Translation of Raffaele Rubini's 1857 'Application of the Theory of Determinants: Note'

Salvatore J. Petrilli, Jr., and Nicole Smolenski
Department of Mathematics & Computer Science
Adelphi University
Garden City, NY 11530

Abstract

Scholars in the Kingdom of Two Sicilies, which united the island Kingdom of Sicily and the mainland Kingdom of Naples (the southernmost regions in modern-day Italy) from 1815 to 1860, were exposed to some of the works of French mathematicians after translations of these works were completed. However, until Francesco Brioschi (1824 – 1897) published his works of algebraic theory, beginning in 1854 with his *Teoria dei determinanti* (*Theory of Determinants*) [O'Connor and Robertson, 2006], these scholars had limited knowledge of algebra, specifically determinants. Their ignorance of this algebraic theory was due not only to the kingdom's geographic isolation, but also to an academic schism between two branches of mathematical thought, synthetic and analytic. This schism had consequences for several Italian mathematicians, including Raffaele Rubini (1817 – 1890), and led Rubini to publish his 1857 article, "Application of the Theory of Determinants: Note." The article, "Analysis and Translation of Raffaele Rubini's 1857 'Application of the Theory of Determinants: Note'," available online in *MAA Convergence,* provides context for and analysis of Rubini's "Note," along with some biographical information about this obscure mathematician and ideas for using his "Note" about determinants in college and high school courses. The present document contains the first English translation of Rubini's article, "Application of the Theory of Determinants: Note."

Translation of Rubini's "Note"

Application of the Theory of Determinants: Note

[179][1] The theory of determinants has, by now, become so important for its concise manner, with which numerous and challenging results can be reached, that it is completely impossible to by pass it in an ordinary Algebra course. I must profess my gratitude to the eminent Professor *Brioschi*,[2] the first in Italy and one of the first in Europe, to have masterfully exposed the main and most important theorems of classical theory, in such a way as to utilize them as a complement to the algebra course, which by virtue of determinants, will certainly change form more than it did with the use of functions. In this note, therefore, we propose to show with some examples how the algorithm of determinants can be useful in the presentation of the theory of equations, and how easily it leads to some formulas, which would be otherwise very difficult to deduce, without making claim to the originality of the subject matter.[3]

1. The determinant is given as

$$P = \begin{vmatrix} a_{1,1}+h_{1,1} & a_{1,2}+h_{1,2} & \dots & a_{1,n}+h_{1,n} \\ a_{2,1}+h_{2,1} & a_{2,2}+h_{2,2} & \dots & a_{2,n}+h_{2,n} \\ \dots & \dots & \dots & \dots \\ a_{n,1}+h_{n,1} & a_{n,2}+h_{n,2} & \dots & a_{n,n}+h_{n,n} \end{vmatrix} \tag{1}$$

and solving it by the determinant of simple elements according to [180] the known rule,[4] we will have:

$$P = M + \Sigma M_1 + \Sigma M_2 + \ldots + \Sigma M_{n-1} + \Sigma M_n, \tag{2}$$

with

$$M = \begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{vmatrix}. \tag{3}$$

In general ΣM_i represents a sum of the determinants, each of which is derived from the previous M, *with changing the corresponding columns of* a *to as many*

[1]The numbers in square brackets represent the original page numbers of Rubini's article (tr.).

[2]Francesco Brioschi, (1824 – 1897). Brioschi brought algebraic thought, including the theory of determinants to Italy (tr.).

[3]At this time it was common in Italian literature for the author to apologize for the insignificance of his work (tr.).

[4]In [Rubini, 1857], the following footnote was given:
Brioschi . . . La teorica dei determinanti e le sue applicazioni; p. 5.
Spottiswoode. – Elementary theorems relating to determinants; p. 13; theorem 1.
See [Brioschi, 1854] and [Spottiswoode, 1851].

Salvatore J. Petrilli, Jr., and Nicole Smolenski, "Analysis and Translation of Raffaele Rubini's 1857 'Application of the Theory of Determinants: Note'," *MAA Convergence* (July 2017): http://www.maa.org/press/periodicals/convergence/analysis-and-translation-of-raffaele-rubinis-1857-application-of-the-theory-of-determinants-note

of h *as* i. Thus these new determinants that enter in ΣM_i and which are, generally speaking, of the n^{th} order, like that M, are, also in general, numerically

$$\frac{n(n-1)(n-2)\dots(n-i+1)}{1\cdot 2\cdot 3\cdot \dots \cdot i}.$$

From this it follows that the last term ΣM_n has just one determinant, that is

$$\Sigma M_n = \begin{vmatrix} h_{1,1} & h_{1,2} & \dots & h_{1,n} \\ h_{2,1} & h_{2,2} & \dots & h_{2,n} \\ \dots & \dots & \dots & \dots \\ h_{n,1} & h_{n,2} & \dots & h_{n,n} \end{vmatrix}. \tag{4}$$

The formula (2) is very important, as we will [181] see. It is sufficient to form the algebraic expansion of only the first determinant M, to quickly obtain the other terms contained in the terms of the summation and since it is treated from the first notions of determinants, it must be considered as a *fundamental* formula of this new algorithm.

2. To come to some application, we assume firstly that in (1) is

$$a_{r,r} = 1; \quad h_{1,1} = 0; \quad a_{r,s} = a_{s,r} = 1; \quad h_{r,s} = h_{s,r} = 0.$$

In this case all of the terms before the penultimate cancel out; because the determinants that enter in them have at least two identical columns and thus, they vanish. Having a column and a row of zeros, the last term also becomes zero.[5] Finally, of the determinants that enter in ΣM_{n-1} only one remains, specifically the following:

$$\begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & h_{2,2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & h_{n,n} \end{vmatrix} = h_{2,2}h_{3,3}\dots h_{n,n}. \tag{5}$$

Therefore we will have:

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1+h_{2,2} & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & \dots & 1+h_{n,n} \end{vmatrix} = h_{2,2}h_{3,3}\dots h_{n,n}. \tag{6}$$

Secondly, supposing that $h_{1,1} = 1$, and that everything else remains the same from the previous case, and we accept this hypothesis as correct, then composing the terms of the formula (2) with the above given rule, all of those that precede ΣM_{n-1} [182] will vanish, for analogous reasons to those mentioned in the previous case. Only the determinants of this last term and the last ΣM_n

[5]Throughout the rest of the text, where Rubini wrote "line," in modern terms we would have the word "row" (tr.).

will remain. Each determinant of ΣM_{n-1} will be of the form (5) and ΣM_n will be (4) reduced to only the elements on the main diagonal.[6]

Therefore we will have:[7]

$$\begin{vmatrix} 1+h_{1,1} & 1 & 1 & \ldots & 1 \\ 1 & 1+h_{2,2} & 1 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & \ldots & \ldots & 1+h_{n,n} \end{vmatrix} = \begin{cases} h_{2,2}h_{3,3}\ldots h_{n,n} \\ +\, h_{1,1}h_{3,3}h_{4,4}\ldots h_{n,n} \\ \ldots\ldots\ldots\ldots \\ +\, h_{1,1}h_{2,2}\ldots h_{n,n} \end{cases} \tag{7}$$

It is important to note that in this formula only the last term contains all of the n elements $h_{1,1}$, etc: every other term always contains one element less.

