English Translation of Servois' 1814 "Essay on a New Method of Exposition of the Principles of Differential Calculus"

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Abstract

Many historians of mathematics consider the nineteenth century to be the Golden Age of mathematics. During this time period many areas of mathematics, such as algebra and geometry, were being placed on rigorous foundations. Another area that experienced fundamental change was analysis. Grabiner [1981] considers Joseph-Louis Lagrange (1736-1813) to be the first mathematician to treat the eighteenth century foundations of calculus as a serious mathematical issue. The publication of his *Fonctions analytiques* [1797] can be seen as the first attempt to resolve these foundational issues. However, many other mathematicians also contributed to the foundational debates of the eighteenth and nineteenth centuries. One such figure was François-Joseph Servois (1767-1847). Servois was a priest, artillery officer during the French Revolutionary period, professor of mathematics, and supporter of Lagrange's algebraic formalism. We give here an English translation of Servois' "Essay on a new method of exposition of the principles of differential calculus" [1814a], in which Servois continued the work of Lagrange by attempting to place calculus on a foundation of algebraic analysis without recourse to infinitesimals. We provide an analysis of Servois' paper and a guide to reading it in our article, "Servois' 1814 Essay on a New Method of Exposition of the Principles of Differential Calculus, with an English Translation," available in the MAA online journal *Loci: Convergence* at http://mathdl.maa.org/mathDL/46/ (DOI: 10.4169/loci003597).

Essay on a new method of exposition of the principles of differential calculus[1]

By Mr. Servois, professor of the artillery schools[2]

> "Insofar as (analysis) is extended and enriched by new methods, it becomes more complicated, and we can only simplify it by simultaneously generalizing and reducing the methods that are susceptible to these benefits."
>
> (*Méchanique analitique*, p. 338.)[3]

[93][4] 1. I begin by establishing some notation and by giving several definitions.

$\mathrm{f}z, fz, Fz, ez, \ldots$ denote functions of an arbitrary quantity of z; I call these simple *monomial functions*.

$\mathrm{f}fz, \mathrm{f}fFz, \ldots$ denote functions of functions of z; these are the *composite monomial functions*.

[94] fz, f^2z, f^3z, $\ldots$, f^nz represent the function denoted by f of the quantity z taken successively 1 time, 2 times, 3 times, $\ldots$, n times. These are the *monomial functions* of the 1st, 2nd, 3rd, $\ldots$, nth order; n is the *exponent* of the order of the function.

$f^{-1}z, f^{-2}z, \ldots, f^{-n}z$ denote functions of z whose complete definitions are given by the general equation

$$f^n f^{-n} z = f^{-n} f^n z = z. \tag{1}$$

These are the *inverse functions* or *functions of negative order*.

If the quantity under the functional symbol, that is to say the *subject of the function*, is polynomial, then we put it between parentheses. Thus, $f(a+z)$ denotes the function f of the binomial $a+z$. Whenever the subject of the function is regarded as a complex,[5] we use commas between the *partial subjects*,

[1] *Essai sur un nouveau mode d'exposition des principes du calcul différentiel*, an article in *Annales des Mathématiques pures et appliquées* **5** (1814-1815), pp. 93-140. In some citations, the title begins with the words "*Analise Transcendante*," because the headline of a title page in Gergonne's *Annales* is the editorial category to which the article was assigned.

[2] The following footnote appeared at this place in the original article: "What we are about to read, in essence, is extracted from two memoirs on the development of functions into series by the differential method, presented to the First Class of the Institute [i.e. the mathematics section of the *Institut National des Sciences et des Arts*], the first, toward the end of 1805 and the second in 1809. They received the approval of the class in a report by Messrs. Legendre and Lacroix, dated October 5, 1812."

[3] Servois inserted this epigram here. It comes from the introductory paragraphs of Section 6 (On the Rotation of Bodies) of [Lagrange 1788].

[4] Numbers in square brackets represent the original page numbers of the article in Gergonne's *Annales*.

[5] That is, a collection, and not to be confused with complex numbers.

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along with parentheses. Thus $f[x,(b+y),z,\ldots]$ denotes the function f of the quantities x, $b+y$, z, $\ldots$.

If $fz=z$, that is, if the subject is taken only one time, the function f is the constant factor 1. If $fz=az$, or if the subject is taken a times, the function f is the constant factor of a.[6]

Suppose that the subject z is a complex, for example $z=\phi(x,y,\ldots)$, where x, y, $\ldots$, are arbitrary or independent *variable quantities* that receive, respectively, invariable or constant increments α, β, $\ldots$. If we have

$$fz=\varphi(x+\alpha,y+\beta,\ldots),$$

the function f is called the *varied state* of z. I propose, following Arbogast (*Calculus des derivations*, no. 442),[7] to denote this particular function by the letter E, and I adopt the following definitions

$$\left.\begin{array}{rcl} Ez &=& \varphi(x+\alpha,y+\beta,\ldots),\\ E^{-1}z &=& \varphi(x-\alpha,y-\beta,\ldots)\quad\text{and}\\ E^{n}z &=& \varphi(x+n\alpha,y+n\beta,\ldots),\end{array}\right\}\tag{2}$$

[95] If $fz=Ez-z$, the function f is called the *difference* of z; it is a time-honored tradition to denote this by the letter Δ. Thus, we have the following definitions

$$\Delta z=Ez-z=\varphi(x+\alpha,y+\beta,\ldots)-\varphi(x,y,\ldots).\tag{3}$$

We conclude immediately from this another expression for the varied state

$$Ez=z+\Delta z.\tag{4}$$

When the subject z is a complex, we frequently need to state that the function f is taken only with respect to a single *partial subject*. Thus, if we wish to state that the function f is taken only with respect to x, we will write $\frac{f}{x}z$. If the function is only to affect y, we will write $\frac{f}{y}z$ and so on. $\frac{f}{x}z, \frac{f}{y}z, \ldots$ are the *partial f functions of z*. Thus, when a is a constant factor, we have the following definition of the partial constant factor a

$$\frac{a}{x}z=\varphi(ax,y,\ldots).$$

Similarly, from (2) and (3), we have the following definition of the *partial varied states* and the *partial differences*

$$\left.\begin{array}{rcl} \dfrac{E^n}{x}z &=& \varphi(x+n\alpha,y,\ldots),\quad \dfrac{E^n}{y}z=\varphi(x,y+n\beta,\ldots),\\ \dfrac{\Delta}{x}z &=& \varphi(x+\alpha,y,\ldots)-\varphi(x,y,\ldots)=\dfrac{E}{x}z-z\quad\text{and}\\ \dfrac{\Delta}{y}z &=& \varphi(x,y+\beta,\ldots)-\varphi(x,y,\ldots)=\dfrac{E}{y}z-z.\end{array}\right\}\tag{5}$$

[6] Servois refers to these functions simply as "factors." We translate this as "constant factors," to avoid confusion in the places where the word "factor" is used in the ordinary sense.

[7] See [Arbogast 1800].

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$f^0 z$ is always equal to z, for the expression itself indicates [96] that we do not take the function f of z and, as a consequence, in this regard and z does not undergo any modification. Thus

$$z = a^0 z = E^0 z = \Delta^0 z = \frac{E^0}{x} z = \frac{E^0}{y} z = \dots. \tag{6}$$

Every inverse function admits an *arbitrary complement*, whenever the direct function of the first order has the property of eliminating certain terms in its subject, or to make certain factors equal to one. Because, for example, the difference Δ eliminates constant terms, among other things, the inverse function $\Delta^{-1} z$ in this case takes an *additive* complement, an arbitrary constant A.

It is customary to denote by $\Sigma z, \Sigma^2 z, \dots, \Sigma^n z$, the function of z that we will call *integrals*, the definition of which is given by the equation

$$\Delta^n \Sigma^n z = \Sigma^n \Delta^n z = z;$$

and, since by equation (1), we also have

$$\Delta^n \Delta^{-n} z = \Delta^{-n} \Delta^n z = z,$$

it follows that

$$\Sigma^n z = \Delta^{-n} z. \tag{7}$$

For the same reason, when ln denotes the natural logarithm[8] and e denotes the base of that system, we have

$$\ln \ln^{-1} z = z = \ln e^z, \quad \ln^2 \ln^{-2} z = z = \ln^2 e^{e^z}, \dots$$

Therefore

$$e^z = \ln^{-1} z, \quad e^{e^z} = \ln^{-2} z, \dots. \tag{8}$$

We also find[9]

$$\sin^{-1} z = \mathrm{Arc}(\sin = z), \ \tan^{-1} z = \mathrm{Arc}(\tan = z), \dots. \tag{9}$$

because we have [97]

$$\begin{aligned} z &= \sin(\sin^{-1} z) = \sin \mathrm{Arc}(\sin = z) \\ &= \tan(\tan^{-1} z) = \tan \mathrm{Arc}(\tan = z). \end{aligned}$$

To prevent any misunderstanding, the product of $\mathrm{f}x$ and fy will be denoted by $\mathrm{f}x \cdot fy$. The expression $\mathrm{f}xfy$ will signify the function f of the product of x by fy. The power n of $\mathrm{f}x$ will be denoted $(\mathrm{f}x)^n$. The expression $\mathrm{f}x^n$ denotes the function f of the power n of x.

[8] Servois used the symbol L for the natural logarithm. We will consistently use ln in its place.

[9] Servois used "Sin." for sine and "Tang." for tangent. We will consistently use sin and tan in place of those.

2. Let

$$Fz = \mathrm{f}z + fz + \varphi z + \dots, \tag{10}$$

that is, let us suppose that to form the function F of z, we must add, to the function f of z a second function f of the same letter, then a third function denoted by φ, and so on. The function F is therefore in the class of *polynomial function*. We may indicate this sense of the function F by a very expressive notation, which has the great advantage of permitting us to treat polynomial functions as monomial functions, without losing sight of the way in which they are composed. We write

$$Fz = (\mathrm{f} + f + \varphi + \dots)z.$$

As a consequence, we also have

$$F^n z = (\mathrm{f} + f + \varphi + \dots)^n z. \tag{11}$$

If F' is another polynomial function of z, given by the equation

$$F'z = (\mathrm{f}' + f' + \varphi' + \dots)z,$$

then we may also express the function F' of Fz by writing

$$F'Fz = (\mathrm{f}' + f' + \varphi' + \dots)(\mathrm{f} + f + \phi + \dots)z, \tag{12}$$

and so on.

[98] There is no reason why some or all of the *composing monomial functions* may not be *constant factors*. In the latter case, we will have, without ambiguity in (11) and (12), that $Fz, F'Fz, \dots$, will be the products of z multiplied by the polynomial $\mathrm{f} + f + \varphi + \dots$ or by the product $(\mathrm{f}' + f' + \varphi' + \dots)(\mathrm{f} + f + \varphi + \dots)$.

3. Let

$$\varphi(x + y + \dots) = \varphi x + \varphi y + \dots, \tag{13}$$

functions which, like φ, are such that the function of the (algebraic) *sum* of any number of quantities is equal to the sum of the same function of each of these quantities, are called *distributive*.

Therefore, because

$$a(x + y + \dots) = ax + by + \dots,\ E(x + y + \dots) = Ex + Ey + \dots,\ \dots$$

the constant factor a, the varied state E, ..., are distributive functions. However, because we do not have

$$\sin(x + y + \dots) = \sin x + \sin y + \dots,\ \ln(x + y + \dots) = \ln x + \ln y + \dots,\ \dots$$

the sine, the natural logarithm, ..., are not distributive functions.

4. Let

$$\mathrm{f}fz = f\mathrm{f}z. \tag{14}$$

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Functions which like f and f, are such that they give identical results, no matter in which order we apply them to the subject, are called *commutative between themselves*.

Therefore, because we have

$$abz = baz,\ aEz = Eaz; , \ldots,$$

the constant factors a and b, the constant factor a and the varied state E, are functions that are commutative between themselves. However, because when a is constant and x is variable we do not have

$$\sin az = a \sin z,\ Exz = xEz,\ \Delta x\, z = x\Delta z \ldots,$$

[99] it follows that the sine and the constant factor, the varied state and the variable constant factor, . . . , do not belong to the class of functions commutative between themselves.

5. We collect together several important theorems from these simple notions.

If two functions ϕ and ψ are distributive, the monomial composed function is also distributive. Because by hypothesis, we have

$$\psi(x+y) = \psi x + \psi y \quad \text{and} \quad \varphi(t+u) = \varphi t + \varphi u,$$

we clearly have[10]

$$\varphi\psi(x+y) = \varphi(\psi x + \psi y) = \varphi(t+u) = \varphi t + \varphi u = \varphi\psi x + \varphi\psi y.$$

From this, it follows immediately that the different orders of a distributive function are also distributive functions.

6. If the monomial functions f, f, φ, . . . making up the polynomial function F are distributive, then the polynomial function F also has the same property because, according to definition (10), we have

$$F(x+y) = \mathrm{f}(x+y) + f(x+y) + \varphi(x+y) + \ldots.$$

However, since f, f, φ, . . . are distributive, this equations becomes

$$F(x+y) = \mathrm{f}x + fx + \varphi x + \ldots + \mathrm{f}y + fy + \varphi y + \ldots = Fx + Fy.$$

We may say the same thing (§5) of the different orders F^n of the same function.

7. Suppose the functions f, $f, \varphi, \ldots$ are commutative between themselves two by two, so that we have

$$\mathrm{f}fz = f\mathrm{f}z, \quad \mathrm{f}\varphi z = \varphi\mathrm{f}z, \quad f\varphi z = \varphi f z, \ldots.$$

Taking a certain number n of these functions, if we form all possible composite monomial functions given by the permutation of these n functional symbols, all of the composite monomial functions that result are equivalent.

[10] The right hand side of the final equality was written as $\varphi\psi t + \varphi\psi u$ in the original text.

