A Generalized Magic Trick from Fibonacci: Designer Decimals

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Recently, Smoak and Osler [5] displayed the Fibonacci numbers within the decimal representation of several remarkable fractions. The most famous example is $1/89 = 0.011235955\ldots$, in which 1, 1, 2, 3, 5 are the first five Fibonacci numbers, denoted $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, with recursion $F_{n+2} = F_{n+1} + F_n$. While the pattern continues, entries after the sixth digit are disguised by carrying.

Our generalized mathematical magician can determine fractions whose decimal representations include successive values of many other sequences [1], [2], [3], [4]. Take any integers that satisfy a recursion of the form

$$c_{n+2} = pc_{n+1} + qc_n, \quad c_n \geq 0, \quad \text{for } n \geq 1. \quad (1)$$

Since $c_{n+2} - pc_{n+1} - qc_n = 0$, the product

$$(1 - px - qx^2) \sum_{n=1}^{\infty} c_n x^n = c_1x + (c_2 - c_1p)x^2 + (c_3 - pc_2 - qc_1)x^3 + (c_4 - pc_3 - qc_2)x^4 + \cdots = c_1x + q c_0 x^2 + 0.$$ 

Thus, $\sum_{n=1}^{\infty} c_n x^n = (c_1x + q c_0 x^2)/(1 - px - qx^2)$. Because we want to look at decimal expansions, change the form by replacing $x$ with $1/x$ to make

$$\sum_{n=1}^{\infty} \frac{c_n}{x^n} = \frac{c_1x + q c_0}{x^2 - px - q}. \quad (2)$$

If $x = 10$, the fraction reads $c_1/10 + c_2/100 + c_3/1000 + \cdots$. If $c_n \geq 0$, successive terms of $\{c_n\}$ appear in the decimal expansion in the form $c_1c_2c_3c_4\ldots$ until disguised by carrying. If $x = 10^k$, successive terms of $\{c_n\}$ appear within blocks of $k$ digits.

Now we can design a decimal for any sequence as in (1) that we choose. Small values for $c_0$, $p$, and $q$ make examples where the pattern is not disguised too soon by carrying. If we want powers of a number $w < 100$, we let $p = w$, $q = 0$, $c_1 = 1$, $c_2 = w$, $c_3 = w^2$, $c_4 = w^3$, $\ldots$.
\[ x = 100 \text{ in (2), and the fraction } 1/(100 - w) \text{ has powers of } w \text{ appearing in groups of two digits. For example, } 1/97 = .01030927\ldots, \text{ the powers of three. The Fibonacci numbers result from } p = q = 1, c_0 = 0, c_1 = 1, x = 10 \text{ in (2) as } 10/89 = .11235\ldots, \text{ or using } x = 100, \text{ we find } 100/9899 = .0101020305081321\ldots \text{ If we make up our own sequence } 2, 5, 7, 12, 19, \ldots, \text{ where } p = q = 1, c_0 = 3, c_1 = 2, \text{ and let } x = 100, \text{ we find } 203/9899 = .0205071219\ldots. \]

We can put an arithmetic progression into the digits of a decimal expansion by taking \( c_1 = a, c_2 = a + d, c_0 = a - d. \) Then \( c_{n+2} = 2c_{n+1} - c_n, \) and \( p = 2, q = -1 \) in (2). Let \( x = 10^2, \) and expand fractions \( y/9801. \) The terms of the chosen arithmetic progression appear in blocks of two digits. Some examples are:

\[
\begin{align*}
100/9801 &= .010203040506\ldots \\
200/9801 &= .020406081012\ldots \\
299/9801 &= .030507111113\ldots \\
301/9801 &= .0307111151923\ldots \\
399/9801 &= .040710131619\ldots \\
799/9801 &= .08152293643\ldots
\end{align*}
\]

Make your own examples and check the results in Mathematica. How many digits before the pattern wears out? What is your favorite example? The possibilities are endless.

References

2. R. H. Hudson and C. F. Winans, A complete characterization of the decimal fractions that can be represented as \( \sum 10^{-k(i+1)} F_{\alpha_i}, \) where \( F_{\alpha_i} \) is the \( \alpha_i \)th Fibonacci Number, *The Fibonacci Quarterly* 19 (1981) 414–421.

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**A Calculus Theorem Motivated by a Statistics Problem**

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Intervals supporting a prescribed area arise in statistics. For example, for random variable \( X \) with probability density function (pdf) \( f(x), \) an interval \((x_1, x_2)\) such that \( \int_{x_1}^{x_2} f(x) \, dx = 0.95 \) is an interval estimate for the next \( x \)-value with degree of confidence 0.95. For the best precision, it is desirable to use the shortest such interval. The following theorem presents conditions for the interval to be the shortest.

**Theorem.** For the non-negative real function \( f(x) \) with continuous first derivative, consider \( x_1 \) and \( x_2 \) in the domain of \( f(x) \) such that \( x_1 < x_2, \) \( f'(x_1) > 0, \) and \( f'(x_2) < 0. \) The interval \((x_1, x_2)\) encloses a relative maximum area among all intervals of the same width and centered in the neighborhood of \((x_1 + x_2)/2\) if and only if \( f(x_1) = f(x_2). \)