The two formulas (6) and (7) are found in another way shown by professor *Ferreres*[8] in the *Quarterly Journal*[9] (March 1856) p. 364.

3. If we change $h_{r,r}$ to x, the same formulas will yield:[10]

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1+x & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & \ldots & 1+x \end{vmatrix}_{n-1} = x^{n-1}; \tag{8}$$

$$\begin{vmatrix} 1+x & 1 & \ldots & 1 \\ 1 & 1+x & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & \ldots & 1+x \end{vmatrix}_{n} = nx^{n-1} + x^n; \tag{9}$$

[183] and changing in these formulas $1+x$ to x, and therefore x to $x-1$, we will have:

$$\begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & x & 1 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & \ldots & \ldots & x \end{vmatrix}_{n-1} = (x-1)^{n-1} = \begin{vmatrix} x & 1 \\ 1 & 1 \end{vmatrix}^{n-1}; \tag{10}$$

$$\begin{vmatrix} x & 1 & 1 & \ldots & 1 \\ 1 & x & 1 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & 1 & \ldots & x \end{vmatrix}_{n} = +n(x-1)^{n-1} + (x-1)^n. \tag{11}$$

Finally setting $1+h_{r,r} = a_{r,r}$, and therefore[11] $h_{r,r} = a_{r,r} - 1$, formula (6) will

[6] In [Rubini, 1857], the original Italian Rubini wrote was *elementi principali* which literally translates to "main elements" but this phrase is known to signify the elements on the main diagonal (tr.).

[7] In [Rubini, 1857], the second summand in the right side of (7) was missing the $h_{1,1}$ term (tr.).

[8] Norman Macleod Ferrers, (1829 – 1903). An English mathematician who made mathematical contributions in the realms of quadriplanar co-ordinates, Lagrange's equations and hydrodynamics [O'Connor and Robertson, 2007] (tr.).

[9] The complete title of this journal is: *Quarterly Journal of Pure and Applied Mathematics* (tr.).

[10] In [Rubini, 1857], the following footnote was given: The index $n-1$, when placed under the determinant, serves to indicate that elements on the main diagonal containing x, and therefore x itself, are only $n-1$. Therefore, like (9), the determinant (8) is of the n^{th} order.

[11] In [Rubini, 1857], the subscript "r, r" was missing from the last a term (tr.).

give:

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & a_{2,2} & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & a_{n,n} \end{vmatrix} = (a_{2,2}-1)(a_{3,3}-1)\dots(a_{n,n}-1); \tag{12}$$

and formula (7) will give:

$$\begin{vmatrix} a_{1,1} & 1 & 1 & \dots & 1 \\ 1 & a_{2,2} & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & \dots & a_{n,n} \end{vmatrix} = \begin{cases} (a_{2,2}-1)(a_{3,3}-1)\dots(a_{n,n}-1) \\ + (a_{3,3}-1)(a_{4,4}-1)\dots(a_{1,1}-1) \\ \dots\dots\dots\dots\dots\dots \\ + (a_{1,1}-1)(a_{2,2}-1)\dots(a_{n-1,n-1}-1)(a_{n,n}-1) \end{cases} \tag{13}$$

In the application of this formula keep in mind [184] that only the last term contains n elements; every other term contains only $n-1$ elements.

According to formula (10) we have:

$$\begin{vmatrix} x & 1 \\ 1 & 1 \end{vmatrix}^m \begin{vmatrix} x & 1 \\ 1 & 1 \end{vmatrix}^n = \begin{vmatrix} x & 1 \\ 1 & 1 \end{vmatrix}^{m+n}; \tag{14}$$

and from the comparison of (11) and (10) results

$$\begin{vmatrix} x & 1 & 1 & \dots & 1 \\ 1 & x & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & x \end{vmatrix}_n = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & x & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & x \end{vmatrix}_n + n \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & x & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & x \end{vmatrix}_{n-1}. \tag{15}$$

4. If in the proposed determinant (1) we suppose the $h_{r,s}$ to be changed to $ih_{r,s}$,[12] i being the imaginary $\sqrt{-1}$, then formula (2) will also divide into two parts, one completely real, the other multiplied by i. Thus, when n is even, we have:[13]

$$P_{=i} = \begin{vmatrix} a_{1,1} \pm ih_{1,1} & a_{1,2} \pm ih_{1,2} & \dots & a_{1,n} \pm ih_{1,n} \\ a_{2,1} \pm ih_{2,1} & a_{2,2} \pm ih_{2,2} & \dots & a_{2,n} \pm ih_{2,n} \\ \dots & \dots & \dots & \dots \\ a_{n,1} \pm ih_{n,1} & a_{n,2} \pm ih_{n,2} & \dots & a_{n,n} \pm ih_{n,n} \end{vmatrix} = \begin{cases} M - \Sigma M_2 + \Sigma M_4 \dots \pm \Sigma M_n \\ \pm (\Sigma M_1 - \Sigma M_3 + \Sigma M_5 \dots \pm \Sigma M_{n-1})i \end{cases} \tag{16}$$

When n is odd, we have:

$$P_{=i} = \begin{cases} M - \Sigma M_2 + \Sigma M_4 \dots \pm \Sigma M_{n-1} \\ \pm (\Sigma M_1 - \Sigma M_3 + \Sigma M_5 \dots \mp \Sigma M_n)i \end{cases} \tag{17}$$

[12] In [Rubini, 1857], this was written as "If in the proposed determinant (1) we suppose the $h_{r,r}$ to be changed to $ih_{r,r}$" (tr.).

[13] In [Rubini, 1857], in the final two lines of the following equation, the ± signs at the end of the expression were missing (tr.).

in which M signifies (3) and ΣM_n signifies (4).

[185] 5. According to formulas (5) and (6) a product of n binomial factors of the form $x - a_{r,r}$ can be written in the determinant form like this:[14]

$$\begin{array}{l}(x-a_{1,1})(x-a_{2,2})(x-a_{3,3})\ldots(x-a_{n,n}) = \\ \begin{vmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & x-a_{1,1} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 0 & 0 & \ldots & x-a_{n,n} \end{vmatrix}_n = \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 1+x-a_{1,1} & 1 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & 1 & \ldots & 1+x-a_{n,n} \end{vmatrix}_n ;\end{array} \tag{18}$$

with each of these determinants being of the $(n+1)^{th}$ order.

