Irrational Roots of Integers

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We all know that if \( n \) is not a perfect square, then \( \sqrt{n} \) is irrational. In many courses we prove this for \( \sqrt{2} \) and then ask students to prove it for \( \sqrt{3} \) or \( \sqrt{5} \). Often this involves altering observations about even and odd integers into comments about divisibility by 3 or 5. While there is some pedagogical benefit to this, it strikes us as inefficient when there is a single concise proof that deals with all \( n \) simultaneously. The proof below resembles ones given in the literature by Bloom [1], Sagher [5], Waterhouse [8], and D’Angelo and West [2]. Bloom also cites Eves [3], who reports it being presented in a 1985 public lecture of Ivan Niven. Sprow [6] warns about Waterhouse’s hidden use of unique factorization. Various other proofs appear in an article by Kalman, Mena, and Shahriari [4], including the following:

**Theorem 1.** If \( k^2 < n < (k + 1)^2 \) for natural numbers \( n \) and \( k \), then \( \sqrt{n} \) is irrational.

**Proof.** (By contradiction.) Suppose that \( \sqrt{n} \) is rational, and let it be written with least positive denominator as \( \frac{a}{b} \). Now \( k < \sqrt{n} = \frac{a}{b} < k + 1 \) implies that \( 0 < a - bk < b \), and so

\[
\frac{a}{b} = \frac{a(a - bk)}{b(a - bk)} = \frac{a^2 - abk}{b(a - bk)}.
\]

But of course \( a^2 = nb^2 \), so we may substitute and cancel to get

\[
\frac{a}{b} = \frac{nb^2 - abk}{b(a - bk)} = \frac{nb - ak}{a - bk}.
\]

This is a new rational representation of \( n \) with a smaller positive denominator, contrary to our supposition.

As a matter of fact, with a little extra effort, we can provide a single proof for the irrationality of \( \sqrt[n]{r} \) for any integer \( r \geq 2 \) and any natural number \( n \) that is not an \( r \)th power. This proof resembles one by Vaughn [7], but we have avoided using induction.

**Theorem 2.** If \( k, r, \) and \( n \) are natural numbers, and \( k^r < n < (k + 1)^r \), then \( \sqrt[n]{r} \) is irrational.

We use Theorem 1 only as motivation, but introduce a lemma to facilitate the proof.

**Lemma.**

(a) If \( x = \frac{a}{b} \) is rational and \( b \) is the minimum possible positive denominator in any rational expression for \( x \), and if \( x = \frac{c}{d} \) is another rational representation, then \( b \mid d \).

(b) If \( \sqrt[n]{r} = \frac{a}{b} \) where \( b \) is the minimum possible positive denominator in any rational expression for \( \sqrt[n]{r} \), then \( \sqrt[n]{r^{-1}} = \frac{a^{r-1}}{b^{r-1}} \) and \( b^{r-1} \) is the minimum possible positive denominator in any rational expression for \( \sqrt[n]{r^{-1}} \).
Proof of Lemma. For \( x = \frac{a}{b} \), the denominator is minimum implies that \( a \) and \( b \) are relatively prime. But then \( x = \frac{a}{b} = \frac{c}{d} \) implies that \( ad = bc \). Since \( b \mid ad \) and \( \gcd(a, b) = 1 \), we have \( b \mid d \) as required.

For part (b) it is evident that

\[
\sqrt[n]{r^{-1}} = \frac{a^{-1}}{b^{-1}}.
\]

If this is not the minimum denominator, then there is another fraction

\[
\frac{c}{d} = \frac{a^{-1}}{b^{-1}}.
\]

We see that \( a^{-1}d = b^{-1}c \). Since \( \gcd(a, b) = 1 \), we observe that \( \gcd(a^{-1}, b^{-1}) = 1 \). (If we have the goal of avoiding the need to use prime factorization here, we can argue that \( \gcd(a, b) = 1 \) implies existence of an integer combination \( ai + bj = 1 \). Then \( (ai + bj)^{2r^{-1}} = 1 \) can be expanded and regrouped to demonstrate that \( \gcd(a^{-1}, b^{-1}) = 1 \) without reference to primes.) Consequently, we have \( b^{-1} \mid d \), so in fact \( d \) cannot be a smaller denominator.

We now prove Theorem 2 by contradiction. Observe that the hypothesis requires that \( k < \sqrt[n]{r} = \frac{a}{b} < k + 1 \), which guarantees that \( a - bk > 0 \). We modify the fraction for \( \sqrt[n]{r^{-1}} \) as follows:

\[
\sqrt[n]{r^{-1}} = \frac{a^{-1}}{b^{-1}} = \frac{a^{-1}(a - bk)}{b^{-1}(a - bk)} = \frac{a^r - a^{-1}bk}{b^{-1}(a - bk)} = \frac{nb^r - a^{-1}bk}{b^{-1}(a - bk)} = \frac{nb^{-1} - a^{-1}k}{b^{-2}(a - bk)}.
\]

Since the minimum denominator for \( \sqrt[n]{r^{-1}} \) is \( b^{-1} \), we get \( b^{-2}(a - bk) \geq b^{-1} \), which reduces to \( a - bk \geq b \). Therefore, \( \frac{a}{b} \geq k + 1 \), which is a contradiction.

References