix}$$

We first calculate the 2×2 determinant in each of the four sub-matrices in the corners of the matrix. The two upper sub-matrices and their determinants are:

$$\begin{vmatrix} 1 & 4 \\ 1 & \mathbf{2} \end{vmatrix} = -2, \begin{vmatrix} 4 & 2 \\ \mathbf{2} & 3 \end{vmatrix} = 8.$$

The two lower sub-matrices and their determinants are:

$$\begin{vmatrix} 1 & \mathbf{2} \\ 1 & 1 \end{vmatrix} = -1, \begin{vmatrix} \mathbf{2} & 3 \\ 1 & 1 \end{vmatrix} = -1.$$

Then we write down the 2×2 matrix containing the determinants from the four sub-matrices and take the determinant of the resultant matrix:

$$\begin{vmatrix} -2 & 8 \\ -1 & -1 \end{vmatrix} = 10$$

Finally, we divide the result by the bolded number in the middle of the top matrix and get $\frac{10}{2} = 5$, the correct answer for the determinant of the original matrix.

Exercise 3.4 The example of the algorithm that Dodgson provides in "Condensation of Determinants, Being a New and Brief Method for Computing their Arithmetical Values" [2] is given below. Use Dodgson's description of the algorithm and Wilson's example to assist you in calculating each step of Dodgson's example. Dodgson's intermediate matrices allow you to check your work.

As an instance of the foregoing rules, let us take the block

$$\begin{vmatrix} -2 & -1 & -1 & -4 \\ -1 & -2 & -1 & -6 \\ -1 & -1 & 2 & 4 \\ 2 & 1 & -3 & -8 \end{vmatrix}$$

By rule (2) this is condensed into $\begin{vmatrix} 3 & -1 & 2 \\ -1 & -5 & 8 \\ 1 & 1 & -4 \end{vmatrix}$ this, again, by rule (3), is condensed into $\begin{vmatrix} 8 & -2 \\ -4 & 6 \end{vmatrix}$ and this, by rule (4), into -8, which is the required value.

The simplest method of working this rule appears to be to arrange the series of blocks one under another, as here exhibited; it will then be found very easy to pick out the divisors required in rules (3) and (4).

$$\begin{vmatrix} -2 & -1 & -1 & -4 \\ -1 & -\mathbf{2} & -1 & -6 \\ -1 & -1 & 2 & 4 \\ 2 & 1 & -3 & -8 \end{vmatrix}$$

$$\begin{vmatrix} \mathbf{3} & -1 & 2 \\ -1 & -5 & 8 \\ 1 & 1 & -4 \end{vmatrix}$$

$$\begin{vmatrix} \mathbf{8} & -2 \\ -4 & 6 \end{vmatrix}$$

$$-8$$

Hint: The values in each cell are coming from the computation of 2×2 sub-matrices. For example the bold 3 in the 3×3 matrix above is found by taking the determinant of the 2×2 sub-matrix $\begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$ in the 4×4 matrix. The bold 8 in the 2×2 matrix above is found by taking the determinant of the sub-matrix $\begin{bmatrix} 3 & -1 \\ -1 & -5 \end{bmatrix}$ in the 3×3 matrix. However, to obtain the value 8, this determinant is divided by the bolded -2 in the 4×4 matrix. This is what is described in Dodgson's rule (3).

Exercise 3.5 Try using the method of condensation to find the following determinants.

a. $\begin{bmatrix} -2 & 3 & 0 \\ 1 & 2 & -1 \\ -2 & 3 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 5 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 3 & 1 & 2 & 4 \\ 4 & 5 & 2 & 3 \\ 3 & 1 & 3 & 2 \\ 4 & 2 & 1 & 3 \end{bmatrix}$

Notice in particular how much less work is involved in finding the determinant of (c).

Exercise 3.6 This exercise gives you some insight into why Dodgson's condensation method works. Compare your answers in Exercise 3.5 to your answers in Exercises 3.2 and 3.3. It is clear that you are getting the same answer using different methods. To understand what is occurring, compute the determinant for $\begin{bmatrix} a & b & c \\ u & v & w \\ x & y & z \end{bmatrix}$ using both the condensation method found in Exercise 3.5 and the computation of minor method found in Exercise 3.2. What do you notice about the equations created by each method? What does this tell us about the methods? A proof showing that Dodgson's method works can be found in [7].