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Therefore, for example, if we take the first three f, f, φ, we have

$$\mathrm{f}f\varphi z = f\mathrm{f}\varphi z = \mathrm{f}\varphi f z = \varphi \mathrm{f} f z = f\varphi \mathrm{f} z = \varphi f \mathrm{f} z.$$

[100] To prove this in general, consider the monomial function

$$f \ldots \mathrm{f}\varphi\psi F \ldots z$$

we may, without changing the value, permute any two consecutive function symbols between themselves, say φ and ψ, for example. For, if

$$F \ldots z = t,$$

then we have

$$\varphi\psi F \ldots z = \varphi\psi t.$$

Now, by hypothesis,

$$\varphi\psi t = \psi\varphi t,$$

so

$$\varphi\psi F \ldots z = \psi\varphi F \ldots z,$$

and taking the composite function of both sides,

$$f \ldots \mathrm{f}\varphi\psi F \ldots z = f \ldots \mathrm{f}\psi\varphi F \ldots z.$$

It follows that each functional symbol may be brought to whatever position we wish in the first combination, and therefore we may apply the functional symbols in all possible permutations, without altering the value of the composed function.

Clearly, we conclude from this theorem that from the functional symbols f, f, φ, ..., that are commutative between themselves two by two, we may form, at will, new functions composed of two, three, ..., symbols, such as, $\mathrm{f}fz, \varphi\psi Fz, \ldots$, all of which are also commutative between themselves and with the first ones.

8. If f and f are commutative between themselves, they are also so with their inverses, which are also commutative between themselves. That is, if we have

$$\mathrm{f}fz = f\mathrm{f}z, \tag{15}$$

we also have

$$\mathrm{f}f^{-1}z = f^{-1}\mathrm{f}z, \quad f\mathrm{f}^{-1}z = \mathrm{f}^{-1}fz, \quad \mathrm{f}^{-1}f^{-1}z = f^{-1}\mathrm{f}^{-1}z. \tag{16}$$

Indeed, by (1) we have [101]

$$f\mathrm{f}\mathrm{f}^{-1}z = \mathrm{f}\mathrm{f}^{-1}fz.$$

Now by (15)

$$f\mathrm{f}\mathrm{f}^{-1}z = \mathrm{f}f\mathrm{f}^{-1}z,$$

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and so[11]

$$\mathrm{f}f\mathrm{f}^{-1}z = \mathrm{ff}^{-1}fz,$$

and taking the function f^{-1} of both sides,

$$f\mathrm{f}^{-1}z = \mathrm{f}^{-1}fz.$$

This is the first of the theorems (16) and the second is proven in the same way. As for the third, we have, by (1),

$$f^{-1}f\mathrm{f}^{-1}z = \mathrm{f}^{-1}f^{-1}fz$$

and, by the first of the theorems (16), we have

$$f^{-1}\mathrm{f}^{-1}fz = \mathrm{f}^{-1}f^{-1}fz,$$

which becomes the third theorem (16), by changing fz to z.

9. From the theorems (§7, 8) we conclude, without further discussion, the following formulas.

When f, f, φ, ..., are commutative between themselves and k, m, n, ..., are positive integers, we have

$$\mathrm{f}^n f^m z = f^m \mathrm{f}^n z. \tag{17}$$

Also, denoting $\mathrm{f}fz$ by φz;

$$\varphi^n z = \mathrm{f}^n f^n z = f^n \mathrm{f}^n z. \tag{18}$$

Finally, denoting $\mathrm{f}^n f^m z$ by ψz;

$$\psi^k z = \mathrm{f}^{kn} f^{km} z = f^{km} \mathrm{f}^{kn} z. \tag{19}$$

10. If the monomial functions making up a polynomial function are, at the same time, distributive and commutative between themselves, all orders of the polynomial functions are distributive functions (we already know this from §6) and commutative, not only with the different orders of the constituent functions, but also with all orders of the distributive functions which are commutative with these latter functions.

Let [102]

$$Fz = \mathrm{f}z + fz + \dots$$

and suppose that the distributive functions $\mathrm{f}, f, \dots$ are commutative both between themselves and with an arbitrary distributive function φ . We have (§6)

$$fFz = \mathrm{f}^2 z + \mathrm{f}fz + \dots = \mathrm{f}^2 z + f\mathrm{f}z + \dots = F\mathrm{f}z.$$

We also find that

$$fFz = Ffz, \dots, \varphi Fz = F\varphi z.$$

[11] The right hand side was $\mathrm{ff}^{-1}\mathrm{f}z$ in the original text.

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Adding to this the consideration given by formula (17), this proposition is completely demonstrated.

11. If the monomial functions of two polynomials are distributive and commutative between themselves, the two polynomial functions will be distributive (by §6) and commutative between themselves.

Indeed, let

$$Fz = \mathrm{f}z + fz + \ldots, \quad F'z = \mathrm{f}'z + f'z + \ldots.$$

We clearly have

$$\left.\begin{aligned} FF'z &= \mathrm{f}\mathrm{f}'z + \mathrm{f}f'z + \ldots + f\mathrm{f}'z + ff'z + \ldots \\ F'Fz &= \mathrm{f}'\mathrm{f}z + \mathrm{f}'fz + \ldots + f'\mathrm{f}z + f'fz + \ldots. \end{aligned}\right\} \tag{20}$$

Now, according to the hypothesis, these two expansions are composed of terms that are identical two by two, and so we have

$$FF'z = F'Fz.$$

If we further let

$$F''z = \mathrm{f}''z + f''z + \ldots,$$

then supposing f'', f'', ... to be distributive and commutative between themselves and with f, f, ..., f', f', ..., then F'' will be commutative with F, F'. As a consequence, by (§7), we have

$$FF'F''z = FF''F'z = F'FF''z = F'F''Fz = F''FF'z = F''F'Fz,$$

and so on.

12. The expansion of composed monomial functions, such [103] as $FF'z$, $FF'F''z, \ldots$ (§11), whose simple functions are themselves polynomial functions, such that the constituent monomial functions are distributive and commutative among themselves, presents no difficulty. In equation (20), we have this for functions of the type $FF'z$. By the same procedure, we pass from those of this type to those of the type $FF'F''z$, and so on. *We therefore know how to expand all functions of the form*

$$FF'\ldots z = (\mathrm{f} + f + \ldots)(\mathrm{f}' + f' + \ldots)\ldots z. \tag{21}$$

The general expansion of an arbitrary order $F^n z$ of a polynomial function Fz, composed of monomial functions that are distributive and commutative highlights the general theory of the expansion of functions into series, whose principles we will now explain.

13. Suppose we have, respectively

$$\left.\begin{array}{lccccc} & x = \alpha, & x = \beta, & x = \gamma, & x = \delta, & \ldots, \\ \text{whenever} & \varphi x = 0, & \varphi' x = 0, & \varphi'' x = 0, & \varphi''' x = 0, & \ldots. \end{array}\right\} \tag{22}$$

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I write the indefinite sequence of equations

$$\left.\begin{aligned} Fx &= F\alpha + \varphi x F'x, \\ F'x &= F'\beta + \varphi' x F''x, \\ F''x &= F''\gamma + \varphi'' x F'''x, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{23}$$

equations, which I render identical, in supposing that

$$F'x = \frac{Fx - F\alpha}{\varphi x},\ F''x = \frac{F'x - F'\beta}{\varphi' x},\ F'''x = \frac{F''x - F''\gamma}{\varphi'' x},\ \ldots. \tag{24}$$

I take the sum of the respective products of the equation (23) by $1, \varphi x, \varphi x \cdot \varphi' x, \varphi x \cdot \varphi' x \cdot \varphi'' x, \ldots$, and obtain, after reducing

$$Fx = F\alpha + \varphi x \cdot F'\beta + \varphi x \cdot \varphi' x \cdot F''\gamma + \varphi x \cdot \varphi' x \cdot \varphi'' x \cdot F'''\delta + \ldots. \tag{25}$$

Equations (24) immediately give [104]

$$\left.\begin{array}{llll} F'\beta = \dfrac{F\beta - F\alpha}{\varphi\beta}, & F'\gamma = \dfrac{F\gamma - F\alpha}{\varphi\gamma}, & F'\delta = \dfrac{F\delta - F\alpha}{\varphi\delta}, & \ldots, \\ F''\gamma = \dfrac{F'\gamma - F'\beta}{\varphi'\gamma}, & F''\delta = \dfrac{F'\delta - F'\beta}{\varphi'\delta}, & F''\varepsilon = \dfrac{F'\varepsilon - F'\beta}{\varphi'\varepsilon}, & \ldots, \\ F'''\delta = \dfrac{F''\delta - F''\gamma}{\varphi''\delta}, & F'''\varepsilon = \dfrac{F''\varepsilon - F''\gamma}{\varphi''\varepsilon}, & F'''\zeta = \dfrac{F''\zeta - F''\gamma}{\varphi''\zeta}, & \ldots, \\ \cdots\cdots\cdots, & \cdots\cdots\cdots, & \cdots\cdots\cdots, & \ldots. \end{array}\right\} \tag{26}$$

Now from these (26) we easily determine the coefficients $F'\beta$, $F''\gamma$, $F'''\delta$, $\ldots$ of equation (25), expressed only in terms of the functions F, φ, φ', φ'', $\ldots$ and the constants α, β, γ, $\ldots$. Indeed, we have

$$\left.\begin{aligned} F'\beta &= \frac{(F\beta - F\alpha)}{\varphi\beta}, \\ F''\gamma &= \frac{(F\gamma - F\alpha)}{\varphi\gamma.\varphi'\gamma} - \frac{(F\beta - F\alpha)}{\varphi\beta.\varphi'\gamma}, \\ F'''\delta &= \frac{(F\delta - F\alpha)}{\varphi\delta.\varphi'\delta.\varphi''\delta} - \frac{(F\gamma - F\alpha)}{\varphi\gamma.\varphi'\gamma.\varphi''\delta} + \frac{(F\beta - F\alpha)(\varphi'\delta - \varphi'\gamma)}{\varphi\beta.\varphi'\gamma.\varphi'\delta.\varphi''\delta}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{27}$$

Here we have series (25), in a very general form, established analytically, by a very natural procedure that has the appearance of the greatest simplicity, so that it seems that nothing remains but to apply it to various particular cases. But we must remark that this procedure also presents serious difficulties. The first is the great difficulty of deducing, even in the simplest cases, the relationship among the coefficients $F'\beta$, $F''\gamma$, $\ldots$. The second, and this is the major difficulty, is that it yields nothing in what is perhaps the most useful case, that of equality among some or all of the constants α, β, , ..., for then some or all of the constant factors take the indeterminate form $\frac{0}{0}$. In particular, this is what occurs when all the functions φx, $\varphi' x$, ..., are equal and consequently when we wish to expand Fx in terms of the powers of another function φx. It also occurs when the functions φx, $\varphi' x$, ..., are different from [105] one another, but are all of the form $x^n \psi x$. However, upon further examination, we recognize that these difficulties are not insurmountable, and that they disappear when we modify the procedure slightly and, in particular, when we don't attack the general problem. Here is what I have found to be simplest in this regard.

14. In $F(x+y)$ I consider only y as variable, having α as an arbitrary and constant increment. I write the identity

$$F(x+y) = Fx + y\left\{\frac{F(x+y)-Fx}{y}\right\},$$

which, in letting

$$\frac{F(x+y)-Fx}{y} = fy \tag{28}$$

becomes

$$F(x+y) = Fx + y\,fy. \tag{29}$$

I take the successive differences of equation (29), with respect to y only, and from this, I observe that in general (3)

$$\Delta(\varphi y \cdot \psi y) = \varphi(y+\alpha)\cdot\psi(y+\alpha) - \varphi y \cdot \psi y,$$

or rather

$$\Delta(\varphi y \cdot \psi y) = \varphi y \cdot \Delta\psi y + \Delta\varphi y \cdot \psi(y+\alpha). \tag{30}$$

Consequently, I have successively

$$\begin{array}{rcl} \Delta F(x+y) & = & \alpha fy + (y+\alpha)\Delta fy, \\ \Delta^2 F(x+y) & = & 2\alpha\Delta fy + (y+2\alpha)\Delta^2 fy, \\ \Delta^3 F(x+y) & = & 3\alpha\Delta^2 fy + (y+3\alpha)\Delta^3 fy, \\ \multicolumn{3}{c}{\dots\dots\dots\dots\dots\dots\dots\dots} \end{array}$$

From this I deduce, by transposition [106]

$$\left.\begin{array}{rcl} fy & = & \dfrac{\Delta F(x+y)}{\alpha} - \dfrac{(y+\alpha)}{\alpha}\Delta fy, \\ 2\Delta fy & = & \dfrac{\Delta^2 F(x+y)}{\alpha} - \dfrac{(y+2\alpha)}{\alpha}\Delta^2 fy, \\ 3\Delta^2 fy & = & \dfrac{\Delta^3 F(x+y)}{\alpha} - \dfrac{(y+3\alpha)}{\alpha}\Delta^3 fy, \\ \multicolumn{3}{c}{\dots\dots\dots\dots\dots\dots\dots\dots} \end{array}\right\} \tag{31}$$

Finally, if we take the sum of the products of these equations (31) by

$$y, \quad -\frac{y(y+\alpha)}{1\cdot 2\cdot\alpha}, \quad +\frac{y(y+\alpha)(y+2\alpha)}{1\cdot 2\cdot 3\cdot\alpha^2}, \quad \dots,$$

respectively, then by reducing and making use of equation (29), we have

$$F(x+y) = Fx + \frac{y}{\alpha}\Delta F(x+y) - \frac{y(y+\alpha)}{1\cdot 2\cdot \alpha^2}\Delta^2 F(x+y) + \ldots.$$

Transposing this, we have

$$\begin{aligned} Fx &= F(x+y) - \frac{y}{\alpha}\Delta F(x+y) + \frac{y(y+\alpha)}{1\cdot 2\cdot \alpha^2}\Delta^2 F(x+y) \\ &\quad - \frac{y(y+\alpha)(y+2\alpha)}{1\cdot 2\cdot 3\cdot \alpha^3}\Delta^3 F(x+y) + \ldots. \end{aligned} \tag{32}$$

We may also express this expansion in several other very remarkable forms.

