

An Interactive Introduction to Complex Numbers

2. Basic Calculations

Complex Conjugates

Example 2.1:

Open the [Applet Basic Calculations](#) and enter $z_1 = 1 + 2 \cdot i$ and $z_2 = 1.5 \cdot e^{150^\circ i}$. Then click on \bar{z}_1 and \bar{z}_2 , respectively.

$\bar{z}_1 = 1 - 2 \cdot i$ is called the “complex conjugate of $z_1 = 1 + 2 \cdot i$ ” and $\bar{z}_2 = 1.5 \cdot e^{210^\circ i} = 1.5 \cdot e^{-150^\circ i}$ is the complex conjugate of $z_2 = 1.5 \cdot e^{150^\circ i}$.

Rule 2.1 a

The complex conjugate of $z = a + b \cdot i$ in cartesian form is $\bar{z} = a - b \cdot i$ with $\operatorname{Re}(\bar{z}) = \operatorname{Re}(z)$ and $\operatorname{Im}(\bar{z}) = -\operatorname{Im}(z)$.

Rule 2.1 b

The complex conjugate of $z = r \cdot e^{\varphi i}$ in exponential form is $\bar{z} = r \cdot e^{-\varphi i}$ with $|z| = |\bar{z}|$ and $\arg(\bar{z}) = -\arg(z)$.

Rule 2.1 c

$$\overline{\bar{z}} = z$$

Graphically, the complex conjugate \bar{z} mirrors z at the real axis and vice versa.

Addition

Example 2.2:

Open the [Applet Basic Calculations](#) and set $z_1 = 2 + i$ and $z_2 = 1 + 2 \cdot i$. Then click on $z_1 + z_2$. Note that both the real parts and the imaginary parts are added up.

Rule 2.2

The sum of two complex numbers $z_1 = a_1 + b_1 \cdot i$ and $z_2 = a_2 + b_2 \cdot i$ in cartesian form is

$$z_1 + z_2 = a_1 + b_1 \cdot i + a_2 + b_2 \cdot i = a_1 + a_2 + (b_1 + b_2) \cdot i \quad \text{with} \quad \operatorname{Re}(z_1 + z_2) = a_1 + a_2 \quad \text{and}$$
$$\operatorname{Im}(z_1 + z_2) = b_1 + b_2.$$

There is no general rule for the addition of complex numbers in exponential form.

Subtraction

Example 2.3:

Open the [Applet Basic Calculations](#) and set $z_1 = 3 + 6 \cdot i$ and $z_2 = 4 + 2 \cdot i$. Then click on $z_1 - z_2$. Note that both the real parts and the imaginary parts are subtracted.

Rule 2.3

The difference between two complex numbers $z_1 = a_1 + b_1 \cdot i$ and $z_2 = a_2 + b_2 \cdot i$ in cartesian form is

$$z_1 - z_2 = a_1 + b_1 \cdot i - (a_2 + b_2 \cdot i) = a_1 - a_2 + (b_1 - b_2) \cdot i \quad \text{with}$$
$$\operatorname{Re}(z_1 - z_2) = a_1 - a_2 \quad \text{and} \quad \operatorname{Im}(z_1 - z_2) = b_1 - b_2.$$

There is no general rule for the subtraction of complex numbers in exponential form.

Multiplication

Example 2.4 a:

Open the [Applet Basic Calculations](#) and set $z_1 = 1 + 2 \cdot i$ and $z_2 = 3 + 4 \cdot i$. Then click on $z_1 \cdot z_2$. Note the multiplication in exponential form first. Obviously, the absolute values are multiplied, whereas the exponents are added up.