Dodgson's condensation method does have problems when encountering 0's (what he calls ciphers) in critical locations. If you look back at the first step of his algorithm:

(1) Arrange the given block, if necessary, so that no ciphers occur in its interior. This may be done either by transposing rows or columns, or by adding to certain rows the several terms of other rows multiplied by certain multipliers.

That takes care of the initial conditions, but what happens when a 0 appears in the middle of the condensation? Here is what Dodgson says:

This process cannot be continued when ciphers occur in the interior of any one of the blocks, since infinite values would be introduced by employing them as divisors. When they occur in the given block itself, it may be rearranged as has been already mentioned; but this cannot be done when they occur in any one of the derived blocks; in such a case the given block must be rearranged as circumstances require, and the operation commenced anew.

The best way of doing this is as follows:

Suppose a cipher to occur in the hth row and kth column of one of the derived blocks (reckoning both row and column from the nearest corner of the block); find the term in the hth row and kth column of the given block (reckoning from the corresponding corner), and transpose rows or columns cyclically until it is left in an outside row or column. When the necessary alterations have been made in the derived blocks, it will be found that the cipher now occurs in an outside row or column, and therefore need no longer be used as a divisor.

The advantage of cyclical transposition is, that most of the terms in the new blocks will have been computed already, and need only be copied; in no case will it be necessary to compute more than one new row or column for each block of the series.

In the following instance it will be seen that in the first series of blocks a cipher occurs in the interior of the third. We therefore abandon the process at that point and begin again, rearranging the given block by transferring the top row to the bottom; and the cipher, when it occurs, is now found in an exterior row. It will be observed that in each block of the new series, there is only one new row to be computed; the other rows are simply copied from the work already done.

$$\begin{vmatrix} 2 & -1 & 2 & 1 & -3 \\ 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & -2 & -1 & -1 \\ 2 & 1 & -1 & -2 & -1 \\ 1 & -2 & -1 & -1 & 2 \end{vmatrix}$$
$$\begin{vmatrix} 5 & -5 & -3 & -1 \\ -3 & -3 & -3 & 3 \\ 3 & 3 & 3 & -1 \\ -5 & -3 & -1 & -5 \end{vmatrix}$$
$$\begin{vmatrix} \mathbf{-15} & 6 & \mathbf{12} \\ 0 & 0 & 6 \\ 6 & -6 & 8 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & -2 & -1 & -1 \\ 2 & 1 & -1 & -2 & -1 \\ 1 & -2 & -1 & -1 & 2 \\ 2 & -1 & 2 & 1 & -3 \end{vmatrix}$$
$$\begin{vmatrix} -3 & -3 & -3 & 3 \\ 3 & 3 & 3 & -1 \\ -5 & -3 & -1 & -5 \\ 3 & -5 & 1 & 1 \end{vmatrix}$$
$$\begin{vmatrix} 0 & 0 & 6 \\ 6 & -6 & 8 \\ -17 & 8 & -4 \end{vmatrix}$$
$$\begin{vmatrix} 0 & 12 \\ 18 & 40 \end{vmatrix}$$
$$36$$

The fact that, whenever ciphers occur in the interior of a derived block, it is necessary to recommence the operation, may be thought a great obstacle to the use of this method; but I believe it will be found in practice that, even though this should occur several times in the course of one operation, the whole amount of labour will still be much less than that involved in the old process of computation.

Note that the two bold numbers in the bottom matrix on the left hand side are corrections of what was in Dodgson's original text which contained an error.

Exercise 3.7 Based on the numerical example above, what does Dodgson mean by "transpose the rows or columns cyclically"?

Exercise 3.8 Walk through both computations in Dodgson's description above. Make sure that you show the 2×2 minor matrices at each step. Mark which 2×2 matrices you are able to reuse in the computation on the right after the matrix has been rearranged.

One question that comes immediately to mind is how do we know that the matrix on the right has the same determinant as the matrix on the left? Dodgson resolves this issue and several other computational issues related to determinants in *Elementary Treatise on Determinants with the Applications to Simultaneous Linear Equations and Algebraical Geometry.* The relevant proposition and corollary are given below.