First of all, I let $x+y=p$, which gives

$$\Delta(x+y) = \Delta y = \Delta p = \alpha,$$

because x is constant. Consequently, the expression $\Delta^n F(x+y)$ clearly becomes $\Delta^n Fp$, the differences being taken with respect to p, which varies by α. Thus, we have

$$\begin{aligned} Fx &= Fp + \frac{(x-p)}{\alpha}\Delta Fp + \frac{(x-p)(x-p-\alpha)}{1\cdot 2\cdot \alpha^2}\Delta^2 Fp \\ &\quad + \frac{(x-p)(x-p-\alpha)(x-p-2\alpha)}{1\cdot 2\cdot 3\cdot \alpha^3}\Delta^3 Fp + \ldots. \end{aligned} \tag{33}$$

[107] In this new expansion, I change x to $x+n\alpha$. Thus by (2), the left hand side becomes

$$F(x+n\alpha) = E^n Fx.$$

In the right hand side, $x-p$ becomes $x-p+n\alpha$. After this, I change p to x, and so Δp becomes Δx and $\Delta^n Fp$ becomes $\Delta^n Fx$. Since the differences are taken with respect to x, which varies by α, we have

$$\begin{aligned} E^n Fx &= F(x+n\alpha) \qquad (34) \\ &= Fx + n\Delta Fx + \frac{n}{1}\cdot\frac{n-1}{2}\Delta^2 Fx + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\Delta^3 Fx + \ldots. \end{aligned}$$

Now I let $n\alpha = m$, from which $n = \frac{m}{\alpha}$, and I have

$$\begin{aligned} F(x+m) &= Fx + \frac{m}{\alpha}\Delta Fx + \frac{m(m-\alpha)}{1\cdot 2\cdot \alpha^2}\Delta^2 Fx \\ &\quad + \frac{m(m-\alpha)(m-2\alpha)}{1\cdot 2\cdot 3\cdot \alpha^3}\Delta^3 Fx + \ldots. \end{aligned} \tag{35}$$

In equation (35), I let $x=0$, which I will express with respect to the functions $Fx, \ldots, \Delta^n Fx$ by writing $Fx_0, ..., \Delta^n Fx_0$. Then I change m to x and I have[12]

$$Fx = Fx_0 + \frac{x}{\alpha}\Delta Fx_0 + \frac{x(x-\alpha)}{1\cdot 2\cdot \alpha^2}\Delta^2 Fx_0 + \frac{x(x-\alpha)(x-2\alpha)}{1\cdot 2\cdot 3\cdot \alpha^3}\Delta^3 Fx_0 + \ldots. \tag{36}$$

[12] This is the Newton Forward Difference Series, more or less.

15. Series (33) is also given by the procedure of §13, where we let

$$\varphi x = x - p, \quad \varphi' x = x - p - \alpha, \quad \varphi'' x = x - p - 2\alpha, \ldots,$$

but it is much more difficult to derive the general and quite simple form of $\Delta^m Fp$ that includes all the constant factors. We conclude immediately from this series the possibility of expanding Fx according to the positive integer powers of $\frac{x-p}{\alpha}$, whereas the procedure of §13 yields nothing in this regard. Indeed, when the products [108]

$$\frac{x-p}{\alpha}, \ \frac{(x-p)(x-p-\alpha)}{\alpha^2}, \ \frac{(x-p)(x-p-\alpha)(x-p-2\alpha)}{\alpha^3}, \ldots,$$

are expanded, all have the form

$$A\left(\frac{x-p}{\alpha}\right) + B\left(\frac{x-p}{\alpha}\right)^2 + C\left(\frac{x-p}{\alpha}\right)^3 + \ldots,$$

so that after this expansion, it will be simply a matter of rearranging the terms with respect to the powers $\left(\frac{x-p}{\alpha}\right), \left(\frac{x-p}{\alpha}\right)^2, \ldots$. Then, without calculating, we already recognize that the coefficient of the first power $\dfrac{x-p}{\alpha}$ will be the series

$$\Delta Fp - \frac{1}{2}\Delta^2 Fp + \frac{1}{3}\Delta^3 Fp - \ldots. \tag{37}$$

Furthermore, it would not be difficult to determine all of the coefficients by means of this consideration alone. However, it will be quicker to investigate this by a procedure analogous to that which was just used (§14).

First of all, I take the sum of the products of equation (31) by $+1, -\frac{1}{2}, +\frac{1}{3}, -\frac{1}{4} + \ldots$, respectively. Multiplying through by α, this gives

$$\begin{aligned} \alpha f y &= \Delta F(x+y) - \frac{1}{2}\Delta^2 F(x+y) + \frac{1}{3}\Delta^3 F(x+y) - \ldots \\ &\quad -y\left\{\Delta f y - \frac{1}{2}\Delta^2 f y + \frac{1}{3}\Delta^3 f y - \ldots\right\}. \end{aligned} \tag{38}$$

Here, I let

$$\Delta F(x+y) - \frac{1}{2}\Delta^2 F(x+y) + \frac{1}{3}\Delta^3 F(x+y) - \ldots = \mathrm{dF}(x+y),$$

according to which notation we have

$$\Delta f y - \frac{1}{2}\Delta^2 f y + \frac{1}{3}\Delta^3 f y - \ldots = \mathrm{d} f y,$$

and in general,

$$\Delta z - \frac{1}{2}\Delta^2 z + \frac{1}{3}\Delta^3 z - \ldots = \mathrm{d}z. \tag{39}$$

This completely defines a new function of z, [109] polynomial or even *infinitinomial*,[13] in general, which I call the *differential* of z.

It follows immediately that

$$\Delta \mathrm{d}z - \frac{1}{2}\Delta^2\mathrm{d}z + \frac{1}{3}\Delta^3\mathrm{d}z - \ldots = \mathrm{d}^2 z,$$

and, in general,

$$\Delta \mathrm{d}^n z - \frac{1}{2}\Delta^2\mathrm{d}^n z + \frac{1}{3}\Delta^3\mathrm{d}^n z - \ldots = \mathrm{d}^{n+1} z. \tag{40}$$

$\mathrm{d}^2 z$, $\mathrm{d}^3 z$, $\ldots$, $\mathrm{d}^n z$, are called the *differentials of z of different orders*.

Given this, equation (38) becomes

$$\alpha f y = \mathrm{d}F(x+y) - y\mathrm{d}fy. \tag{41}$$

Taking the successive differences of this, I have, by formula (30)

$$\begin{array}{rcl}
\alpha\Delta fy &=& \Delta\mathrm{d}F(x+y) - \alpha\mathrm{d}fy - (y+\alpha)\Delta\mathrm{d}fy, \\
\alpha\Delta^2 fy &=& \Delta^2\mathrm{d}F(x+y) - 2\alpha\Delta\mathrm{d}fy - (y+2\alpha)\Delta^2\mathrm{d}fy, \\
\alpha\Delta^3 fy &=& \Delta^3\mathrm{d}F(x+y) - 3\alpha\Delta^2\mathrm{d}fy - (y+3\alpha)\Delta^3\mathrm{d}fy, \\
\ldots & \ldots & \ldots
\end{array}$$

I take the sum of the products of these equations by $+1$, $-\frac{1}{2}$, $+\frac{1}{3}$, $-\frac{1}{4}$, $\ldots$, respectively, and, in reducing, I have

$$\begin{aligned}
&\alpha\left(\Delta fy - \tfrac{1}{2}\Delta^2 fy + \tfrac{1}{3}\Delta^3 fy - \ldots\right) \\
&\quad = \Delta\mathrm{d}F(x+y) - \tfrac{1}{2}\Delta^2\mathrm{d}F(x+y) + \tfrac{1}{3}\Delta^3\mathrm{d}F(x+y) - \ldots \\
&\qquad -\alpha\mathrm{d}fy - y\left(\Delta\mathrm{d}fy - \tfrac{1}{2}\Delta^2\mathrm{d}fy + \tfrac{1}{3}\Delta^3\mathrm{d}fy - \ldots\right),
\end{aligned}$$

an equation which, using the notation of (39) and (40), becomes

$$\alpha\mathrm{d}fy = \mathrm{d}^2F(x+y) - \alpha\mathrm{d}fy - y\mathrm{d}^2fy,$$

or rather

$$2\alpha\mathrm{d}fy = \mathrm{d}^2F(x+y) - y\mathrm{d}^2fy. \tag{42}$$

I perform the same operations on this equation as on equation (41); that is, I take the sum of the products of the successive differences with $+1, -\frac{1}{2}, +\frac{1}{3}, -\frac{1}{4}, \ldots$, respectively. In reducing, this gives me [110]

$$3\alpha\mathrm{d}fy = \mathrm{d}^3F(x+y) - y\mathrm{d}^3fy, \tag{43}$$

using the notation of equations (39) and (40).

[13] Servois uses the term *infinitinôme* here. Both "infinitinome" and "infinitinomial" were occasionally used in English by Newton and others in the 18th century.

The procedures given for deriving equation (42) from (41) clearly can be used to derive equation (43) from (42), and then to a new equation, and so on. Thus, by a rigorous induction, we obtain the following indefinite sequence of equations

$$\begin{aligned} \alpha f y &= \mathrm{d}F(x+y) - y\mathrm{d}fy, \\ 2\alpha\mathrm{d}fy &= \mathrm{d}^2F(x+y) - y\mathrm{d}^2fy, \\ 3\alpha\mathrm{d}^2fy &= \mathrm{d}^3F(x+y) - y\mathrm{d}^3fy, \\ 4\alpha\mathrm{d}^3fy &= \mathrm{d}^4F(x+y) - y\mathrm{d}^4fy, \\ &\cdots\cdots\cdots \end{aligned}$$

Taking the sum of the products of these by

$$\frac{y}{\alpha}, \quad -\frac{y^2}{1\cdot 2\cdot \alpha^2}, \quad +\frac{y^3}{1\cdot 2\cdot 3\cdot \alpha^3}, \quad -\frac{y^4}{1\cdot 2\cdot 3\cdot 4\cdot \alpha^4}, \quad \ldots,$$

respectively, we come, by means of the defining equation (29), to

$$F(x+y) = Fx + \frac{y}{\alpha}\mathrm{d}F(x+y) - \frac{y^2}{1\cdot 2\cdot \alpha^2}\mathrm{d}^2F(x+y) + \frac{y^3}{1\cdot 2\cdot 3\cdot \alpha^3}\mathrm{d}^3F(x+y) - \ldots,$$

or, by transposition

$$\begin{aligned} Fx &= F(x+y) - \frac{y}{\alpha}\mathrm{d}F(x+y) + \frac{y^2}{1\cdot 2\cdot \alpha^2}\mathrm{d}^2F(x+y) \\ &\quad - \frac{y^3}{1\cdot 2\cdot 3\cdot \alpha^3}\mathrm{d}^3F(x+y) + \ldots, \end{aligned} \tag{44}$$

This series is very much analogous to series (32) and, like this latter, by the same procedures, may take various different forms, namely:[14]

$$Fx = Fp + \frac{(x-p)}{\alpha}\mathrm{d}Fp + \frac{(x-p)^2}{1\cdot 2\cdot \alpha^2}\mathrm{d}^2Fp + \frac{(x-3)^3}{1\cdot 2\cdot 3\cdot \alpha^3}\mathrm{d}^3Fp + \ldots, \tag{45}$$

$$E^nFx = F(x+n\alpha) = Fx + \frac{n}{1}\mathrm{d}Fx + \frac{n^2}{1\cdot 2}\mathrm{d}^2Fx + \frac{n^3}{1\cdot 2\cdot 3}\mathrm{d}^3Fx + \ldots, \tag{46}$$

$$F(x+m) = Fx + \frac{m}{\alpha}\mathrm{d}Fx + \frac{m^2}{1\cdot 2\cdot \alpha^2}\mathrm{d}^2Fx + \frac{m^3}{1\cdot 2\cdot 3\cdot \alpha^3}\mathrm{d}^3Fx + \ldots, \tag{47}$$

and

$$Fx = Fx_0 + \frac{x}{\alpha}\mathrm{d}Fx_0 + \frac{x^2}{1\cdot 2\cdot \alpha^2}\mathrm{d}^2Fx_0 + \frac{x^3}{1\cdot 2\cdot 3\cdot \alpha^3}\mathrm{d}^3Fx_0 + \ldots. \tag{48}$$

16. I'm eager to apply these formulas to the expansion of the various orders of the same function.

Let

$$Fx = \varphi^x z,$$

and let the constant increment in x be α. Then we have, by (3),

$$\Delta Fx = \varphi^{x+\alpha}z - \varphi^x z.$$

[14] The last of these is Taylor's Theorem, when we understand α to be dx.

