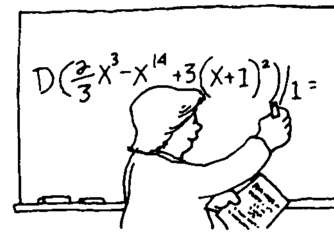
CLASSROOM CAPSULES

EDITOR

Warren Page
30 Amberson Ave.
Yonkers, NY 10705



Classroom Capsules consists primarily of short notes (1–3 pages) that convey new mathematical insights and effective teaching strategies for college mathematics instruction. Please submit manuscripts prepared according to the guidelines on the inside front cover to the Editor, Warren Page, 30 Amberson Ave., Yonkers, NY 10705-3613.

On the Indeterminate Form 0^0

Leonard J. Lipkin (llipkin@unf.edu), University of North Florida, Jacksonville, FL 32224

All calculus books have a section on L'Hôpital's Rule and its application to study various indeterminate forms. One of the forms is "0^0." In most books, examples such as $\lim_{x\to 0^+} x^x$ or $\lim_{x\to 0^+} (\tan x)^{\sin x}$ invoke logarithms to show that the limit is 1. Typically the same thing happens in the exercises given for this indeterminate form. So the question arises: Why is this form indeterminate if the limit is always 1?

In this capsule, we point out why the typical examples have limit 1, and we show how to construct examples with limits other than 1. As we will see below, the key is that most examples in calculus courses are either (real) analytic functions or are products of analytic functions with x^α for some $\alpha > 0$. The two theorems below show why most of the limits that appear in the calculus books are 1.

Theorem 1. Let $f(x) = x^\alpha \phi(x)$, where $\alpha > 0$ and $0 < m \le \phi(x) \le M < \infty$ for constants α, m, and M. Let $g(x)$ be any function that satisfies $\lim_{x\to 0^+} g(x) = 0$ and $\frac{g(x)}{x}$ bounded on $(0, a)$ for some $a > 0$. Then $\lim_{x\to 0^+} f(x)^{g(x)} = 1$.

Proof. Using standard techniques, we obtain

$$\begin{aligned}\log f(x)^{g(x)} &= g(x)\log f(x)\\ &= g(x)\,(\log x^\alpha + \log\phi(x))\\ &= \frac{g(x)}{x}\alpha x\log x + g(x)\log\phi(x).\end{aligned}$$

Since $\frac{g(x)}{x}$ is bounded and $x\log x \longrightarrow 0$ as $x \longrightarrow 0$, the first term has limit 0. Since $\lim_{x\to 0^+} g(x) = 0$ and $\log\phi(x)$ is bounded (by hypothesis on ϕ), the second term has limit 0. Exponentiation yields the conclusion $\lim_{x\to 0^+} f(x)^{g(x)} = 1$. ■

In Theorem 1, we assumed that $g(x)/x$ is bounded near 0. This assumption rules out, for example, $g(x) = \sqrt{x}$. Theorem 1 is actually a special case of the following more general theorem. The proof is similar to that given before.

Theorem 2. Let $f(x)$ be as in Theorem 1 and let $g(x) = x^\beta \gamma(x)$, where $\beta > 0$ and $\gamma(x)$ is bounded on $(0, a)$. Then $\lim_{x\to 0^+} f(x)^{g(x)} = 1$.

Theorems 1 and 2 take care of the analytic functions and much more. Notice that differentiability is not assumed. For example, we can take $g(x) = x^\beta Q(x)$, where $\beta > 0$ and $Q(x)$ is the characteristic function of the rationals. We now know where *not* to look in order to find cases in which $\lim_{x\to 0^+} f(x)^{g(x)} \neq 1$. The following examples show that any limit may occur.

Example 1. Let $f(x) = x$ and $g(x) = 1/\ln(x)$ for $x > 0$. Since the function

$$f(x)^{g(x)} = x^{1/\ln(x)}$$

has the property that $\ln(x^{1/\ln(x)}) = 1$ for all $x > 0$, we see that $\lim_{x\to 0^+}(x^{1/\ln(x)}) = e$. Therefore, $\lim_{x\to 0^+} x^{1/\ln(x^\tau)} = e^{1/\tau}$ for every $\tau > 0$. More generally, let $f(x) = x^\alpha \phi(x)$ where $\alpha > 0$ and $0 < m \leq \phi(x) \leq M < \infty$ for some positive constants α, m, and M. Then

$$\lim_{x\to 0^+} f(x)^{1/\ln(x^\tau)} = e^{\alpha/\tau}.$$

Example 2. Let $f(x) = e^{-1/x}$ and $g(x) = -1/\ln(x)$. Then

$$\lim_{x\to 0^+} f(x)^{g(x)} = \lim_{x\to 0^+} e^{1/(x\ln(x))} = 0.$$

On the Work to Fill a Water Tank

Robert R. Rogers (rogers@cs.fredonia.edu), SUNY College at Fredonia, Fredonia, NY 14063

In a second semester calculus class, a problem arose in which one had to compute the work required to fill a cylindrical water tank from a water source below the tank through a filler pipe at the bottom of the tank (Figure 1). See [**1**, p. 426] for a similar problem.

The typical solution (Figure 2) is to divide the water in the tank into (infinitely) thin horizontal slices and compute the work to lift each slice of water into place (as though it was a solid). In particular, the work to lift a slice of water of thickness dy to the level y units above the base of the tank is $(w\pi R^2\,dy)(y + a)$, where w is the weight density of water. Therefore, the total work required is $\int_{y=0}^{h}(w\pi R^2(y + a))\,dy$.

After we did the problem, a student asked essentially: "In computing the work to lift the slice of water, we said that the force was the weight of the slice of water. Isn't it true that as the tank fills up, the force at the top of the filler pipe becomes greater which means that it takes more work to fill slices of the tank as the water level rises?" I am sure that he was not the first person to have raised this question. My explanations why