If we apply formula (2) to either of these and the above established rule, we immediately find the expansion of the proposed product. Indeed, more simply, we can consider the determinant of the n^{th} order,[15]

$$\begin{vmatrix} x-a_{1,1} & 0 & 0 & \ldots & 0 \\ 0 & x-a_{2,2} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & x-a_{n,n} \end{vmatrix},$$

to be equal to each of the previous determinants. Forming the first term M of the mentioned formula, we have

$$M = \begin{vmatrix} x & 0 & 0 & \ldots & 0 \\ 0 & x & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & x \end{vmatrix}_n = xxx\ldots x = x^n.$$

Therefore following the above-stated rule, we will readily have [186] the term ΣM_1, *successively changing in the previous product* $xxx\ldots x$, *the first* x *to* $-a_{1,1}$ and the second to $-\ a_{2,2}$ etc., *the* n^{th} *to* $-\ a_{n,n}$. Similarly we will have the second term ΣM_2, *successively changing in the aforementioned product two* x*'s to two corresponding* a*'s*, and like this hereafter. Therefore we will have:[16]

$$\left.\begin{array}{l}(x-a_{1,1})(x-a_{2,2})(x-a_{3,3})\ldots(x-a_{n,n}) \\ = x^n + A_1x^{n-1} + A_2x^{n-2} + \ldots + A_{n-1}x + A_n,\end{array}\right\} \tag{19}$$

[14]In [Rubini, 1857], the open parenthesis of the last term in the first line was missing and the ellipses in the first determinant in between the quantities in the third and fourth columns were missing in the equation below (tr.).

[15]In [Rubini, 1857], in the following equation the ellipses in between the quantities in the third and fourth columns were missing (tr.).

[16]In [Rubini, 1857], in the equation below the + sign was missing in the second term of A_2 (tr.).

with:

$$\left.\begin{array}{l} A_1 = -(a_{1,1} + a_{2,2} + a_{3,3} + \ldots + a_{n,n}) \\ A_2 = +(a_{1,1}a_{2,2} + \ldots + a_{1,1}a_{n,n} + a_{2,2}a_{3,3} + \ldots \\ \qquad +a_{3,3}a_{n,n} + \ldots + a_{n-1,n-1}a_{n,n}) \\ A_3 = -(a_{1,1}a_{2,2}a_{3,3} + a_{1,1}a_{2,2}a_{4,4} + \ldots \\ \qquad +a_{1,1}a_{n-1,n-1}a_{n,n} + \ldots + a_{n-2,n-2}a_{n-1,n-1}a_{n,n}) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ A_n = (-1)^n a_{1,1}a_{2,2}a_{3,3}\ldots a_{n,n}; \end{array}\right\} \quad (20)$$

which we already knew.

This method of showing formula (19) seems preferable to those ordinarily used in Algebra. It is also important to note that the formula of the binomial may be deduced directly from formula (2) without using (19), if the theory of combinations is premised.

6. As the first member of (19) cancels for each value[17] $x = a_{1,1}, x = a_{2,2}, \ldots, x = a_{n,n}$, but not for others, and is divisible by each of the factors[18] $x - a_{1,1}, \ldots, x - a_{n,n}$, this will also occur for the second member. The converse proposition, namely, that every polynomial of the n^{th} degree is factored into n factors of the first degree, is already known in Algebra.

Therefore we can represent an equation of the n^{th} degree with any one of the three following formulas:[19]
[187]

$$f(x) = x^n + A_1x^{n-1} + A_2x^{n-2} + \ldots + A_{n-1}x + A_n$$
$$\begin{vmatrix} x - a_1 & 0 & 0 & \ldots & 0 \\ 0 & x - a_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & x - a_n \end{vmatrix} = 0,$$

the first and last of which being only symbolic. This being the case, we consider separately the polynomial:

$$f(x) = x^n + A_1x^{n-1} + A_2x^{n-2} + \ldots + A_{n-1}x + A_n$$
$$= \begin{vmatrix} x - a_1 & 0 & 0 & \ldots & 0 \\ 0 & x - a_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & x - a_{nn} \end{vmatrix}$$

[17]In [Rubini, 1857], the commas separating the terms were missing (tr.).
[18]In [Rubini, 1857], the commas separating the terms were missing (tr.).
[19]In [Rubini, 1857], the subscript of the a in the final term in the determinant below was given as "n, n"; Rubini explains the change of notation from "$a_{1,1}$" to "a_1" on page 190 in the article telling the reader $a_1, a_2, \ldots a_n$ are solutions to the following equation $f(x) = x^n + A_1x^{n-1} + A_2x^{n-2} + \ldots + A_{n-1}x + A_n$ (tr.).

Salvatore J. Petrilli, Jr., and Nicole Smolenski, "Analysis and Translation of Raffaele Rubini's 1857 'Application of the Theory of Determinants: Note'," *MAA Convergence* (July 2017): http://www.maa.org/press/periodicals/convergence/analysis-and-translation-of-raffaele-rubinis-1857-application-of-the-theory-of-determinants-note

In it we place $x+y$ in place of x and we will have:[20]

$$f(x+y) = \begin{vmatrix} x-a_1+y & 0 & 0 & \dots & 0 \\ 0 & x-a_2+y & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x-a_n+y \end{vmatrix}. \tag{21}$$

According to formula (2), the first term of the expansion will be:

$$M = \begin{vmatrix} x-a_1 & 0 & 0 & \dots & 0 \\ 0 & x-a_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x-a_n \end{vmatrix} = f(x). \tag{22}$$

We will have the other terms, operating here, as done [188] for the expansion of formula (19), or, more simply, with the rule in section 5. Thus, we will at once have:

$$\begin{aligned} \Sigma M_1 &= \left\{ \begin{array}{c} (x-a_2)(x-a_3)\dots(x-a_n) \\ +\quad (x-a_1)(x-a_3)\dots(x-a_n) \\ \dots\dots\dots\dots\dots\dots \\ +\quad (x-a_1)(x-a_2)\dots(x-a_{n-1}) \end{array} \right\} y \\ \Sigma M_2 &= \left\{ \begin{array}{c} (x-a_3)(x-a_4)\dots(x-a_n) \\ +\quad (x-a_1)(x-a_4)\dots(x-a_n) \\ \dots\dots\dots\dots\dots\dots \\ +\quad (x-a_1)(x-a_2)\dots(x-a_{n-2}) \end{array} \right\} y^2 \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ \Sigma M_n &= y^n. \end{aligned} \tag{23}$$

Now to have the expansions of the coefficients of the various powers of y, contained in the previous terms, we consider firstly the term where y enters in the first degree. Since every term of that coefficient is a product of $n-1$ factors, we may understand it to be expanded according to (19) and it will give as the expansion a polynomial of degree $n-1$. Let us consider the product that forms the r^{th} term. As the coefficient of x, in the place of the $i+1$ term, must be the sum of the products of the second terms taken i by i, missing the $x-a_r$ factor in that product, that coefficient will be equal to the sum of the products contained in the coefficient A_i of the proposed polynomial, minus the products of a_r by all of the remaining terms $a_1 \dots a_n$ taken $i-1$ by $i-1$. Since all of the terms contained in ΣM_1 are n (**Section 1**), thus in the complete expansion of all the products [189] of ΣM_1 the coefficient of x in the $(i+i)^{th}$ term of the expansion will be:

$$A'_i = (nA_i - iA_i) = (n-i)A_i. \tag{24}$$

[20]In [Rubini, 1857], equation numbers $21-23$ were missing. We believe they should be placed with the subsequent formulas (tr.).