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If the function φ is *distributive*, this expression will become

$$\Delta Fx = \varphi^x \left(\varphi^\alpha z - z\right). \tag{49}$$

Admitting this hypothesis for the time being, let

$$\varphi^\alpha z - z = fz. \tag{50}$$

By the theorems of §5 and §6, φ^α and f will be distributive functions and, because of (49), we have

$$\Delta Fx = \varphi^x fz.$$

Taking the difference of this, we have

$$\Delta^2 Fx = \varphi^{x+\alpha} fz - \varphi^x fz = \varphi^x \left(\varphi^\alpha fz - fz\right). \tag{51}$$

If the function φ is *commutative* with constant factors, it will also be commutative with the binomial function f in (50), by virtue of the theorem of §10. That is, we have

$$\varphi^\alpha fz = f\varphi^\alpha z.$$

Admitting this hypothesis in addition, then because f is distributive, we will have, from (50),

$$\varphi^\alpha fz - fz = f\varphi^\alpha z - fz = f\left(\varphi^\alpha z - z\right) = f^2 z,$$

and so equation (51) becomes [112]

$$\Delta^2 Fx = \varphi^x f^2 z.$$

We also find that

$$\Delta^3 Fx = \varphi^x f^3 z, \quad \Delta^4 Fx = \varphi^x f^4 z, \quad \ldots.$$

By an obvious induction, we have

$$\Delta^m Fx = \varphi^x f^m z,$$

an expression which, using the notation of §2, becomes

$$\Delta^m Fx = \varphi^x \left(\varphi^\alpha - 1\right)^m z. \tag{52}$$

Now, by (6) we have

$$Fx_0 = \varphi^0 z = z \quad \text{and} \quad \Delta^m Fx_0 = \left(\varphi^\alpha - 1\right)^m z.$$

Hence, by formula (36), we have

$$\begin{aligned} \varphi^x z \quad = \quad & z + \frac{x}{\alpha}\left(\varphi^\alpha - 1\right) z + \frac{x(x-\alpha)}{1\cdot 2\cdot \alpha^2}\left(\varphi^\alpha - 1\right)^2 z \\ & + \frac{x(x-\alpha)(x-2\alpha)}{1\cdot 2\cdot 3\cdot \alpha^3}\left(\varphi^\alpha - 1\right)^3 z + \ldots. \end{aligned} \tag{53}$$

Finally, combining definition (39) and formula (52), we find that

$$\begin{aligned} \mathrm{d}Fx &= \Delta Fx - \frac{1}{2}\Delta^2 Fx + \ldots \\ &= \varphi^x \left[(\varphi^\alpha - 1)\, z - \frac{1}{2}(\varphi^\alpha - 1)^2 z + \frac{1}{3}(\varphi^\alpha - 1)^3 z - \ldots \right]. \end{aligned} \quad (54)$$

I usually denote the polynomial function contained above within the square brackets by $\ln \varphi^\alpha z$. The notation ln therefore denotes a definite function of $\varphi^\alpha z$, whose complete definition is given by the equation[15]

$$\ln \varphi^\alpha z = (\varphi^\alpha - 1)\, z - \frac{1}{2}(\varphi^\alpha - 1)^2 z + \frac{1}{3}(\varphi^\alpha - 1)^3 z - \ldots. \quad (55)$$

The function ln is called the *Logarithm* and $\ln \varphi^\alpha z$ is a monomial composed function called the *Logarithm of* φ^α *of* z. Clearly (by §10) the function $\ln \varphi^\alpha$ is not only distributive, but commutative with the function φ and with constant factors. The same is not true for the simple function ln.

Therefore, equation (54) becomes [113]

$$\mathrm{d}Fx = \varphi^x \ln \varphi^\alpha z.$$

From this, we conclude immediately that

$$\mathrm{d}^2 Fx = \varphi^x (\ln \varphi^\alpha)^2 z,\ \mathrm{d}^3 Fx = \varphi^x (\ln \varphi^\alpha)^3 z,\ \ldots,\ \mathrm{d}^m Fx = \varphi^x (\ln \varphi^\alpha)^m z. \quad (56)$$

Consequently, letting $x = 0$ in $Fx, \mathrm{d}Fx, \ldots, \mathrm{d}^m Fx$, we have, by formula (48), this other expansion for $\varphi^x z$,

$$\varphi^x z = z + \frac{x}{\alpha} \ln \varphi^\alpha z + \frac{x^2}{1 \cdot 2 \cdot \alpha^2} (\ln \varphi^\alpha)^2 z + \frac{x^3}{1 \cdot 2 \cdot 3 \cdot \alpha^2} (\ln \varphi^\alpha)^3 z + \ldots. \quad (57)$$

Let us draw several important consequences. Because the increment α in (57) is arbitrary, I make it equal to 1 and I have[16]

$$\varphi^x z = z + \frac{x}{1} \ln \varphi z + \frac{x^2}{1 \cdot 2} (\ln \varphi)^2 z + \frac{x^2}{1 \cdot 2 \cdot 3} (\ln \varphi)^3 z + \ldots \quad (58)$$

Comparing this expression term by term with that of equation (57), then because x is completely arbitrary we obtain the relation

$$\alpha \ln \varphi z = \ln \varphi^\alpha z. \quad (59)$$

Let f be a distributive function, commutative with φ and with constant factors. Applying the function f^x to both sides of equation (58) we have, by formula (18) of §9,

$$f^x \varphi^x z = (f\varphi)^x z = f^x z + \frac{x}{1} f^x \ln \varphi z + \frac{x^2}{1 \cdot 2} f^x (\ln \varphi)^2 z + \ldots.$$

[15] We use the modern ln to denote the natural logarithm where Servois used L.
[16] In [Servois 1814a], the "+" between the terms of order 2 and order 3 was missing in the equation below.

Expanding each term of the right hand side of this using formula (58), we clearly have[17]

$$\left.\begin{aligned}(f\varphi)^x z = z\ &+ x\ln fz + \frac{x^2}{1\cdot 2}(\ln f)^2 z + \dots\dots\dots\\ &+ x\ln\varphi z + 2\frac{x^2}{1\cdot 2}(\ln f)(\ln\varphi)z + \dots\\ &+\frac{x^2}{1\cdot 2}(\ln\varphi)^2 z + \dots\dots\dots\\ &+\dots\dots\dots\end{aligned}\right\} \tag{60}$$

[114] from which, using (58) again, we have this other expression

$$(f\varphi)^x z = z + x(\ln f\varphi)z + \frac{x^2}{1\cdot 2}(\ln f\varphi)^2 z + \dots .$$

Therefore, comparing this term by term to (60), we have the relation[18]

$$\ln f\varphi z = \ln fz + \ln\varphi z, \tag{61}$$

because of the indeterminacy of x.

Suppose that

$$\ln\varphi z = \psi z$$

and take the inverse function $\ln^{-1}$ on both sides. Then we have

$$\varphi z = \ln^{-1}\psi z \quad\text{and}\quad \varphi^x z = \left(\ln^{-1}\psi\right)^x z.$$

As a consequence, using formula (58),

$$\left(\ln^{-1}\psi\right)^x z = z + \frac{x}{1}\psi z + \frac{x^2}{1\cdot 2}\psi^2 z + \frac{x^3}{1\cdot 2\cdot 3}\psi^3 z + \dots . \tag{62}$$

Once again, let f and φ be two distributive functions that are commutative both with each other and with constant factors. Since u and x are arbitrary exponents, we immediately have by (1),

$$f^u\varphi^x z = \ln^{-1}\ln f^u\varphi^x z, \tag{63}$$

but, by (61) and (59), we also have

$$\ln f^u\varphi^x z = \ln f^u z + \ln\varphi^x z = u\ln fz + x\ln\varphi z.$$

Therefore, by (63), using the notation of §2,

$$f^u\varphi^x z = \ln^{-1}(u\ln f + x\ln\varphi)z, \tag{64}$$

[17] In [Servois 1814a], the variable z was missing from the last term of the first line below.

[18] In [Servois 1814a], the last term of this equation was given as $\mathrm{L}\varphi x$, but the variable z was almost certainly what was intended.

and, because of (62),

$$\begin{aligned} f^u\varphi^x z &= z + (u\ln f + x\ln\varphi)\,z + \frac{1}{1\cdot 2}\,(u\ln f + x\ln\varphi)^2\,z \\ &\quad + \frac{1}{1\cdot 2\cdot 3}\,(u\ln f + x\ln\varphi)^3\,z + \ldots. \end{aligned} \tag{65}$$

Let us make some particular assumptions about the form of the function φ. First of all, let [115]

$$\varphi z = z + fz + (1+f)z.$$

Taking $\alpha = 1$, we immediately have from (53), (58), and (55)

$$\left.\begin{aligned} (1+f)^x z &= z + \frac{x}{1}fz + \frac{x}{1}\frac{x-1}{2}f^2 z + \frac{x}{1}\frac{x-1}{2}\frac{x-2}{3}f^3 z + \ldots \\ (1+f)^x z &= z + \frac{x}{1}\ln(1+f)z + \frac{x^2}{1\cdot 2}\left[\ln(1+f)\right]^2 z + \ldots \\ \ln(1+f)z &= fz - \frac{1}{2}f^2 z + \frac{1}{3}f^3 z - \frac{1}{4}f^4 z + \ldots. \end{aligned}\right\} \tag{66}$$

Let

$$\varphi z = fz + \mathrm{f}z.$$

Taking the inverse function f^{-1} of both sides, I have

$$f^{-1}\varphi z = z + f^{-1}\mathrm{f}z,$$

which, by letting

$$f^{-1}\varphi z = \psi z \quad \text{and} \quad f^{-1}\mathrm{f}z = Fz,$$

becomes

$$\psi z = z + Fz.$$

Using formula (66), I obtain

$$\begin{aligned} \psi^x z &= z + \frac{x}{1}Fz + \frac{x}{1}\cdot\frac{x-1}{2}F^2 z + \frac{x}{1}\cdot\frac{x-1}{2}\cdot\frac{x-2}{3}F^3 z + \ldots \\ \psi^x z &= z + \frac{x}{1}\ln(1+F)z + \frac{x}{1\cdot 2}\left[\ln(1+F)\right]^2 + \ldots \\ \ln(1+F)z &= Fz - \frac{1}{2}F^2 z + \frac{1}{3}F^3 z - \frac{1}{4}F^4 z + \ldots. \end{aligned}$$

In these formulas, I substitute the defining expression for ψz and Fz and I also apply the function f to both sides of the first two formulas. Then I have [116]

$$\left.\begin{aligned} \varphi^x = (f+\mathrm{f})^x z &= f^x z + \frac{x}{1}f^{x-1}\mathrm{f}z + \frac{x}{1}\frac{x-1}{2}f^{x-2}\mathrm{f}^2 z + \ldots \\ \varphi^x = (f+\mathrm{f})^x z &= f^x z + \frac{x}{1}\ln\left(1+\mathrm{f}f^{-1}\right)f^x z \\ &\quad + \frac{x^2}{1\cdot 2}\left[\ln\left(1+\mathrm{f}f^{-1}\right)\right]^2 f^x z + \ldots \\ \ln\left(1+\mathrm{f}f^{-1}\right)z &= \mathrm{f}f^{-1}z - \frac{1}{2}\mathrm{f}^2 f^{-2}z + \frac{1}{3}\mathrm{f}^3 f^{-3}z - \ldots \end{aligned}\right\} \tag{67}$$

Let

$$\varphi z = fz + \mathrm{f}z + \psi z.$$

We let $\mathrm{f}z + \psi z = Fz$ and by (67) we have the expansions related to

$$\varphi^x z = (f + F)^x z.$$

In these, we substitute for the different orders $F^2 z, F^3 z, \ldots$, their expansions as given by the same equation (67), using

$$F^x z = (\mathrm{f} + \psi)^x z.$$

We see, without needing to dwell on it, how to arrive at two expansions of order x of an arbitrary polynomial function of distributive and commutative functions. That is to say, *we know how to expand* the function

$$\varphi^x z = (f + \mathrm{f} + F + \psi + \ldots)^x z. \tag{68}$$

17. I will apply these generalities to functions given by the consideration of differences of variable quantities, functions which I call *differential functions*.

Suppose z is a function of two variables x and y only (what we say may be generalized without trouble to functions of more variables). Its differential functions, both *total* and *partial* are, by (§2)

$$Ez,\ \frac{E}{x}z,\ \frac{E}{y}z;\quad \Delta z,\ \frac{\Delta}{x}z,\ \frac{\Delta}{y}z;\quad \mathrm{d}z,\ \frac{\mathrm{d}}{x}z,\ \frac{\mathrm{d}}{y}z.$$

Here, we use the general notation of (§1) for partial functions [117] to express the partial differentials $\frac{\mathrm{d}}{x}z,\ \frac{\mathrm{d}}{y}z,\ \ldots$.

The definitions of the total differential functions (3), (4), and (39), expressed in terms of the notation of §2 for polynomial functions, are

$$\left.\begin{array}{c} E^n z = (1+\Delta)^n z, \qquad \Delta^n z = (E-1)^n z; \\ \mathrm{d}^n z = (\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \ldots)^n z = \left[(E-1) - \frac{1}{2}(E-1)^2 + \ldots\right]^n z. \end{array}\right\} \tag{69}$$

These may be used to express the partial differential functions, simply by changing E, Δ, and d to $\frac{E}{x}$, $\frac{\Delta}{x}$, and $\frac{\mathrm{d}}{x}$ or to $\frac{E}{y}$, $\frac{\Delta}{y}$, and $\frac{\mathrm{d}}{y}$, respectively.

Let us also add the formula which establishes the connection between total and partial functions, that is

$$Ez = \frac{E}{x}\frac{E}{y}z. \tag{70}$$

This is evidently true, since to have $\varphi(x+\alpha, y+\beta) = Ez$, it is sufficient to change, first of all, y to $y+\beta$ – that is, to take $\frac{E}{y}$ first – and then, in the result, to change x to $x+\alpha$ – that is, to take the varied state $\frac{E}{x}$, with respect to x, of $\frac{E}{y}z$.

Given this, it is easy to see, first of all, that all differential functions are *distributive*. Indeed, the varied states, E, $\frac{E}{x}$ and $\frac{E}{y}$ are clearly distributive,

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as are the constant factors. Now, using definition (69), the total or partial difference and differentials are polynomial functions whose composing functions are the orders of the varied states and of the constant factors. Thus, by virtue of the theorem of §6, they are themselves distributive.

[118] Secondly, all the varied states are *commutative* with constant factors. Indeed, it is quite remarkable that any varied state is commutative with any function of constant order. That is, we have

$$E\varphi z = \varphi E z, \quad \frac{E}{x}\varphi z = \varphi \frac{E}{x} z \quad \text{and} \quad \frac{E}{y}\varphi z = \varphi \frac{E}{y} z.$$

Indeed, it makes no difference at all whether we change x to $x + \alpha$ first, for example, in the function z and then take the function φ of z, or rather we take the function φ first, and then change x to $x + \alpha$ in it. It follows from this that the varied states are commutative, both with each other and with all the differences and differentials.

In the third place, differences and differentials, being commutative with the varied states, and being polynomial functions composed of varied states, which are commutative with constant factors, are themselves commutative with constant factors, by virtue of the theorem of §10.