Rule 2.4 a

The product of two complex numbers $z_1 = r_1 \cdot e^{\varphi_1 \cdot i}$ and $z_2 = r_2 \cdot e^{\varphi_2 \cdot i}$ in exponential form is

$$z_1 \cdot z_2 = r_1 \cdot e^{\varphi_1 \cdot i} \cdot r_2 \cdot e^{\varphi_2 \cdot i} = r_1 \cdot r_2 \cdot e^{\varphi_1 \cdot i + \varphi_2 \cdot i} = r_1 \cdot r_2 \cdot e^{(\varphi_1 + \varphi_2) \cdot i} \quad \text{with} \quad |z_1 \cdot z_2| = r_1 \cdot r_2 \quad \text{and}$$
$$\arg(z_1 \cdot z_2) = \varphi_1 + \varphi_2.$$

Example 2.4 b:

How would you multiply $z_1 = 1 + 2 \cdot i$ and $z_2 = 3 + 4 \cdot i$ in cartesian form intuitively?

Probably by calculating $z_1 \cdot z_2 = (1 + 2 \cdot i) \cdot (3 + 4 \cdot i) = 1 \cdot 3 + 1 \cdot 4 \cdot i + 2 \cdot 3 \cdot i + 2 \cdot i \cdot 4 \cdot i = 1 \cdot 3 + (1 \cdot 4 + 2 \cdot 3) \cdot i + 2 \cdot 4 \cdot i^2 = 1 \cdot 3 - 2 \cdot 4 + (1 \cdot 4 + 2 \cdot 3) \cdot i = -5 + 10 \cdot i$. The applet in example 2.4 a shows that the result is correct.

Rule 2.4 b

The product of two complex numbers $z_1 = a_1 + b_1 \cdot i$ and $z_2 = a_2 + b_2 \cdot i$ in cartesian form is $z_1 \cdot z_2 = (a_1 + b_1 \cdot i) \cdot (a_2 + b_2 \cdot i) = a_1 \cdot a_2 + b_1 \cdot a_2 \cdot i + a_1 \cdot b_2 \cdot i + b_1 \cdot b_2 \cdot i^2 = a_1 \cdot a_2 - b_1 \cdot b_2 + (a_1 \cdot b_2 + a_2 \cdot b_1) \cdot i$ with $\text{Re}(z_1 \cdot z_2) = a_1 \cdot a_2 - b_1 \cdot b_2$ and $\text{Im}(z_1 \cdot z_2) = a_1 \cdot b_2 + a_2 \cdot b_1$.

Rule 2.4 c

$$z \cdot \bar{z} = (a + b \cdot i) \cdot (a - b \cdot i) = a^2 - (b \cdot i)^2 = a^2 - b^2 \cdot (-1) = a^2 + b^2 = |z|^2$$

$$z \cdot \bar{z} = r \cdot e^{\varphi \cdot i} \cdot r \cdot e^{-\varphi \cdot i} = r^2 \cdot e^{(\varphi - \varphi) \cdot i} = r^2 \cdot e^{0 \cdot i} = r^2 = |z|^2$$

Division

Example 2.5 a:

Open the [Applet Basic Calculations](#) and set $z_1 = 1 + 2 \cdot i$ and $z_2 = 3 + 4 \cdot i$. Then click on z_1/z_2 . Note the division in exponential form first. Obviously, the absolute values are divided, whereas the exponents are subtracted.

Rule 2.5 a

The quotient of two complex numbers $z_1 = r_1 \cdot e^{\varphi_1 \cdot i}$ and $z_2 = r_2 \cdot e^{\varphi_2 \cdot i}$ in exponential form

$$\text{is } \frac{z_1}{z_2} = \frac{r_1 \cdot e^{\varphi_1 \cdot i}}{r_2 \cdot e^{\varphi_2 \cdot i}} = \frac{r_1}{r_2} \cdot e^{\varphi_1 \cdot i - \varphi_2 \cdot i} = \frac{r_1}{r_2} \cdot e^{(\varphi_1 - \varphi_2) \cdot i} \quad r_2 \neq 0 \text{ with } \left| \frac{z_1}{z_2} \right| = \frac{r_1}{r_2} \text{ and } \arg\left(\frac{z_1}{z_2}\right) = \varphi_1 - \varphi_2.$$