Salvatore J. Petrilli, Jr., and Nicole Smolenski, "Analysis and Translation of Raffaele Rubini's 1857 'Application of the Theory of Determinants: Note'," *MAA Convergence* (July 2017): http://www.maa.org/press/periodicals/convergence/analysis-and-translation-of-raffaele-rubinis-1857-application-of-the-theory-of-determinants-note

Therefore, denoting with $f'(x)$ the mentioned coefficient of y, we will have:

$$\left.\begin{array}{l}\Sigma M_1 = f'(x) = nx^{n-1} + (n-1)A_1x^{n-2} \\ \quad + (n-2)A_2x^{n-3} + \ldots + A_{n-2}x + A_{n-1}.\end{array}\right\} \tag{25}$$

To have the expansion of ΣM_2, we will observe that if from the first line of ΣM_1 we remove one factor at a time, we will have $n-1$ products of those that enter in ΣM_2. Therefore, according to the formula (24), the sum of these expanded products would have for the coefficient of the $(i+1)^{th}$ term:

$$A''_i = (n-1-i)A'_i = (n-1-i)(n-1)A_i. \tag{26}$$

Now if we did the same with each of the following lines of ΣM_1, we would have all of the other products that enter in ΣM_2, but, however, gathering all the products formed in the indicated way, each of those contained in ΣM_2 would be doubled. Therefore in the complete expansion, the coefficient (26) will have to be reduced in half. It will be:

$$A''_i = \frac{(n-1)[n-(i+1)]}{1\cdot 2}A_i.$$

Therefore denoting with $f''(x)$ the polynomial in terms of x that results from the expansion, with the exception of the coefficient $\frac{1}{1\cdot 2}$, we will have:

$$\left.\begin{array}{l}\Sigma M_2 = \frac{1}{1\cdot 2}f''(x) = \frac{1}{1\cdot 2}[n(n-1)x^{n-2} \\ \quad + (n-1)(n-2)A_1x^{n-3} + (n-2)(n-3)A_2x^{n-4} + \ldots].\end{array}\right\} \tag{27}$$

Continuing this reasoning, we arrive at the known formula:[21]
[190]

$$\left.\begin{array}{l}f(x+y) = f(x) + f'(x)y + \frac{1}{1\cdot 2}f''(x)y^2 \\ \quad + \frac{1}{1\cdot 2\cdot 3}f'''(x)y^3 + \ldots + y^n.\end{array}\right\} \tag{28}$$

Deducing formula (28) in the above stated manner, we have the advantage of having the expressions of the derived functions expanded as products of factors of the first degree all at once. Thus, denoting the n roots of the equation

$$f(x) = x^n + A_1x^{n-1} + A_2x^{n-2} + \ldots + A_{n-1}x + A_n = 0, \tag{29}$$

[21] In [Rubini, 1857], the second term of the following equation was written as $f'(x)y^2$ (tr.).

with $a_1, a_2, a_3, \dots a_n$, we have:[22]

$$f'(x) = \left\{ \begin{array}{ll} & (x-a_2)(x-a_3)\dots(x-a_n) \\ + & (x-a_1)(x-a_3)\dots(x-a_n) \\ \multicolumn{2}{l}{\dots\dots\dots\dots\dots\dots\dots\dots} \\ + & (x-a_1)(x-a_2)\dots(x-a_{n-1}) \end{array} \right\};$$

$$\frac{1}{2}f''(x) = \left\{ \begin{array}{ll} & (x-a_3)(x-a_4)\dots(x-a_n) \\ + & (x-a_1)(x-a_4)\dots(x-a_n) \\ \multicolumn{2}{l}{\dots\dots\dots\dots\dots\dots\dots\dots} \\ + & (x-a_1)(x-a_2)\dots(x-a_{n-2}) \end{array} \right\};$$

etc...

whence we immediately deduce the known expressions of[23]

$$f'(a_1), f'(a_2), \dots, f'(a_n); \frac{1}{2}f''(a_1), \frac{1}{2}f''(a_2), \dots, \frac{1}{2}f''(a_n); etc.$$

7. Also, we very simply deduce the rule to make the second term vanish from an equation. If in (29), placed under the form:
[191]

$$\left| \begin{array}{lllll} x-a_1 & 0 & 0 & \dots & 0 \\ 0 & x-a_2 & 0 & \dots & 0 \\ \multicolumn{5}{l}{\dots\dots\dots\dots\dots\dots\dots\dots} \\ 0 & 0 & 0 & \dots & x-a_n \end{array} \right| = 0,$$

we set $x = y + h$, then the fundamental formula (2), or better yet the rule in section 5, immediately gives the coefficient of the second term as:

$$h + a_1 + h + a_2 + \dots h + a_n = nh + A_1,$$

which setting $h = -\dfrac{A_1}{n}$ will vanish,[24] as we knew.

8. Now we move to another type of application of determinants, and therefore we return to formulas (16), (17) namely:

$$\begin{aligned} P_{a+ih} &= \left| \begin{array}{llll} a_{1,1}+ih_{1,1} & a_{1,2}+ih_{1,2} & \dots & a_{1,n}+ih_{1,n} \\ a_{2,1}+ih_{2,1} & a_{2,2}+ih_{2,2} & \dots & a_{2,n}+ih_{2,n} \\ \multicolumn{4}{l}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ a_{n,1}+ih_{n,1} & a_{n,2}+ih_{n,2} & \dots & a_{n,n}+ih_{n,n} \end{array} \right| \\ &= \left\{ \begin{array}{c} M - \Sigma M_2 + \Sigma M_4 \dots \pm \Sigma M_n \\ + (\Sigma M_1 - \Sigma M_3 + \Sigma M_5 \dots \mp \Sigma M_{n-1})i \end{array} \right\}; \end{aligned} \tag{30}$$

$$P_{a-ih} = \left\{ \begin{array}{c} M - \Sigma M_2 + \Sigma M_4 \dots \pm \Sigma M_n \\ - (\Sigma M_1 - \Sigma M_3 + \Sigma M_5 \dots \mp \Sigma M_{n-1})i \end{array} \right\} \tag{31}$$

[22]In [Rubini, 1857], the two + signs of the second equation below were missing (tr.).
[23]In [Rubini, 1857], the commas between the penultimate and the last term were missing in both of the following expressions (tr.).
[24]In [Rubini, 1857], the subscript of the numerator in the formula for h was missing (tr.).

when n is even; and[25]

$$P_{a+ih} = \left\{ \begin{array}{c} M - \Sigma M_2 + \Sigma M_4 \ldots \pm \Sigma M_{n-1} \\ + (\Sigma M_1 - \Sigma M_3 + \Sigma M_5 \ldots \pm \Sigma M_n)i \end{array} \right\}, \tag{32}$$

[192]

$$P_{a-ih} = \left\{ \begin{array}{c} M - \Sigma M_2 + \Sigma M_4 \ldots \pm \Sigma M_{n-1} \\ - (\Sigma M_1 - \Sigma M_3 + \Sigma M_5 \ldots \pm \Sigma M_n)i \end{array} \right\}, \tag{33}$$

when n is odd.