In the fourth place, because of its definition, the partial difference $\frac{\Delta}{x} z$ is commutative with $\frac{\Delta}{y} z$ and $\frac{\mathrm{d}}{y} z$, by §10, because these latter two are commutative with $\frac{E}{x} z$ and with constant factors.

In the fifth place, because of its definition, the partial differential $\frac{\mathrm{d}}{x} z$ is commutative with $\frac{\mathrm{d}}{y} z$, by §10, because this latter is commutative with the various orders of $\frac{\Delta}{x} z$ and with constant factors.

Taking all these observations together it follows that all differential functions and their various orders, both positive and negative, are commutative functions, both among themselves and with constant factors. We may add to these the integral functions

$$\Sigma, \ \frac{\Sigma}{x}, \ \frac{\Sigma}{y}, \ \int, \ \frac{\int}{x}, \ \frac{\int}{y},$$

[119] as well as their various orders, as these functions are nothing but the negative orders of differences and differentials.

Therefore, all the formulas given in the previous article are immediately applicable to all of these functions. We immediately obtain several abbreviated expressions, of which the following are the most remarkable.

In formula (46), I substitute z in place of Fx. Comparing this to equation (62), I have

$$E^n z = \left(\ln^{-1} \mathrm{d}\right)^n z, \tag{71}$$

and consequently

$$\frac{E^n}{x} z = \left(\ln^{-1} \frac{\mathrm{d}}{x}\right)^n z \quad \text{and} \quad \frac{E^n}{y} z = \left(\ln^{-1} \frac{\mathrm{d}}{y}\right)^n z. \tag{72}$$

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From the previous expressions and the definition $\Delta^n z = (E-1)^n z$ (69), we immediately have

$$\Delta^n z = \left(\ln^{-1} d - 1\right)^n z, \qquad \frac{\Delta^n}{x} z = \left(\ln^{-1} \frac{\mathrm{d}}{x} - 1\right)^n z,$$
$$\frac{\Delta^n}{y} z = \left(\ln^{-1} \frac{\mathrm{d}}{y} - 1\right)^n z. \tag{73}$$

Comparing definitions (69) of the differential with formula (55), we have

$$\begin{aligned}
\mathrm{d}^n z &= [\ln(1+\Delta)]^n x = (\ln E)^n z, \\
\frac{\mathrm{d}^n}{x} z &= \left[\ln\left(1+\frac{\Delta}{x}\right)\right]^n z = \left(\ln \frac{E}{x}\right)^n z, \\
\frac{\mathrm{d}^n}{y} z &= \left[\ln\left(1+\frac{\Delta}{y}\right)\right]^n z = \left(\ln \frac{E}{y}\right)^n z.
\end{aligned} \tag{74}$$

If, in the formula $\Delta^n z = (E-1)^n z$, we substitute Ez for the equivalent expression $\frac{E}{x}\frac{E}{y} z$ which is itself equivalent to $\left(1+\frac{\Delta}{x}\right)\left(1+\frac{\Delta}{y}\right) z$ by (69), we have [120]

$$\begin{aligned}
\Delta^n z &= \left[\left(1+\frac{\Delta}{x}\right)\left(1+\frac{\Delta}{y}\right) - 1\right]^n z \\
&= \left[\frac{E}{y}\left(1+\frac{\Delta}{x}\right) - 1\right]^n z = \left[\frac{E}{x}\left(1+\frac{\Delta}{y}\right) - 1\right]^n z.
\end{aligned} \tag{75}$$

If we substitute the expression (70) in place of Ez in $\mathrm{d}^n z = (\ln E)^n z$ (74), we have

$$\mathrm{d}^n z = \left(\ln \frac{E}{x}\frac{E}{y}\right)^n z. \tag{76}$$

Now, using formula (61) and the expressions of (72), we have

$$\ln \frac{E}{x}\frac{E}{y} z = \ln \frac{E}{x} z + \ln \frac{E}{y} z = \frac{\mathrm{d}}{x} z + \frac{\mathrm{d}}{y} z = \left(\frac{\mathrm{d}}{x} + \frac{\mathrm{d}}{y}\right) z.$$

Therefore, in place of (76), we have

$$\mathrm{d}^n z = \left(\frac{\mathrm{d}}{x} + \frac{\mathrm{d}}{y}\right)^n z. \tag{77}$$

If, in equation (64), we change u, f, x, and φ to m, $\frac{E}{x}$, n, and $\frac{E}{y}$, respectively, we have

$$\frac{E^m}{x}\frac{E^n}{y} z = \varphi(x + m\alpha, y + n\beta) = \ln^{-1}\left(m \ln \frac{E}{x} + n \ln \frac{E}{y}\right) z.$$

By equation (62),[19] this becomes

$$\frac{E^m}{x}\frac{E^n}{y}z = \varphi\left(x + m\alpha, y + n\beta\right) = \ln^{-1}\left(m\frac{\mathrm{d}}{x} + n\frac{\mathrm{d}}{y}\right)z. \tag{78}$$

From §11 and §18, *we know how to expand* all such abbreviated expressions.

This is the appropriate place to make the observation that by combining differential functions among themselves and with constant factors, we may form an infinitude of new differential functions, all of which, by our general theorems (§5 through §10), will be distributive [121] and commutative, both among themselves and with constant factors. Therefore, if we assign particular expressions to polynomial functions, such as, for example,

$$az + bEz, \quad az + bEz + cE^2z, \quad \mathrm{d}z + a\mathrm{d}^2z + b\mathrm{d}^3z + \ldots,$$

we may form new algorithms, for which all of the theoretical and practical rules are in the formulas of §16. The *Calculus of Variations*, in particular, is the result of a consideration of this kind.

Constant factors, being functions that are eminently distributive and commutative among themselves, are clearly included as particular cases in our formulas. Therefore, the expression $\ln\varphi^\alpha z$ is the *natural logarithm of the factor* φ^α *multiplying* z. The other expression $\ln^{-1}\psi z$ is the same thing as the common expression $e^\psi z$; see §1. It is not even necessary to look elsewhere for a theory of logarithms - it is all there in definition (55) and in formulas (59), (61), and (62). For the same reason, the means of expansion given by the basic properties, for raising an arbitrary polynomial to an arbitrary power are all particular cases of those which lead to the expansion of formula (68).

18. In the preceding article, we have sketched the set of laws that brings together and unites all the differential functions, that is, the most general theory of the *differential calculus*. The practice of this calculus, which is nothing other than the execution of the operations given in the definitions, would not form a separate branch, had we not remarked that, for certain classes of variable functions, the reduced differential functions present themselves in much simpler forms than we might have expected. Moreover, in view of the current state of analysis, it is sufficient to recognize a small number of functions that we call *elementary*. If we know the differential functions of these, then following the ordinary rules of calculation, we are able to find the differentials of the variable functions composed of them. [122] It would be unnecessary[20] here to enter into any detail concerning the *varied states* and *differences* of the elementary functions. I will restrict myself to researching their *differentials*.

The *elementary simple* functions of a single variable x are the monomial functions

$$x^m, \quad a^x, \quad \ln x, \quad \sin x, \quad \cos x,$$

[19] Although Servois cites equation (62) here, the result seems in fact to follow from an application of (74). We are grateful to the referee who pointed this out.

[20] Servois says "*déplacé*," literally "inappropriate."

in which we apply a constant difference to x. The *elementary composed* functions are

$$\varphi x \cdot \psi x, \quad (\varphi x)^m, \quad a^{\varphi x}, \quad \ln \varphi x, \quad \sin \varphi x, \quad \cos \varphi x.$$

To express the differentials of these and, in general, any composite functions, in terms of the differentials of the simple functions, there exists an important theorem that we must first establish.

Let $y = \varphi x$ and $Fy = F\varphi x$, where φ and F are arbitrary functions. Supposing that the difference in y is the constant β, we have, by formula (47),

$$F(y+m) = Fy + \frac{m}{\beta}\mathrm{d}Fy + \frac{m^2}{1\cdot 2\cdot \beta^2}\mathrm{d}^2Fy + \frac{m^3}{1\cdot 2\cdot 3\cdot \beta^3}\mathrm{d}^3Fy + \ldots.$$

Here m is arbitrary, so we may let

$$m = n\mathrm{d}\varphi x + \frac{n^2}{1\cdot 2}\mathrm{d}^2\varphi x + \frac{n^3}{1\cdot 2\cdot 3}\mathrm{d}^3\varphi x + \ldots, \tag{79}$$

and we have

$$\left.\begin{aligned} F(y+m) = Fy + \frac{n}{\beta}\mathrm{d}Fy\cdot \mathrm{d}\varphi x &+ \frac{n^2}{1\cdot 2\cdot \beta}\mathrm{d}Fy\cdot \mathrm{d}^2\varphi x \quad && + \ldots \\ &+ \frac{n^2}{1\cdot 2\cdot \beta^2}\mathrm{d}^2Fy(\mathrm{d}\varphi x)^2 && + \ldots \\ & && + \ldots \end{aligned}\right\} \tag{80}$$

However, by formula (46), taking the assumption (79) into account, we have

$$\varphi(x+n\alpha) = \varphi x + \frac{n}{1}\mathrm{d}\varphi x + \frac{n^2}{1\cdot 2}\mathrm{d}^2\varphi x + \ldots = y + m.$$

Thus, [123]

$$F(y+m) = F\varphi(x+n\alpha).$$

I expand the right hand side of this, using the same formula (46), and I have the following expression for $F(y+m)$

$$F(y+m) = F\varphi x + \frac{n}{1}\mathrm{d}F\varphi x + \frac{n^2}{1\cdot 2}\mathrm{d}^2F\varphi x + \ldots.$$

Comparing, this to the previous formula (80), we immediately have that[21]

$$\mathrm{d}F\varphi x = \frac{\mathrm{d}Fy}{\beta}\cdot \mathrm{d}\varphi x, \tag{81}$$

because n is indeterminate. If we let $x = \psi t$ and give x the constant difference α, it is clear that we have

$$\mathrm{d}F\varphi\psi t = \frac{\mathrm{d}Fy}{\beta}\cdot \frac{\mathrm{d}\varphi x}{\alpha}\cdot \mathrm{d}\psi t,$$

[21] This is a form of the Chain Rule.

by formula (81), and so on.

Given this, and supposing the function φ in formula (56) to be the constant factor a, and z to be equal to 1, we have

$$\mathrm{d}a^x = a^x \ln a^\alpha, \tag{82}$$

where α is the constant increment in x. Under this assumption, we have $\alpha = \Delta x$, $0 = \Delta^2 x = \Delta^3 x = \ldots$. Consequently, it follows from definition (39) that

$$\mathrm{d}x = \Delta x = \alpha.$$

However, from (59) we have

$$\ln a^\alpha = \alpha \ln a,$$

so, in place of (82) we have

$$\mathrm{d}a^x = a^x \mathrm{d}x \ln a. \tag{83}$$

Now suppose that [124]

$$F\varphi x = Fy = a^{\varphi x} = a^y.$$

Then from theorem (81), we have

$$\mathrm{d}a^{\varphi x} = \frac{\mathrm{d}a^y}{\beta} \cdot \mathrm{d}\varphi x.$$

However, since $\mathrm{d}y = \beta$ by assumption, we have, by (83),

$$\mathrm{d}a^y = a^y \ln a = a^{\varphi x} \ln a.$$

Thus, we have

$$\mathrm{d}a^{\varphi x} = a^{\varphi x} \cdot \mathrm{d}\varphi x \cdot \ln a, \tag{84}$$

that is, the formula for differentiating exponentials.

If we note that $\ln a^{\varphi x} = \varphi x \ln a$, and consequently that $\mathrm{d}\varphi x \cdot \ln a = \mathrm{d} \ln a^{\varphi x}$, formula (84) becomes

$$\mathrm{d}a^{\varphi x} = a^{\varphi x} \cdot \mathrm{d} \ln a^{\varphi x},$$

in which, if we let $Fx = a^{\varphi x}$, which is permissible, we have[22]

$$\mathrm{d}Fx = Fx \cdot \mathrm{d} \ln Fx. \tag{85}$$

This is an expression of the following theorem: the differential of a variable function is always equal to the function multiplied by the differential of its logarithm.

[22] This is logarithmic differentiation.

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We conclude immediately that

$$\mathrm{d}\ln Fx = \frac{\mathrm{d}Fx}{Fx}, \tag{86}$$

which is the formula for differentiating natural logarithms. Taking note that $\ln(Fx)^m = m\ln Fx$, then by formulas (85) and (86) we have

$$\mathrm{d}(Fx)^m = (Fx)^m \cdot \mathrm{d}\ln(Fx)^m = m(Fx)^m \cdot \mathrm{d}\ln Fx = m(Fx)^{m-1} \cdot \mathrm{d}Fx, \tag{87}$$

which is the formula for differentiating powers.

Since $\ln(\varphi x \cdot Fx) = \ln\varphi x + \ln Fx$, we have by (85) [125]

$$\mathrm{d}\left(\varphi x \cdot Fx\right) = \varphi x \cdot Fx \cdot \mathrm{d}\ln\left(\varphi x \cdot Fx\right) = \varphi x \cdot Fx\left(\mathrm{d}\ln\varphi x + \mathrm{d}\ln Fx\right).$$

Therefore, from (86) we have

$$\mathrm{d}\left(\varphi x \cdot Fx\right) = Fx \cdot \mathrm{d}\varphi x + \varphi x \cdot \mathrm{d}Fx, \tag{88}$$

the formula for the differentiation of products.