Proposition II.

If, in a square block, 2 rows, or 2 columns, be interchanged: the Determinant of the new Block has the same absolute value as that of the first, but the opposite sign.

Footnote for Proposition II:

Thus the Determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & l \end{vmatrix} = - \begin{vmatrix} c & b & a \\ f & e & d \\ l & h & g \end{vmatrix}$

Corollary to Proposition II.

If, in a square Block, a row, or a column, be made to pass over the next r rows, or columns, either way: the Determinant of the new Block has the same sign as that of the first, or the opposite sign, according as r is even or odd: that is, it is equal to the Determinant of the first Block multiplied by $(-1)^r$. For this may be effected by interchanging it with each of these r rows, or columns, in turn; and after one such interchange, the sign of the Determinant is changed, after two, it is the same sign again, and so on.

Footnote for the Corollary to Proposition II:

Thus the Determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = - \begin{vmatrix} b & a & c \\ e & d & f \\ h & g & k \end{vmatrix}$, where the first column has been passed over one column: but the same Determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = + \begin{vmatrix} b & c & a \\ e & f & d \\ h & k & g \end{vmatrix}$, where the first column has been passed over two columns.

The corollary tells us that the right side determinant of Dodgson's 5×5 matrix given above is equal to $(-1)^4$ times the determinant show on the left side of the same example. This comes from the four row swaps to get the top row of the original matrix on the left hand side to the bottom row of the starting matrix on the right hand side. Since $(-1)^4 = 1$ the two determinants are the same.

5 Bibliography

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Notes to the Instructor

This project in the computation of determinants was designed to be incorporated into the middle of a sophomore level linear algebra course. The project assumes that students have learned a limited amount about linear systems and their connection to augmented matrices. It also assumes that they have used basic row reduction to solve systems of equations. In addition, students should have seen simple matrix computations and have experimented with computing matrix inverses. By the time that students encounter the material in this project they should know that the system of equations

$$\begin{aligned} ax_1 + bx_2 &= u_1 \\ cx_1 + dx_2 &= u_2 \end{aligned}$$

can be written as the matrix equation $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ which we denote as $\mathbf{Ax} = \mathbf{u}$, that $\mathbf{Ax} = \mathbf{u}$ has a solution if and only if the matrix A is invertible, and the solution is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{u}$. They should also have enough experience to know that computing an inverse can be time consuming and that it would be useful to have an easy test of whether or not a matrix is invertible.

This project is meant to be used as an alternative to traditional textbook approaches to teaching students how to compute the determinant. In the course of completing this project, the students will read Cauchy's explanation of how to compute the determinant (the standard method) as well as learning Dodgson's efficient method of condensation. Most students appreciate the efficiency of Dodgson's rarely taught method and they enjoy the connection with Lewis Carroll.

Lower division students often need to be taught to slow down and read carefully when encountering mathematical texts. One of the goals of this project is to have students wrestle with somewhat unfamiliar language and symbols so that they begin to develop good mathematical habits. These habits include reading the text slowly, reading the text more than once, making marginal notes as they begin to understand the text, looking for gaps, and filling in missing steps. Many of the exercises encourage that type of reading and explicitly ask students to fill in steps.

This project typically takes one week of class time. It can be given out as an individual assignment but is well suited to being used as a small group project. Students appear to be more successful in extracting meaning from unfamiliar mathematical language when working with a partner. It is particularly effective if the class time for the week in which the project is assigned is devoted to group work on the project so that the professor can provide assistance as needed and the class can collectively discuss some of the more challenging parts of the project such as trying to extend Cauchy's method to the 4×4 case (Exercise 2.3). There is generally a vigorous class conversation about the details of the computations in Exercise 3.4 and 3.8 (large condensations).

Certainly one question that arises in working on this project is "Why does Dodgson's method yield the same determinant as the familiar method?" The project does not attempt to answer that question on a theoretical level, but Exercise 3.6 asks students to compute the 3×3 determinant for a matrix of variables using both methods and then compare the algebraic output. "'Shuttling up like a telescope': Lewis Carroll's 'Curious' Condensation Method for Evaluating Determinants," [7] does include a proof and is a good supplemental reading for this project.