Therefore, multiplying (30) by (31) and (32) by (33), we will have, for when n is even:

$$P_{a+ih}P_{a-ih} = \left\{ \begin{array}{c} (M - \Sigma M_2 + \Sigma M_4 \ldots \pm \Sigma M_n)^2 \\ + (\Sigma M_1 - \Sigma M_3 + \Sigma M_5 \ldots \mp \Sigma M_{n-1})^2 \end{array} \right\}; \tag{34}$$

and when n is odd, it will be:

$$P_{a+ih}P_{a-ih} = \left\{ \begin{array}{c} (M - \Sigma M_2 + \Sigma M_4 \ldots \pm \Sigma M_{n-1})^2 \\ + (\Sigma M_1 - \Sigma M_3 + \Sigma M_5 \ldots \pm \Sigma M_n)^2 \end{array} \right\}. \tag{35}$$

On the other hand we know that the product of the two determinants (30) and (31) is a third determinant

$$Q = \begin{vmatrix} h_{1,1} & h_{1,2} & \ldots & h_{1,n} \\ h_{2,1} & h_{2,2} & \ldots & h_{2,n} \\ \ldots & \ldots & \ldots & \ldots \\ h_{n,1} & h_{n,2} & \ldots & h_{n,n} \end{vmatrix} \tag{36}$$

whose elements are derived, as we know,[26] by multiplying those of (30) and (31) either by rows or columns, or by rows and columns.

9. Carrying out the multiplication by rows, we easily find the elements on the main diagonal to be:

$$h_{r,r} = \sum_{s'=1}^{s'=n} ({a^2}_{r,s'} + {h^2}_{r,s'}) = \alpha_{r,r} \tag{37}$$

[193] and, if r and s are two different indices then:

$$h_{r,s} = \alpha_{r,s} - i\beta_{r,s} \qquad h_{s,r} = \alpha_{r,s} + i\beta_{r,s}, \tag{38}$$

with:

$$\left. \begin{array}{l} \alpha_{r,s} = \sum_{s'=1}^{s'=n} (a_{r,s'}a_{s,s'} + h_{r,s'}h_{s,s'}); \\ \\ \beta_{r,s} = \sum_{s'=1}^{s'=n} (a_{r,s'}h_{s,s'} - h_{r,s'}a_{s,s'}). \end{array} \right\} \tag{39}$$

[25] In [Rubini, 1857], there was a closed parenthesis in the following formula after ΣM_{n-1} (tr.).

[26] In [Rubini, 1857], the following footnote was given: Brioschi − Oper. cit.; p. 23.

It is important to note that, according to these formulas, the values of β are derived from those of α, changing in α the second a to h, the second h to a, and the + sign to − in the second term. Therefore the determinant (36) will be in the form[27]

$$Q = \begin{vmatrix} \alpha_{1,1} & \alpha_{1,2} - i\beta_{1,2} & \dots & \alpha_{1,n} - i\beta_{1,n} \\ \alpha_{1,2} + i\beta_{1,2} & \alpha_{2,2} & \dots & \alpha_{2,n} - i\beta_{2,n} \\ \dots & \dots & \dots & \dots \\ \alpha_{1,n} + i\beta_{1,n} & \alpha_{2,n} + i\beta_{2,n} & \dots & \alpha_{n,n} \end{vmatrix} \tag{40}$$

and we will be able to expand it, according to the same formulas (30) and following. It is useful to note that in the expansion, the last term being of the following form:

$$\begin{vmatrix} 0 & -i\beta_{1,2} & \dots & -i\beta_{1,n} \\ +i\beta_{1,2} & 0 & \dots & -i\beta_{2,n} \\ \dots & \dots & \dots & \dots \\ +i\beta_{1,n} & +i\beta_{2,n} & \dots & 0 \end{vmatrix} \tag{41}$$

[194] will be a skew-symmetric determinant,[28] and therefore equal to zero or to a square, depending on whether n is even or odd.[29]
Setting:

$$A = \begin{vmatrix} \alpha_{1,1} & \alpha_{1,2} & \dots & \alpha_{1,n} \\ \alpha_{2,1} & \alpha_{2,2} & \dots & \alpha_{2,n} \\ \dots & \dots & \dots & \dots \\ \alpha_{n,1} & \alpha_{n,2} & \dots & \alpha_{n,n} \end{vmatrix}, \tag{42}$$

if n is even, it will be

$$Q = \left\{ \begin{array}{c} A - \Sigma A_2 + \Sigma A_4 \dots \pm \Sigma A_n \\ +(\Sigma A_1 - \Sigma A_3 + \Sigma A_5 \dots \pm \Sigma A_{n-1})i; \end{array} \right. \tag{43}$$

and if n is odd, it will be

$$Q = \left\{ \begin{array}{c} A - \Sigma A_2 + \Sigma A_4 \dots \pm \Sigma A_{n-1} \\ +(\Sigma A_1 - \Sigma A_3 + \Sigma A_5 \dots \pm \Sigma A_n)i. \end{array} \right. \tag{44}$$

In these formulas the term ΣA_n represents the determinant (41) and therefore it will be a square or it will be zero, depending on whether n is even or odd.

Now since, by the theory of determinants, it is

$$P_{n+ih} P_{n-ih} = Q,$$

if n is even, we will have:[30]

$$\left\{ \begin{array}{c} (M - \Sigma M_2 + \Sigma M_4 \dots \pm \Sigma M_n)^2 \\ +(\Sigma M_1 - \Sigma M_3 + \Sigma M_5 \dots \pm \Sigma M_{n-1})^2 \end{array} \right\} \\ = A - \Sigma A_2 + \Sigma A_4 \dots \mp \Sigma A_n; \tag{45}$$

[27]In [Rubini, 1857], in (40) the second term in the last row was given as "$+i\beta_{n,2}$" (tr.).
[28]In [Rubini, 1857], the original Italian Rubini wrote was "*gobbo simmetrico*", which literally means "hunchback symmetric" (tr.).
[29]In [Rubini, 1857], the following footnote was given: Brioschi − Oper. cit. pp. 88 and 87
[30]In [Rubini, 1857], the M was missing from the last term of the second row in (45) (tr.).

$$\Sigma A_1 - \Sigma A_3 + \Sigma A_5 \ldots \pm \Sigma A_{n-1} = 0 \tag{46}$$

and when n is odd, it will be:[31]
[195]

$$\left\{ \begin{array}{c} (M - \Sigma M_2 + \Sigma M_4 \ldots \pm \Sigma M_{n-1})^2 \\ +(\Sigma M_1 - \Sigma M_3 + \Sigma M_5 \ldots \pm \Sigma M_n)^2 \end{array} \right\} \\ = A - \Sigma A_2 + \Sigma A_4 \ldots \pm \Sigma A_{n-1}; \tag{47}$$

$$\Sigma A_1 - \Sigma A_3 + \Sigma A_5 \ldots \pm \Sigma A_{n-2} = 0. \tag{48}$$

When[32]

$$\sum_{s'=1}^{s'=n} (a_{r,s'} h_{s,s'} - h_{r,s'} a_{s,s'}) = 0, \tag{49}$$

the determinant (40) reduces to just the determinant (42). As in this case only the first term A will remain in the second members of (45) and (47).