Let

$$Fx = \frac{\cos\alpha x + \sqrt{-1}\sin\alpha x}{\cos^x\alpha} \tag{89}$$

where α is a constant and x is variable, with a constant difference of 1. We have

$$\Delta Fx = \frac{\cos\alpha(x+1) + \sqrt{-1}\sin\alpha(x+1)}{\cos^{x+1}\alpha} - \frac{\cos\alpha x + \sqrt{-1}\sin\alpha x}{\cos^x\alpha}.$$

Now, expanding the cosine and sine of $\alpha x+\alpha$ using the well-known trigonometric formulas, this reduces to

$$\Delta Fx = Fx \cdot \sqrt{-1}\tan\alpha.$$

Consequently, we have

$$\Delta^m Fx = Fx \cdot \left(\sqrt{-1}\tan\alpha\right)^m,$$

in general. Thus, according to definition (39), we have

$$\mathrm{d}Fx = Fx \cdot \left[\sqrt{-1}\tan\alpha - \frac{1}{2}\left(\sqrt{-1}\tan\alpha\right)^2 + \ldots\right].$$

Comparing this to formula (55), we have

$$\mathrm{d}Fx = Fx \cdot \ln\left(1 + \sqrt{-1}\tan\alpha\right). \tag{90}$$

However, by (88),

$$\begin{aligned}\mathrm{d}Fx \quad = \quad & \left(\frac{1}{\cos\alpha}\right)^m \cdot \mathrm{d}\left(\cos\alpha x + \sqrt{-1}\sin\alpha x\right) \\ & + \left(\cos\alpha x + \sqrt{-1}\sin\alpha x\right) \cdot \mathrm{d}\left(\frac{1}{\cos\alpha}\right)^x.\end{aligned} \tag{91}$$

On the one hand, by differentiating the well-known formula [126]

$$\cos^2 \alpha x + \sin^2 \alpha x = 1$$

using formula (87), we find that

$$\mathrm{d}\cos \alpha x = -\frac{\sin \alpha x}{\cos \alpha x}\mathrm{d}\sin \alpha x, \tag{92}$$

and consequently that

$$\mathrm{d}\left(\cos \alpha x + \sqrt{-1}\sin \alpha x\right) = \mathrm{d}\sin \alpha x \cdot \frac{\sqrt{-1}}{\cos \alpha x}\left(\cos \alpha x + \sqrt{-1}\sin \alpha x\right). \tag{93}$$

On the other hand, in recalling that $\mathrm{d}x = 1$ we have, by formula (83),

$$\mathrm{d}\left(\frac{1}{\cos \alpha}\right)^x = -\frac{1}{\cos \alpha}^x \ln \cos \alpha.$$

Therefore, substituting this expression and the one in (93) into (91), then comparing to (90), we have

$$\mathrm{d}\sin \alpha x \cdot \frac{\sqrt{-1}}{\cos \alpha x} - \ln \cos \alpha = \ln\left(1 + \sqrt{-1}\tan \alpha\right).$$

From this, if we let[23]

$$A\sqrt{-1} = \ln\left[\cos \alpha\left(1 + \sqrt{-1}\tan \alpha\right)\right],$$

we conclude that

$$\mathrm{d}\sin \alpha x = A\cos \alpha x. \tag{94}$$

Also, by substituting this expression in (92), we have

$$\mathrm{d}\cos \alpha x = -A\sin \alpha x. \tag{95}$$

If we change αx in these to x, we have the formulas

$$\mathrm{d}\sin x = \frac{A}{\alpha}\cos x \quad \text{and} \quad \mathrm{d}\cos x = -\frac{A}{\alpha}\sin x$$

Here, the difference in x is 1. If x were a function of a different variable, we would have, by virtue of theorem (81),

$$\mathrm{d}\sin x = \frac{A}{\alpha}\mathrm{d}x\cos x \quad \text{and} \quad \mathrm{d}\cos x = -\frac{A}{\alpha}\mathrm{d}x\sin x. \tag{96}$$

In these formulas, the quantity α is an arbitrary arc.[24]

[23] In the original the following equation was given as $A\sqrt{-1} = \ln\left(\cos \alpha + \sqrt{-1}\tan \alpha\right)$.

[24] Servois is using α to represent an arbitrary real number, but because it is the argument of sine, cosine and tangent, he also chooses to refer to it as an arc.

[127] The constant A, although implicitly imaginary,[25] is easily put into the form of a real number. Indeed, it follows from the well-known formula

$$\cos^2\alpha = \frac{1}{1+\tan^2\alpha} = \frac{1}{\left(1+\sqrt{-1}\tan\alpha\right)\left(1-\sqrt{-1}\tan\alpha\right)},$$

that

$$A\sqrt{-1} = \frac{1}{2}\ln\left[\cos^2\alpha\cdot\left(1+\sqrt{-1}\tan\alpha\right)^2\right] = \frac{1}{2}\ln\left(\frac{1+\sqrt{-1}\tan\alpha}{1-\sqrt{-1}\tan\alpha}\right).$$

Expanding the last expression according to a well-known logarithmic formula, and dividing by $\sqrt{-1}$,

$$A = \tan\alpha - \frac{1}{3}\tan^3\alpha + \frac{1}{5}\tan^5\alpha - \dots. \tag{97}$$

Therefore, had we not otherwise known that this expression for A was equal to α, we would still have the means, according to formula (96) and (97), to differentiate trigonometric functions. Furthermore, we may show that $\frac{A}{\alpha} = 1$ by elementary means alone (see *Theory of Analytic Functions*, No. 28 in the first edition, No. 23 in the second.)[26]

19. We have seen the differential calculus arise from the simple expansion of functions of one variable according to the powers of this variable. This calculus will now assist us in ascending to something even more general.

Suppose that, relating the variables x and y, we give the equation $V = 0$ and the equation $z = Fx$. We may suppose, at least, that we have deduced $y = \varphi x$ from the first of these, and that between this and the second equation, we have eliminated x, to give $z = fy$, so that the hypothesis amounts to giving us the three equations

$$y = \varphi x, \quad z = Fx, \quad \text{and} \quad z = fy. \tag{98}$$

Then, according to formula (45) we have [128]

$$Fx = fy = fp + \frac{(y-p)}{1}\frac{\mathrm{d}fp}{\beta} + \frac{(y-p)^2}{1\cdot 2}\frac{\mathrm{d}^2fp}{\beta^2} + \dots. \tag{99}$$

In this expression, p is arbitrary and has β as its constant difference. I differentiate equation (99) with respect to x alone and I have

$$\mathrm{d}Fx = \mathrm{d}y\cdot\frac{\mathrm{d}fp}{\beta} + \frac{(y-p)}{1}\mathrm{d}y\cdot\frac{\mathrm{d}^2fp}{\beta^2} + \dots. \tag{100}$$

Now I suppose that by letting $y = p$ in $V = 0$, we find that $x = \theta$, among others, and reciprocally. By (98), we have

$$p = \varphi\theta, \quad \mathrm{d}p = \mathrm{d}\varphi\theta, \quad \text{and} \quad fp = f\varphi\theta = F\theta.$$

[25] Such a use of "imaginary" by Servois and his contemporaries corresponds more or less to the modern word "complex." It does not mean a purely imaginary number.

[26] [Lagrange 1797].

Next, I let $y = p$ in (100) and this equation becomes

$$\mathrm{d}F\theta = \mathrm{d}\varphi\theta \cdot \frac{\mathrm{d}fp}{\beta}, \tag{101}$$

from which

$$\frac{\mathrm{d}fp}{\beta} = \frac{\mathrm{d}F\theta}{\mathrm{d}\varphi\theta}.$$

Equation (101) is the same as (81), derived in a different manner. I divide equation (100) by $\mathrm{d}y$, I differentiate with respect to x, and I have

$$\mathrm{d}\left(\frac{\mathrm{d}Fx}{\mathrm{d}y}\right) = \mathrm{d}y \cdot \frac{\mathrm{d}^2 fp}{\beta^2} + \frac{(y-p)}{1}\mathrm{d}y \cdot \frac{\mathrm{d}^3 fp}{\beta^3} + \ldots. \tag{102}$$

I let $y = p$ in this, and I have

$$\frac{\mathrm{d}^2 fp}{\beta^2} = \frac{1}{\mathrm{d}\varphi\theta}\mathrm{d}\left(\frac{\mathrm{d}F\theta}{\mathrm{d}\varphi\theta}\right).$$

I manipulate equation (102) as I did (99) and (100). That is, I divide by $\mathrm{d}y$, I differentiate, I let $y = p$, and I have

$$\frac{\mathrm{d}^3 fp}{\beta^3} = \frac{1}{\mathrm{d}\varphi\theta}\mathrm{d}\left[\frac{1}{\mathrm{d}\varphi\theta}\mathrm{d}\left(\frac{\mathrm{d}F\theta}{\mathrm{d}\varphi\theta}\right)\right].$$

The induction is obvious, and we see that, in general, I have [129][27]

$$\frac{\mathrm{d}^n fp}{\beta^n} = \frac{1}{\mathrm{d}\varphi\theta}\mathrm{d}\left\{\frac{1}{\mathrm{d}\varphi\theta}\mathrm{d}\left\{\frac{1}{\mathrm{d}\varphi\theta}\mathrm{d}\left\{\cdots\frac{1}{\mathrm{d}\varphi\theta}\mathrm{d}\left\{\frac{\mathrm{d}F\theta}{\mathrm{d}\varphi\theta}\right\}\cdots\right\}\right\}\right\}. \tag{103}$$

In this expression, there are $n-1$ subordinate differentials. This is very simple, but we may derive another expression from it that lends itself better to expansions as needed in practice, by employing a procedure which is not devoid of elegance.

I let

$$\frac{\mathrm{d}fp}{\beta} = A, \quad \frac{\mathrm{d}^2 fp}{\beta^2} = B, \ldots, \quad \frac{\mathrm{d}^n fp}{\beta^n} = N,$$

for short. I successively multiply equation (99)[28] by $\dfrac{x-\theta}{y-p}$, $\left(\dfrac{x-\theta}{y-p}\right)^2$, $\ldots$. What's more, noting that in general

$$\frac{\mathrm{d}y}{(y-p)^m} = -\frac{1}{m-1}\mathrm{d}(y-p)^{-(m-1)},$$

[27] The original bracketing of equation (103) was incorrect.
[28] Servois almost certainly means equation (100).

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a relation that is easily verified using formula (87), I have[29]

$$\left.\begin{array}{rcl}\left(\dfrac{x-\theta}{y-p}\right)\mathrm{d}Fx & = & A(x-\theta)\cdot\dfrac{\mathrm{d}y}{y-p}+B(x-\theta)\cdot\mathrm{d}y\\ & & +\dfrac{C}{1\cdot 2}(x-\theta)(y-p)\mathrm{d}y+\ldots\\ \left(\dfrac{x-\theta}{y-p}\right)^2\mathrm{d}Fx & = & -A(x-\theta)^2\cdot\mathrm{d}(y-p)^{-1}+B(x-\theta)^2\cdot\dfrac{\mathrm{d}y}{y-p}\\ & & +\dfrac{C}{1\cdot 2}(x-\theta)^2\mathrm{d}y+\ldots\\ \left(\dfrac{x-\theta}{y-p}\right)^3\mathrm{d}Fx & = & -\dfrac{A}{2}(x-\theta)^3\cdot\mathrm{d}(y-p)^{-2}\\ & & -B(x-\theta)^3\cdot\mathrm{d}(y-p)^{-1}\\ & & +\dfrac{C}{1\cdot 2}(x-\theta)^3\dfrac{\mathrm{d}y}{y-p}+\ldots\\ \ldots\ldots & & \ldots\ldots\end{array}\right\}\quad(104)$$

Now, it follows from (45) that we have

$$y-p=(x-\theta)\mathrm{d}\varphi\theta+\frac{(x-\theta)^2}{1\cdot 2}\mathrm{d}^2\varphi\theta+\ldots,\qquad(105)$$

and, differentiating with respect to x,

$$\mathrm{d}y=\mathrm{d}\varphi\theta+(x-\theta)\mathrm{d}^2\varphi\theta+\ldots.\qquad(106)$$

It therefore follows from (106) that $(y-p)^{-m}$ and $\mathrm{d}(y-p)^{-m}$ have the forms [130]

$$\begin{array}{rcl}(y-p)^{-m} & = & A(x-\theta)^{-m}+B(x-\theta)^{-(m-1)}+\ldots\\ & & +\,G(x-\theta)^{-1}+H+K(x-\theta)+L(x-\theta)^2+\ldots\\ \mathrm{d}(y-p)^{-m} & = & A'(x-\theta)^{-m-1}+B'(x-\theta)^{-m}+\ldots\\ & & +\,G'(x-\theta)^{-2}+0+K'+L'(x-\theta)+\ldots,\end{array}\qquad(107)$$

respectively. From this last formula, we conclude that, for m a whole number greater than 0, the expansion of $\mathrm{d}(y-p)^{-m}$ following the powers of $(x-\theta)$ is missing only the term multiplied by $(x-\theta)^{-1}$. It further follows that, for a whole number n also greater than 0, the expansion $(x-\theta)^{n+1}\mathrm{d}(y-p)^{-m}$ is missing the term multiplied by $(x-\theta)^n$. Furthermore, it is evident from (107) that, for n greater than or equal to m, this expansion will include no negative

[29] In [Servois 1814a] there were many typographical errors in equations (104). The first right-hand side read $A(x-\theta)\cdot\frac{\mathrm{d}y}{y-p}+B(x-\theta)\cdot\mathrm{d}y+\frac{C}{1\cdot 2}(x-\theta)\mathrm{d}y+\ldots$, the second $-A(x-\theta)\cdot\mathrm{d}(x-\theta)^{-1}+B(x-\theta)\cdot\frac{\mathrm{d}y}{y-p}+\frac{C}{1\cdot 2}(x-\theta)^2\mathrm{d}y+\ldots$ and the third $-\frac{A}{2}(x-\theta)\cdot\mathrm{d}(y-p)^{-2}-B(x-\theta)\cdot\mathrm{d}(y-p)^{-1}+\frac{C}{1\cdot 2}(x-\theta)^3\frac{\mathrm{d}y}{y-p}+\ldots$.

powers of $(x-\theta)$. However, it follows from formula (87) that $\mathrm{d}^n(x-\theta)^q = 0$ when $n > q$ and has the form $R(x-\theta)^r$, where r is positive, when $n < q$. Thus, taking the differential, d^n, of the expression $(x-\theta)^{n+1}\mathrm{d}(y-p)^{-m}$, all the terms in which $(x-\theta)$ has an exponent less than n will be destroyed, and all the others will have the form $R(x-\theta)^r$. Because the term in $(x-\theta)^n$ is absent, in all of the others, the exponent of $(x-\theta)$ is greater than n. Consequently, when we let $x=\theta$, we always have

$$\mathrm{d}^n\left[(x-\theta)^{n+1}\cdot \mathrm{d}(y-p)^{-m}\right] = 0. \tag{108}$$

Secondly, it follows from equation (106) that the expression $(x-\theta)^{n+1}\cdot\frac{\mathrm{d}y}{y-p}$ is always of the form

$$(x-\theta)^{n+1}\frac{\mathrm{d}y}{y-p} = (x-\theta)^n + P(x-\theta)^{n+1} + \ldots.$$

But by (87), $\mathrm{d}^n(x-\theta)^n = 1\cdot 2\cdot 3\cdots n$. Thus, when we let $x=\theta$, we always have

$$\mathrm{d}^n\left[(x-\theta)^{n+1}\cdot\frac{\mathrm{d}y}{y-p}\right] = 1\cdot 2\cdot 3\cdots n. \tag{109}$$

I will now apply these two important observations to the series of equations (104).