Example 2.5 b:

Although not quite obvious one can also calculate the quotient of $z_1 = 1 + 2 \cdot i$ and $z_2 = 3 + 4 \cdot i$ in cartesian form. To get a result, expand the quotient with $\bar{z}_2 = 3 - 4 \cdot i$ and consider rule 2.4 c.

$$\text{Then } \frac{z_1}{z_2} = \frac{1+2 \cdot i}{3+4 \cdot i} = \frac{1+2 \cdot i}{3+4 \cdot i} \cdot \frac{3-4 \cdot i}{3-4 \cdot i} = \frac{3-4 \cdot i+6 \cdot i-8 \cdot i^2}{9-16 \cdot i^2} = \frac{11+2 \cdot i}{25} = 0.44+0.08 \cdot i.$$

Rule 2.5 b

The quotient of two complex numbers $z_1 = a_1 + b_1 \cdot i$ and $z_2 = a_2 + b_2 \cdot i$ in cartesian form is

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a_1 + b_1 \cdot i}{a_2 + b_2 \cdot i} = \frac{(a_1 + b_1 \cdot i) \cdot (a_2 - b_2 \cdot i)}{(a_2 + b_2 \cdot i) \cdot (a_2 - b_2 \cdot i)} = \frac{a_1 \cdot a_2 - a_1 \cdot b_2 \cdot i + a_2 \cdot b_1 \cdot i - b_1 \cdot b_2 \cdot i^2}{a_2^2 - (b_2 \cdot i)^2} \\ &= \frac{a_1 \cdot a_2 + b_1 \cdot b_2 + (a_2 \cdot b_1 - a_1 \cdot b_2) \cdot i}{a_2^2 + b_2^2} = \frac{a_1 \cdot a_2 + b_1 \cdot b_2}{|z_2|^2} + \frac{a_2 \cdot b_1 - a_1 \cdot b_2}{|z_2|^2} \cdot i \quad a_2 \neq 0 \vee b_2 \neq 0 \end{aligned}$$

$$\text{with } \operatorname{Re}\left(\frac{z_1}{z_2}\right) = \frac{a_1 \cdot a_2 + b_1 \cdot b_2}{a_2^2 + b_2^2} = \frac{a_1 \cdot a_2 + b_1 \cdot b_2}{|z_2|^2} \quad \text{and} \quad \operatorname{Im}\left(\frac{z_1}{z_2}\right) = \frac{a_2 \cdot b_1 - a_1 \cdot b_2}{a_2^2 + b_2^2} = \frac{a_2 \cdot b_1 - a_1 \cdot b_2}{|z_2|^2}.$$

Rule 2.5 c

$$\frac{z_1}{z_2} = \frac{z_1 \cdot \bar{z}_2}{z_2 \cdot \bar{z}_2} = \frac{z_1 \cdot \bar{z}_2}{|z_2|^2}$$

Exercise 2.1:

Determine the complex conjugates of $z_1 = 1 - 4 \cdot i$ and $z_2 = 0.5 \cdot e^{117^\circ i}$. Use the [Applet Basic Calculations](#) to check your answers.

Exercise 2.2:

We have $z_1 = 1 - 4 \cdot i$ and $z_2 = -1 + 3 \cdot i$. Determine $z_1 + z_2$, $z_1 - z_2$, $z_1 \cdot z_2$ and $\frac{z_1}{z_2}$. Use

the [Applet Basic Calculations](#) to check your answers.

Jens Siebel: An Interactive Introduction to Complex Numbers
2. Basic Calculations

Exercise 2.3:

We have $z_1 = 1.5 \cdot e^{60^\circ \cdot i}$ and $z_2 = e^{320^\circ \cdot i}$. Determine $z_1 + z_2$, $z_1 - z_2$, $z_1 \cdot z_2$ and $\frac{z_1}{z_2}$. If necessary, turn the numbers into cartesian form. Use the [Applet Basic Calculations](#) to check your answers.