10. If we carry out the multiplication of the two determinants $P_{a+ih} \cdot P_{a-ih}$ by rows and columns, we easily find that any element $h_{r,s}$ of the product is of the form

$$h_{r,s} = \alpha_{r,s} - i\beta_{r,s}, \tag{50}$$

where[33]

$$\left. \begin{array}{l} \alpha_{r,s} = \sum\limits_{s'=1}^{s'=n} (a_{r,s'} a_{s',s} + h_{r,s'} h_{s',s}); \\ \\ \beta_{r,s} = \sum\limits_{s'=1}^{s'=n} (a_{r,s'} h_{s',s} - h_{r,s'} a_{s',s}). \end{array} \right\} \tag{51}$$

Here too β is deduced from α, as in the formulas of (39). The conditions that allow the imaginary part to vanish on its own from product Q, are given in this formula:

$$\sum_{s'=1}^{s'=n} (a_{r,s'} h_{s',s} - h_{r,s'} a_{s',s}) = 0 \tag{52}$$

[196] giving to r all of the integer values from 1 to n, and therefore these conditions are the same as those deduced from (49), as it should be.

11. Let us now apply the previous formulas to some particular cases and firstly:

$$\begin{array}{l} P_{a+ih} = \left| \begin{array}{cc} a_{1,1} + ih_{1,1} & a_{1,2} + ih_{1,2} \\ a_{2,1} + ih_{2,1} & a_{2,2} + ih_{2,2} \end{array} \right|, \\ P_{a-ih} = \left| \begin{array}{cc} a_{1,1} - ih_{1,1} & a_{1,2} - ih_{1,2} \\ a_{2,1} - ih_{2,1} & a_{2,2} - ih_{2,2} \end{array} \right|. \end{array} \tag{53}$$

[31] In [Rubini, 1857], in (47) the last term of the first row has a subscript of n, the last term of the second row has a subscript of $n-1$ and the third term of the last row was given as ΣM_4 (tr.).
[32] In [Rubini, 1857], the lower bound for the following summation was $s' = n$ (tr.).
[33] In [Rubini, 1857], the lower bound for the α summation was $s' = n$ (tr.).

According to the formula (34) and the rule of the first section, it will be:[34]

$$P_{a+ih} \cdot P_{a-ih} =$$
$$\left\| \begin{array}{cc} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{array} \right| - \left| \begin{array}{cc} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{array} \right\|^2 + \left\| \begin{array}{cc} h_{1,1} & a_{1,2} \\ h_{2,1} & a_{2,2} \end{array} \right| + \left| \begin{array}{cc} a_{1,1} & h_{1,2} \\ a_{2,1} & h_{2,2} \end{array} \right\|^2 = \tag{54}$$
$$\begin{cases} (a_{1,1}a_{2,2} - a_{1,2}a_{2,1} + h_{1,2}h_{2,1} - h_{1,1}h_{2,2})^2 \\ +(h_{1,1}a_{2,2} - a_{1,2}h_{2,1} + a_{1,1}h_{2,2} - h_{1,2}a_{2,1})^2. \end{cases}$$

At the same time the formulas (37), (38) and (39) give:

$$\begin{aligned}
\alpha_{1,1} &= a^2{}_{1,1} + h^2{}_{1,1} + a^2{}_{1,2} + h^2{}_{1,2}; \beta_{1,1} = 0; \\
\alpha_{2,2} &= a^2{}_{2,1} + h^2{}_{2,1} + a^2{}_{2,2} + h^2{}_{2,2}; \beta_{2,2} = 0; \\
\alpha_{1,2} &= a_{1,1}a_{2,1} + h_{1,1}h_{2,1} + a_{1,2}a_{2,2} + h_{1,2}h_{2,2} = \alpha_{2,1}; \\
\beta_{1,2} &= a_{1,1}h_{2,1} - h_{1,1}a_{2,1} + a_{1,2}h_{2,2} - h_{1,2}a_{2,2} = -\beta_{2,1}.
\end{aligned}$$

Then by formula (42)
[197]

$$A = \left| \begin{array}{cc} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{array} \right| = \begin{cases} (a^2{}_{1,1} + h^2{}_{1,1} + a^2{}_{1,2} + h^2{}_{1,2}) \\ (a^2{}_{2,1} + h^2{}_{2,1} + a^2{}_{2,2} + h^2{}_{2,2}) \\ -\ (a_{1,1}a_{2,1} + h_{1,1}h_{2,1} + a_{1,2}a_{2,2} + h_{1,2}h_{2,2})^2; \end{cases}$$
$$\Sigma A_1 = \left| \begin{array}{cc} \beta_{1,1} & \alpha_{1,2} \\ \beta_{2,1} & \alpha_{2,2} \end{array} \right| + \left| \begin{array}{cc} \alpha_{1,1} & \beta_{1,2} \\ \alpha_{1,2} & \beta_{2,2} \end{array} \right| = 0;$$
$$\Sigma A_2 = \left| \begin{array}{cc} 0 & \beta_{1,2} \\ \beta_{2,1} & 0 \end{array} \right| = (a_{1,1}h_{2,1} - h_{1,1}a_{2,1} + a_{1,2}h_{2,2} - h_{1,2}a_{2,2})^2$$

and then by formula (43):[35]

$$Q = \begin{cases} (a^2{}_{1,1} + h^2{}_{1,1} + a^2{}_{1,2} + h^2{}_{1,2})(a^2{}_{2,1} + h^2{}_{2,1} + a^2{}_{2,2} + h^2{}_{2,2}) \\ -\ (a_{1,1}a_{2,1} + h_{1,1}h_{2,1} + a_{1,2}a_{2,2} + h_{1,2}h_{2,2})^2 \\ -\ (a_{1,1}h_{2,1} - h_{1,1}a_{2,1} + a_{1,2}h_{2,2} - h_{1,2}a_{2,2})^2. \end{cases} \tag{55}$$