[131] I let $x=\theta$ in the first of these. Then the first term becomes A, because of (109), and the following terms vanish. Thus,

$$A = \left\{\frac{x-\theta}{y-p}\mathrm{d}Fx\right\}_0.$$

I use the 0 placed on the side of our expression to indicate that one takes $x-\theta=0$ in the expansion.

I differentiate the second equation of (104) once, then I let $x=\theta$. The first term, $-A\mathrm{d}\left[(x-\theta)^2\mathrm{d}(y-p)^{-1}\right]$, is equal to zero, by (108). The second term, $B\mathrm{d}\left[(x-\theta)^2\frac{\mathrm{d}y}{y-p}\right]$ becomes B, by (109). All the following terms vanish, so that

$$B = \mathrm{d}\left\{\left(\frac{x-\theta}{y-p}\right)^2\mathrm{d}Fx\right\}_0.$$

I differentiate the third equation of (104) twice and I let $x=\theta$. The first two terms of the right-hand side are zero, by (108). The third term reduces to C, by (109). The following terms are clearly zero, so that

$$C = \mathrm{d}^2\left\{\left(\frac{x-\theta}{y-p}\right)^3\mathrm{d}Fx\right\}_0.$$

It's unnecessary to go any further to conclude with all rigor that in general

$$N = \frac{\mathrm{d}^n fp}{\beta^n} = \mathrm{d}^{n-1}\left\{\left(\frac{x-\theta}{y-p}\right)^n\mathrm{d}Fx\right\}_0. \tag{110}$$

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Thus, equation (99) becomes

$$\begin{aligned} Fx &= F\theta + \frac{(y-p)}{1}\left\{\frac{x-\theta}{y-p}\mathrm{d}Fx\right\}_0 + \frac{(y-p)^2}{1\cdot 2}\mathrm{d}\left\{\left(\frac{x-\theta}{y-p}\right)^2\mathrm{d}Fx\right\}_0 \\ &\quad + \frac{(y-p)^3}{1\cdot 2\cdot 3}\mathrm{d}^2\left\{\left(\frac{x-\theta}{y-p}\right)^3\mathrm{d}Fx\right\}_0 + \ldots. \end{aligned} \tag{111}$$

Alternately, if we wish to replace y and p with their corresponding expressions φx and $\varphi\theta$,[30]

$$\begin{aligned} Fx &= F\theta + (\varphi x - \varphi\theta)\left\{\frac{(x-\theta)\mathrm{d}Fx}{\varphi x - \varphi\theta}\right\}_0 + \frac{(\varphi x-\varphi\theta)^2}{1\cdot 2}\mathrm{d}\left\{\frac{(x-\theta)^2\mathrm{d}Fx}{(\varphi x-\varphi\theta)^2}\right\}_0 \\ &\quad + \frac{(\varphi x-\varphi\theta)^3}{1\cdot 2\cdot 3}\mathrm{d}^2\left\{\frac{(x-\theta)^3\mathrm{d}Fx}{(\varphi x-\varphi\theta)^3}\right\}_0 + \ldots. \end{aligned} \tag{112}$$

This is the formula of Professor Burman (see *Memoirs de l'Institut* 1^{st} Class, Vol. II, p. 16).[31] In the second of the two memoirs of which this is an extraction, I have deduced it from the celebrated formula of Lagrange on the recurrence of series.

In expression (110) of the general term of the coefficients of formula (111), we may, before differentiating, replace $y-p$ with its expression in x, if the form of the equation $V=0$ makes this possible. If not, then after differentiating, we must substitute for $\frac{x-\theta}{y-p}, \mathrm{d}y, \mathrm{d}^2y, \ldots$, what these functions become when $x-\theta$ and $y-p$ vanish at the same time. This will be possible, in general, using the equation $V=0$.

If the given equation between x and y is simply $y=\varphi x$, then by (105) we have

$$\left(\frac{x-\theta}{y-p}\right)_0 = \frac{1}{\mathrm{d}\varphi\theta},$$

supposing that all the while the equation $\varphi x = 0$ gives x but a single value, equal to θ. It is this that we must substitute in place of $\frac{x-\theta}{y-p}$ after the expansions.

If the given equation between x and y is, for example,

$$x-\theta = (y-p)\psi x,$$

which does indeed give $x=\theta$ when $y=p$ and reciprocally, then equation (111) becomes

$$\begin{aligned} Fx &= F\theta + (y-p)\psi\theta\cdot\mathrm{d}F\theta + \frac{(y-p)^2}{1\cdot 2}\mathrm{d}\left[(\psi\theta)^2\cdot\mathrm{d}F\theta\right] \\ &\quad + \frac{(y-p)^3}{1\cdot 2\cdot 3}\mathrm{d}^2\left[(\psi\theta)^3\cdot\mathrm{d}F\theta\right] + \ldots. \end{aligned} \tag{113}$$

[30] In [Servois 1814a] the d^2 was absent in the last term of (112).

[31] This reference is not to Burman's paper itself, but to a report on it by Legendre; see [Grattan-Guinness 1990, pp. 167-168] for more on Burman's formula.

[133] When we let $p = 0$, then this is the formula of Lagrange that we have just mentioned.

Let the relation between x and y be

$$x - \theta = (y - \lambda)\psi(x, y), \tag{114}$$

which gives $x = \theta$ when $y = \lambda$, and reciprocally.

In the given function $F(x, y)$ and in (114), I consider x to be the only variable and, by formula (113), I have

$$\begin{aligned} F(x,y) \quad &= \quad F(\theta,y) + (y-\lambda)\frac{\mathrm{d}}{\theta}F(\theta,y)\cdot\psi(\theta,y) + \ldots \\ &\quad + \frac{(y-\lambda)^n}{1\cdot 2\cdots n}\frac{\mathrm{d}^n}{\theta}\left\{\frac{\mathrm{d}}{\theta}F(\theta,y)\cdot[\psi(\theta,y)]^n\right\} + \ldots . \end{aligned} \tag{115}$$

$F(\theta, y)$ and the coefficients of $(y - \lambda)$ are functions of y that I expand following the powers of $(y - \lambda)$, by means of formula (45) and, letting $u = F(\theta, \lambda)$ and $v = \psi(\theta, \lambda)$ for short, I have

$$F(\theta,y) = u + (y-\lambda)\frac{\mathrm{d}}{\lambda}u + \frac{(y-\lambda)^2}{1\cdot 2}\frac{\mathrm{d}^2}{\lambda}u + \frac{(y-\lambda)^3}{1\cdot 2\cdot 3}\frac{\mathrm{d}^3}{\lambda}u + \ldots \quad \text{and}$$

$$\begin{aligned} \frac{\mathrm{d}^{n-1}}{\theta}\left\{\frac{\mathrm{d}}{\theta}F(\theta,y)\cdot[\psi(\theta,y)]^n\right\} &= \frac{\mathrm{d}^{n-1}}{\theta}\left(\frac{\mathrm{d}}{\theta}u\cdot v^n\right) \\ &\quad + (y-\lambda)\frac{\mathrm{d}}{\lambda}\frac{\mathrm{d}^{n-1}}{\theta}\left(\frac{\mathrm{d}}{\theta}u\cdot v^n\right) + \ldots . \end{aligned}$$

I substitute these results in (115), re-order according to the powers of $(y - \lambda)$, and I have

$$F(x,y) = u + A(y-\lambda) + B\frac{(y-\lambda)^2}{1\cdot 2} + \ldots + N\frac{(y-\lambda)^n}{1\cdot 2\cdots n} + \ldots . \tag{116}$$

In this equation, the general term of the coefficients is[32]

$$\begin{aligned} N \quad &= \quad \frac{\mathrm{d}^n}{\lambda}u + n\frac{\mathrm{d}^{n-1}}{\lambda}\left(\frac{\mathrm{d}}{\theta}uv\right) + \frac{n}{1}\frac{n-1}{2}\frac{\mathrm{d}^{n-2}}{\lambda}\frac{\mathrm{d}}{\theta}\left(\frac{\mathrm{d}}{\theta}uv^2\right) + \ldots \\ &\quad + n\frac{\mathrm{d}}{\lambda}\frac{\mathrm{d}^{n-2}}{\theta}\left(\frac{\mathrm{d}}{\theta}uv^{n-1}\right) + \frac{\mathrm{d}^{n-1}}{\theta}\left(\frac{\mathrm{d}}{\theta}uv^n\right). \end{aligned} \tag{117}$$

[134] Such a formula as (116) has a wide range of applications, of which I have made a number in my two memoirs. I was brought to them immediately, and by a very different method: that of the elimination of arbitrary functions, by partial differentiation, a method which, in the hands of Laplace, Lagrange, etc., has produced brilliant results and which, in the matter with which we are occupying ourselves, permits us to broach the following very general problem with success: given an equation among several variables, to expand a given

[32]In [Servois 1814a], the coefficient n was missing from the penultimate term.

function of one or several of these variables into a series arranged according to powers of one or several of the variables. I can give here but an idea of the way to proceed, by making an application to a case that is not very complicated.

Let the equation

$$ft = u\varphi(x+t) + v\psi(x+t) \tag{118}$$

be given. We wish to expand $F(x+t)$ according to the powers and products of u and v.

The solution of equation (118) gives an expression for t of the form $t = f(u,v,x)$, where there are no other equations constraining u, v, and x. Thus, we many consider t to be a function of three independent variables, u, v, and x, whose differences are constant and equal to one. Given this, we know, and it is furthermore easy to conclude it from formula (78, §17) that, letting $x+t$ be p for simplicity, we have

$$\left.\begin{array}{rcllll} Fp & = & Fp_0 & + u\frac{\mathrm{d}}{u}Fp_0 & + \frac{u^2}{1\cdot 2}\frac{\mathrm{d}^2}{u}Fp_0 & + \dots \\ & & & + v\frac{\mathrm{d}}{v}Fp_0 & + 2\frac{uv}{1\cdot 2}\frac{\mathrm{d}}{u}\frac{\mathrm{d}}{v}Fp_0 & + \dots \\ & & & & + \frac{v^2}{1\cdot 2}\frac{\mathrm{d}^2}{v}Fp_0 & + \dots \\ & & & & & + \dots . \end{array}\right\} \tag{119}$$

[135] The zero, on the sides of $Fp, \frac{\mathrm{d}}{u}Fp, \frac{\mathrm{d}}{v}Fp, \dots$, signify that the variables u and v are to be set to zero after expanding.

I successively differentiate Fp with respect to u, v, and x, and making use of theorem (81), I have

$$\frac{\mathrm{d}}{u}Fp = \mathrm{d}Fp\cdot\frac{\mathrm{d}}{u}t, \quad \frac{\mathrm{d}}{v}Fp = \mathrm{d}Fp\cdot\frac{\mathrm{d}}{v}t \quad \text{and} \quad \frac{\mathrm{d}}{x}Fp = \mathrm{d}Fp\left(1+\frac{\mathrm{d}}{x}t\right).$$

Eliminating $\mathrm{d}Fp$ from among these, I have

$$\frac{\mathrm{d}}{u}Fp = \frac{\mathrm{d}}{x}Fp\cdot\frac{\frac{\mathrm{d}}{u}t}{1+\frac{\mathrm{d}}{x}t} \quad \text{and} \quad \frac{\mathrm{d}}{v}Fp = \frac{\mathrm{d}}{x}Fp\cdot\frac{\frac{\mathrm{d}}{v}t}{1+\frac{\mathrm{d}}{x}t}. \tag{120}$$

I successively differentiate equation (118) with respect to u, v and x, and I write the results as follows

$$\frac{\mathrm{d}}{u}t\,(\mathrm{d}ft - u\mathrm{d}\varphi p - v\mathrm{d}\psi p) = \varphi p, \tag{121}$$

$$\frac{\mathrm{d}}{v}t\,(\mathrm{d}ft - u\mathrm{d}\varphi p - v\mathrm{d}\psi p) = \psi p \quad \text{and} \tag{122}$$

$$\left(1+\frac{\mathrm{d}}{x}t\right)(\mathrm{d}ft - u\mathrm{d}\varphi p - v\mathrm{d}\psi p) = \mathrm{d}ft. \tag{123}$$

I eliminate from these three the common polynomial factor on the left-hand sides, and I have

$$\frac{\mathrm{d}}{u}t = \frac{\varphi p}{\mathrm{d}ft}\left(1+\frac{\mathrm{d}}{x}t\right) \quad \text{and} \quad \frac{\mathrm{d}}{v}t = \frac{\psi p}{\mathrm{d}ft}\left(1+\frac{\mathrm{d}}{x}t\right). \tag{124}$$