From the comparison of (54) and (55) the following formula immediately results:[36]

$$\left. \begin{array}{l} (a^2{}_{1,1} + h^2{}_{1,1} + a^2{}_{1,2} + h^2{}_{1,2})(a^2{}_{2,1} + h^2{}_{2,1} + a^2{}_{2,2} + h^2{}_{2,2}) = \\ \quad (a_{1,1}a_{2,1} + h_{1,1}h_{2,1} + a_{1,2}a_{2,2} + h_{1,2}h_{2,2})^2 \\ +\ (a_{1,1}a_{2,2} - a_{1,2}a_{2,1} + h_{1,2}h_{2,1} - h_{1,1}h_{2,2})^2 \\ +\ (a_{1,1}h_{2,1} - h_{1,1}a_{2,1} + a_{1,2}h_{2,2} - h_{1,2}a_{2,2})^2 \\ +\ (h_{1,1}a_{2,2} - a_{1,2}h_{2,1} + a_{1,1}h_{2,2} - h_{1,2}a_{2,1})^2. \end{array} \right\} \tag{56}$$

[34]In [Rubini, 1857], in (54) the first term of the second row in the second determinant was written as $h_{1,1}$ (tr.).
[35]In [Rubini, 1857], within the first set of parenthesis of the first line the second term was written as $h^2{}_{2,1}$, the second term of the second line was written as $h_{1,2}h_{2,1}$ and the last term of the third line was written as $h_{1,2}h_{2,2}$ in the equation below (tr.).
[36]In [Rubini, 1857], the last line was missing from the following equation (tr.).

which constitutes the fundamental proposition for the proof of the theorem: *every number is the sum of four squares.*[37]

The same formula contains, as a particular case, the other formula already noted by Mr. CAUCHY.[38]

$$\left.\begin{aligned} &(\alpha^2{}_1+\beta^2{}_1+\gamma^2{}_1)(\alpha^2{}_2+\beta^2{}_2+\gamma^2{}_2)-(\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2)^2 \\ =\ &(\alpha_2\beta_1-\alpha_1\beta_2)^2+(\gamma_2\alpha_1-\gamma_1\alpha_2)^2+(\beta_2\gamma_1-\beta_1\gamma_2)^2, \end{aligned}\right\} \tag{57}$$

which is derived from (56),[39] setting:

$$\begin{array}{ccc} h_{1,2}=h_{2,2}=0; & a_{1,1}=\alpha_1, & h_{1,1}=\beta_1; \\ a_{2,1}=\alpha_2,\ h_{2,1}=\beta_2; & a_{1,2}=\gamma_1, & a_{2,2}=\gamma_2. \end{array}$$

[198] 12. It is noteworthy that the same formula (56) or also more simply formula (57) leads to an elegant geometric theorem. In fact, setting

$$\alpha_2=x,\quad \beta_2=y,\quad \gamma_2=z,$$

formula (57) can be put in the following form

$$\left.\begin{aligned} &\left(\frac{\alpha_1x+\beta_1y+\gamma_1z}{\sqrt{(\alpha^2{}_1+\beta^2{}_1+\gamma^2{}_1)}}\right)^2+\left(\frac{\alpha_1z-\gamma_1x}{\sqrt{(\alpha^2{}_1+\beta^2{}_1+\gamma^2{}_1)}}\right)^2 \\ +\ &\left(\frac{\beta_1z-\gamma_1y}{\sqrt{(\alpha^2{}_1+\beta^2{}_1+\gamma^2{}_1)}}\right)^2+\left(\frac{\alpha_1y-\beta_1x}{\sqrt{(\alpha^2{}_1+\beta^2{}_1+\gamma^2{}_1)}}\right)^2 \\ =\ &x^2+y^2+z^2. \end{aligned}\right\} \tag{58}$$

Now the terms of the first member sequentially represent the distances squared

of a point $M(x,y,z)$ from a plane (P), $\alpha_1x'+\beta_1y'+\gamma_1z'=0$,
" " $N'(x,0,z)$ " " (Π'), $-\gamma_1x'+\beta_1y'+\alpha_1z'=0$,
" " $N''(0,y,z)$ " " (Π''), $\alpha_1x'-\gamma_1y'+\beta_1z'=0$,
" " $N(x,y,0)$ " " (Π), $-\beta_1x'+\alpha_1y'+\gamma_1z'=0$.

It is clear that these last three points are the projections of the point $M(x,y,z)$ on the three coordinate planes. The second member (57) denotes the distance squared of the same point M from the origin.

[37] In [Rubini, 1857], the following footnote was given: LEGENDRE. Théorie des nombres Volume 1 p. 213

[38] Augustin-Louis Cauchy (1789 – 1857). Cauchy was the first mathematician to prove this proposition. See [Cauchy, 1813 – 1815] (tr.).

[39] In [Rubini, 1857], it was written as being derived from (55) (tr.).

Furthermore, assuming the axes to be rectangular, we have:

$$\cos(P,\Pi) = \frac{\gamma^2{}_1}{\alpha^2{}_1+\beta^2{}_1+\gamma^2{}_1} = \cos^2[P,(x'y')],$$
$$\cos(P,\Pi') = \frac{\beta^2{}_1}{\alpha^2{}_1+\beta^2{}_1+\gamma^2{}_1} = \cos^2[P,(x'z')],$$
$$\cos(P,\Pi'') = \frac{\alpha^2{}_1}{\alpha^2{}_1+\beta^2{}_1+\gamma^2{}_1} = \cos^2[P,(y'z')].$$

Therefore: *If from the vertex O of a rectangular parallelepiped,* [199] *whose adjacent sides are* a, b, c, *one takes an arbitrary plane* (P), and then *from the same vertex three other planes* (Π),(Π'), (Π'') *are built, oriented so that*[40]

$$\cos(P,\Pi) = \cos^2[P,(ab)];$$
$$\cos(P,\Pi') = \cos^2[P,(ac)];\ \cos(P,\Pi'') = cos^2[P,(bc)],$$

and *dropping a perpendicular from the vertex* M (*opposite to* O) and the three vertices N, N', N", *adjacent to M respectively on the planes* (P), (Π), (Π'), (Π''), *then the sum of the squares of these perpendiculars will be equal to the square of the diagonal* O M.[41]

13. Lets apply to the determinants from (54) the formulas from (51) and we will have:[42]

$$A - \Sigma A_2 = \begin{cases} \quad (a^2{}_{1,1}+h^2{}_{1,1}+a_{1,2}a_{2,1}+h_{1,2}h_{2,1}) \\ \times\ (a^2{}_{2,2}+h^2{}_{2,2}+a_{1,2}a_{2,1}+h_{1,2}h_{2,1}) \\ -\ [a_{1,2}(a_{1,1}+a_{2,2})+h_{1,2}(h_{1,1}+h_{2,2})] \\ \times\ [a_{2,1}(a_{1,1}+a_{2,2})+h_{2,1}(h_{1,1}+h_{2,2})] \\ -\ [a_{1,2}(h_{1,1}-h_{2,2})-h_{1,2}(a_{1,1}-a_{2,2})] \\ \times\ [a_{2,1}(h_{1,1}-h_{2,2})+h_{2,1}(a_{1,1}-a_{2,2})] \\ +\ (a_{1,2}h_{2,1}-h_{1,2}a_{2,1})^2; \end{cases} \tag{59}$$

$$\Sigma A_1 = \begin{cases} -\ [a_{1,2}(a_{1,1}+a_{2,2})+h_{1,2}(h_{1,1}+h_{2,2})] \\ \times\ [a_{2,1}(h_{1,1}-h_{2,2})-h_{2,1}(a_{1,1}-a_{2,2})] \\ -\ [a_{2,1}(a_{1,1}+a_{2,2})+h_{2,1}(h_{1,1}+h_{2,2})] \\ \times\ [h_{1,2}(a_{1,1}-a_{2,2})-a_{1,2}(h_{1,1}-h_{2,2})] \\ -\ (a_{1,2}h_{2,1}-h_{1,2}a_{2,1})(a^2{}_{1,1}+h^2{}_{1,1}-a^2{}_{2,2}-h^2{}_{2,2}). \end{cases} \tag{60}$$

[200] Therefore setting (59) equal to (54) or (55) we will have two other formulas of numerical transformation that we avoid writing in full; content to only point out that in the case when we assume

$$a_{1,1} = a_{2,2};\quad h_{1,1} = h_{2,2};\quad a_{1,2} = a_{2,1},\quad h_{1,2} = h_{2,1},$$

[40] In [Rubini, 1857], the open parenthesis was missing from the *bc* in the equation below (tr.).

[41] This is a very analytical approach to the extension of the Pythagorean Theorem to the three-dimensionsal rectangular parallelepiped, with the diagonal squared being equal to the sum of the squares of the figure's width, length and height (tr.).

[42] In [Rubini, 1857], in formula 60 the last line is missing the $+h^2{}_{1,1}$ term (tr.).

the comparison of (59) with (54) leads to the following formula

$$\left.\begin{aligned}({a^2}_{1,1} + {h^2}_{1,1} + {a^2}_{1,2} + {h^2}_{1,2})^2 = ({a^2}_{1,1} - {h^2}_{1,1} + {h^2}_{1,2} - {a^2}_{1,2})^2 \\ +[2(a_{1,1}a_{1,2} + h_{1,1}h_{1,2})]^2 + [2(a_{1,1}h_{1,1} - a_{1,2}h_{1,2})]^2,\end{aligned}\right\} \tag{61}$$

which we equally obtain from (56) and indicates that the "*square of a number is always the sum of only three squares.*"[43]

Similarly, since formula (60) has to be equal to zero, according to formula (48), we will have:[44]

$$\begin{aligned}&[a_{1,2}(a_{1,1} + a_{2,2}) + h_{1,2}(h_{1,1} + h_{2,2})] \\ &[h_{2,1}(a_{1,1} - a_{2,2}) - a_{2,1}(h_{1,1} - h_{2,2})] \\ -&[a_{2,1}(a_{1,1} + a_{2,2}) + h_{2,1}(h_{1,1} + h_{2,2})] \\ &[h_{1,2}(a_{1,1} - a_{2,2}) - a_{1,2}(h_{1,1} - h_{2,2})] \\ =&(a_{1,2}h_{2,1} - h_{1,2}a_{2,1})({a^2}_{1,1} + {h^2}_{1,1} - {a^2}_{2,2} - {h^2}_{2,2})\end{aligned} \tag{62}$$

With the use of the formulas that we have shown and without the need for multiplication, we can arrive to other formulas of reduction and numerical transformation, and therefore to other theorems, by considering determinants of a greater number of elements.

[43]The following theorem had been worked on by various mathematicians including Leonhard Euler (1707 – 1783) and Christian Goldbach (1690 – 1764) [Lemmermeyer, n.d.] (tr.).

[44]In [Rubini, 1857], in the following formula the second term of the fist line was given as $b_{1,2}(h_{1,1} + h_{2,2})$ (tr.).

Acknowledgments

We are extremely grateful to Associate Dean Briziarelli for her assistance in the translation of Rubini's article, as well as to Professor Rosaria for a primary source providing additional biographical information about Rubini. We would also like to thank Professor R. Bradley, Professor E. De Freitas, and Professor L. Stemkoski for their helpful suggestions on this paper. All five are professors at Adelphi University. The authors are also extremely grateful to the referees for their many helpful suggestions and corrections.

About the Authors

Salvatore J. Petrilli, Jr., Ed.D., is an Associate Professor of Mathematics and Department Chair at Adelphi University. He has a B.S. in mathematics from Adelphi University and an M.A. in mathematics from Hofstra University. He received an Ed.D. in mathematics education from Teachers College, Columbia University, where his advisor was J. Philip Smith. His general research interests include history of mathematics, mathematics education, and applied statistics. However, the majority of his research has been devoted to the life and mathematical contributions of François-Joseph Servois.

Nicole Smolenski is pursuing a graduate certificate in International Education from Florence University of the Arts. She will next be pursuing her JD and MA in Public Policy to become an Educational Policy lawyer. She taught for the past three years as a middle school mathematics teacher in the New York City public schools and taught an Italian elective for a year. She has a B.S. in Mathematics from Adelphi University and earned her M.A. in Mathematics Education at Teachers College, Columbia University. Her research interests are in Mathematics Education and its history.

References

[Brioschi, 1854] Brioschi, F. (1854). *La teorica dei determinanti e le sue applicazioni*. Pavia: Eredi Bizzoni. 44.

[Cauchy, 1813 – 1815] Cauchy, A. (1813 – 1815). Démonstration du théorème général de Fermat sur les nombres polygones. *Mém. Sci. Math. Phys. Inst. France, 1*(14). 177 – 220.

[Lemmermeyer, n.d.] Lemmermeyer, F. (n.d.). *Euler, Goldbach, and "Fermat's Theorem"*. Retrieved April 3, 2013, from http://www.rzuser.uni-heidelberg.de/~hb3/publ/eu4-2.pdf

[O'Connor and Robertson, 2006] O'Connor, J., and Robertson, E. (Aug. 2006). *Francesco Brioschi.*. Retrieved April 29, 2013, from http://www-history.mcs.st-andrews.ac.uk/Biographies/Brioschi.html

[O'Connor and Robertson, 2007] O'Connor, J., and Robertson, E. (Aug. 2007). *Norman Macleod Ferrers*. Retrieved July 18, 2017, from http://www-history.mcs.st-andrews.ac.uk/Biographies/Ferrers.html

[Rubini, 1857] Rubini, R. (1857). Applicazione Della Teorica Dei Determinanti: Nota. *Annali di Scienze Matematiche e Fisiche, 8*. 179 – 200.

[Spottiswoode, 1851] Spottiswoode, W. (1851). *Elementary Theorems Relating to Determinants*. London: Spottiswoode and Shaw.

Salvatore J. Petrilli, Jr., and Nicole Smolenski, "Analysis and Translation of Raffaele Rubini's 1857 'Application of the Theory of Determinants: Note'," *MAA Convergence* (July 2017): http://www.maa.org/press/periodicals/convergence/analysis-and-translation-of-raffaele-rubinis-1857-application-of-the-theory-of-determinants-note