I substitute the expressions of (124) into equations (120), and I have

$$\frac{\mathrm{d}}{u}Fp = \frac{\mathrm{d}}{x}Fp \cdot \frac{\varphi p}{\mathrm{d}ft} \quad \text{and} \quad \frac{\mathrm{d}}{v}Fp = \frac{\mathrm{d}}{x}Fp \cdot \frac{\psi p}{\mathrm{d}ft} \tag{125}$$

Because the function F is arbitrary, these give [136]

$$\left.\begin{aligned} \frac{\mathrm{d}}{u}\varphi p &= \frac{\mathrm{d}}{x}\varphi p \cdot \frac{\varphi p}{\mathrm{d}ft}, & \frac{\mathrm{d}}{v}\varphi p &= \frac{\mathrm{d}}{x}\varphi p \cdot \frac{\psi p}{\mathrm{d}ft}, \\ \frac{\mathrm{d}}{u}\psi p &= \frac{\mathrm{d}}{x}\psi p \cdot \frac{\varphi p}{\mathrm{d}ft}, \quad \text{and} & \frac{\mathrm{d}}{v}\psi p &= \frac{\mathrm{d}}{x}\psi p \cdot \frac{\psi p}{\mathrm{d}ft}. \end{aligned}\right\} \tag{126}$$

When we let $u = v = 0$ in (118), it follows that $ft = 0$. Supposing that this equation gives $t = \theta$, then we have $Fp_0 = F(x+\theta)$ and, by equation (125),

$$\frac{\mathrm{d}}{u}Fp_0 = \frac{\mathrm{d}}{x}F(x+\theta) \cdot \frac{\varphi(x+\theta)}{\mathrm{d}f\theta} \quad \text{and} \quad \frac{\mathrm{d}}{v}Fp_0 = \frac{\mathrm{d}}{x}F(x+\theta) \cdot \frac{\psi(x+\theta)}{\mathrm{d}f\theta}.$$

Here, already, are the first three terms of the expansion (119) completely determined. To go beyond this, we differentiate equations (125), the first one with respect to u and v, and the second with respect to v, and we have expressions for $\frac{\mathrm{d}^2}{u}Fp$, $\frac{\mathrm{d}}{u}\frac{\mathrm{d}}{v}Fp$ and $\frac{\mathrm{d}^2}{v}Fp$ that contain, linearly, the differentials of Fp, φp, ψp, and t with respect to u, v, and x. We eliminate the differentials with respect to u and v by means of equations (124), (125), and (126). Having made these reductions, it follows that

$$\left.\begin{aligned} \frac{\mathrm{d}^2}{u}Fp &= \frac{\frac{\mathrm{d}}{x}\left[\frac{\mathrm{d}}{x}Fp \cdot (\varphi p)^2\right]}{(\mathrm{d}ft)^2} \\ &\quad - \frac{\frac{\mathrm{d}}{x}Fp \cdot (\varphi p)^2 \cdot \mathrm{d}^2 ft\left(1 + 2\frac{\mathrm{d}}{x}t\right)}{(\mathrm{d}ft)^3}, \\ \frac{\mathrm{d}}{u}\frac{\mathrm{d}}{v}Fp &= \frac{\frac{\mathrm{d}}{x}\left[\frac{\mathrm{d}}{x}Fp \cdot \varphi p \cdot \psi p\right]}{(\mathrm{d}ft)^2} \\ &\quad - \frac{\frac{\mathrm{d}}{x}Fp \cdot \varphi p \cdot \psi p \cdot \mathrm{d}^2 ft\left(1 + 2\frac{\mathrm{d}}{x}t\right)}{(\mathrm{d}ft)^3} \quad \text{and} \\ \frac{\mathrm{d}^2}{v}Fp &= \frac{\frac{\mathrm{d}}{x}\left[\frac{\mathrm{d}}{x}Fp \cdot (\psi p)^2\right]}{(\mathrm{d}ft)^2} \\ &\quad - \frac{\frac{\mathrm{d}}{x}Fp \cdot (\psi p)^2 \cdot \mathrm{d}^2 ft\left(1 + 2\frac{\mathrm{d}}{x}t\right)}{(\mathrm{d}ft)^3}. \end{aligned}\right\} \tag{127}$$

In these equations, we satisfy the hypothesis $u = v = 0$, which gives $t = \theta$, [137] $p = x + \theta$ and, by (123), $\frac{\mathrm{d}}{x}t = 0$, and we have the three differential coefficients $\frac{\mathrm{d}^2}{u}Fp_0$, $\frac{\mathrm{d}}{u}\frac{\mathrm{d}}{v}Fp_0$ and $\frac{\mathrm{d}^2}{v}Fp_0$.

We continue in the same manner. That is, we differentiate equation (127) with respect to u and v, to get $\frac{\mathrm{d}^3}{u}Fp$, $\frac{\mathrm{d}^2}{u}\frac{\mathrm{d}}{v}Fp$, $\frac{\mathrm{d}}{u}\frac{\mathrm{d}^2}{v}Fp$ and $\frac{\mathrm{d}^3}{v}Fp$. In the resulting expressions, the differentials with respect to u and v of Fp, φp and ψp are

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eliminated by the equations (125) and (126). $\frac{\mathrm{d}}{u}t$ and $\frac{\mathrm{d}}{v}t$ are eliminated using (124). We eliminate the other two, $\frac{\mathrm{d}}{u}\frac{\mathrm{d}}{x}t$ and $\frac{\mathrm{d}}{v}\frac{\mathrm{d}}{x}t$, which are the same as $\frac{\mathrm{d}}{x}\frac{\mathrm{d}}{u}t$ and $\frac{\mathrm{d}}{x}\frac{\mathrm{d}}{v}t$, respectively, after having differentiated equation (124) with respect to x. Then we satisfy the hypothesis $u = v = 0$, which gives $0 = \frac{\mathrm{d}}{x}t = \frac{\mathrm{d}^2}{x}t$ and in general $\frac{\mathrm{d}^n}{x}t = 0$, which is worthy of note and is easily deduced from equation (123). We now have the four coefficients

$$\frac{\mathrm{d}^3}{u}Fp_0, \quad \frac{\mathrm{d}^2}{u}\frac{\mathrm{d}}{v}Fp_0, \quad \frac{\mathrm{d}}{u}\frac{\mathrm{d}^2}{v}Fp_0 \quad \text{and} \quad \frac{\mathrm{d}^3}{v}Fp_0.$$

The route to follow in order to continue indefinitely is easily recognized. It is clear that everything reduces to differentiation, with respect to u and v, of the last results obtained, and the elimination of the differentials of $Fp, \varphi p$, and ψp, with respect to u and v, using (125), and of the differentials of the form $\frac{\mathrm{d}^n}{x}\frac{\mathrm{d}}{u}t$ and $\frac{\mathrm{d}^n}{x}\frac{\mathrm{d}}{v}t$ from equation (124), differentiated with respect to x as many times as necessary.

Supposing now, in particular, that $ft = t$ and from this $\mathrm{d}ft = 1$, and applying this hypothesis in (125) and (126) we have, in the first place, [138]

$$\frac{\mathrm{d}}{u}\left\{\frac{\mathrm{d}}{x}Fp \cdot (\varphi p)^m\right\} = (\varphi p)^m \cdot \frac{\mathrm{d}}{x}\frac{\mathrm{d}}{u}Fp + \frac{\mathrm{d}}{x}Fp \cdot \frac{\mathrm{d}}{u}(\varphi p)^m.$$

Since, by (125) and (126)

$$\begin{aligned}
\frac{\mathrm{d}}{x}\frac{\mathrm{d}}{u}Fp &= \varphi p \cdot \frac{\mathrm{d}^2}{x}Fp + \frac{\mathrm{d}}{x}Fp \cdot \frac{\mathrm{d}}{x}\varphi p \quad \text{and} \\
\frac{\mathrm{d}}{u}(\varphi p)^m &= m(\varphi p)^{m-1} \cdot \frac{\mathrm{d}}{u}\varphi p = m(\varphi p)^m \cdot \frac{\mathrm{d}}{x}\varphi p,
\end{aligned}$$

it follows, by reducing, that

$$\frac{\mathrm{d}}{u}\left\{\frac{\mathrm{d}}{x}Fp \cdot (\varphi p)^m\right\} = \frac{\mathrm{d}}{x}\left\{\frac{\mathrm{d}}{x}Fp \cdot (\varphi p)^{m+1}\right\}. \tag{128}$$

We find, in the same way, that

$$\frac{\mathrm{d}}{v}\left\{\frac{\mathrm{d}}{x}Fp \cdot (\varphi p)^m \cdot (\psi p)^n\right\} = \frac{\mathrm{d}}{x}\left\{\frac{\mathrm{d}}{x}Fp \cdot (\varphi p)^m \cdot (\psi p)^{n+1}\right\}. \tag{129}$$

Given this, by differentiating the first equation of (125) successively with respect to u, we have, by (128)

$$\begin{aligned}
\frac{\mathrm{d}^2}{u}Fp &= \frac{\mathrm{d}}{u}\left(\frac{\mathrm{d}}{x}Fp \cdot \varphi p\right) = \frac{\mathrm{d}}{x}\left\{\frac{\mathrm{d}}{x}Fp \cdot (\varphi p)^2\right\}, \\
\frac{\mathrm{d}^3}{u}Fp &= \frac{\mathrm{d}}{x}\frac{\mathrm{d}}{u}\left[\frac{\mathrm{d}}{x}Fp \cdot (\varphi p)^2\right] = \frac{\mathrm{d}^2}{x}\left\{\frac{\mathrm{d}}{x}Fp \cdot (\varphi p)^3\right\},
\end{aligned}$$

and, in general,

$$\frac{\mathrm{d}^m}{u}Fp = \frac{\mathrm{d}^{m-1}}{x}\left\{\frac{\mathrm{d}}{x}Fp \cdot (\varphi p)^m\right\}. \tag{130}$$

[139] Next, we differentiate (130) successively with respect to v. Making use of (129), we have

$$\frac{\mathrm{d}}{v}\frac{\mathrm{d}^m}{u}Fp = \frac{\mathrm{d}^m}{u}\frac{\mathrm{d}}{v}Fp = \frac{\mathrm{d}^{m-1}}{x}\frac{\mathrm{d}}{v}\left\{\frac{\mathrm{d}}{x}Fp\cdot(\varphi p)^m\right\} = \frac{\mathrm{d}^m}{x}\left\{\frac{\mathrm{d}}{x}Fp\cdot(\varphi p)^m\psi p\right\},$$

$$\frac{\mathrm{d}^m}{u}\frac{\mathrm{d}^2}{v}Fp = \frac{\mathrm{d}^{m+1}}{x}\left\{\frac{\mathrm{d}}{x}Fp\cdot(\varphi p)^m\cdot(\psi p)^2\right\},$$

and, in general,

$$\frac{\mathrm{d}^m}{u}\frac{\mathrm{d}^n}{v}Fp = \frac{\mathrm{d}^{m+n-1}}{x}\left\{\frac{\mathrm{d}}{x}Fp\cdot(\varphi p)^m\cdot(\psi p)^n\right\}. \tag{131}$$

This is the general term of the coefficients of the expansion we seek, where nothing remains except to satisfy the hypothesis $u = v = 0$, which gives $t = 0$ by (118).[33] Therefore, in our general term (131), p changes to x, the partial differentials with respect to x become total, and so

$$\frac{\mathrm{d}^m}{u}\frac{\mathrm{d}^n}{v}Fp_0 = \mathrm{d}^{m+n-1}\left\{\mathrm{d}Fx\cdot(\varphi x)^m\cdot(\psi x)^n\right\}. \tag{132}$$

Finally, we have by (119)

$$\left.\begin{aligned} F(x+t) = Fx + {} & u\,\mathrm{d}Fx\cdot\varphi x + \frac{u^2}{1\cdot 2}\mathrm{d}\left\{\mathrm{d}Fx\cdot(\varphi x)^2\right\} + \dots \\ + {} & v\,\mathrm{d}Fx\cdot\psi x + 2\frac{uv}{1\cdot 2}\mathrm{d}\left\{\mathrm{d}Fx\cdot\varphi x\cdot\psi x\right\} + \dots \\ & + \frac{v^2}{1\cdot 2}\mathrm{d}\left\{\mathrm{d}Fx\cdot(\psi x)^2\right\} + \dots \\ & + \dots \end{aligned}\right\} \tag{133}$$

I will not make applications of the expansion formulas [140] that we have just read, so as not to exceed the limits I have set myself. Indeed, my project was simply to offer a somewhat detailed outline of the manner in which I have treated the principles of the differential calculus in the first part of the work, which I had the honor of presenting to the 1st Class of the Institute. The applications of the formulas of the expansion of functions into series are the object of a second part. I succeeded in deducing from these formulas, without needing recourse to any new notation, the principle formulas that have up to now been based on *combinatorial analysis* or the *calculus of derivations*. The Commissioners of the Class were willing to say on this matter, in their report:

> "In thereby recalling to the differential calculus several methods, some of which don't seem very appropriate to the current state of analysis, (the author) has done something that is very useful for the

[33] Strictly speaking, the hypothesis $u = v = 0$ gives $ft = 0$, as Servois noted on p. 35. However, because Servois is now considering the case that $ft = t$, it follows that $t = 0$.

science. It is necessary that all new facts, whenever they make up an ensemble, even if they don't seem individually to be of very great importance, be reconciled to the theories that form the body of the science, and it is most appropriate to encourage this in the culture."

It would be even more foreign to my design to enter into any detail about the third part, in which I am concerned with the research on the simplest practical means of expanding the differentials of composed functions later on, only once we had considered constant differences, so that the entirety is given immediately by a single expansion; that is, by the methods of the second part.

But perhaps it is not without value at this point to cast a general glance on the various systems which, up to this point, have been followed in the exposition of the principles of the differential calculus. The reflections that this examination will give rise to would be entirely appropriate to underscore the advantages of the theory which has just been described, to prevent false interpretations, and finally to refute the objections which this theory could and may yet give rise to.

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